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ON MEASURING THE SENSITIVITY OF THE OPTIMAL PORTFOLIO ALLOCATION

In this paper we consider the sensitivity problem connected with portfolio optimization results when different measures of risk such as portfolio rates of return standard deviation, portfolio VaR, CVaR are minimized. Conditioning the data (represented by spectral condition index of the rates of return correlation matrix) plays, as it is shown, a crucial role in describing the properties of the models. We report on the research conducted for 13 largest firms on Warsaw Stock Exchange.

Keywords: *portfolio selection, Value-at-Risk, conditional Value-at-Risk*

1. Introduction – some remarks on consequences of data matrix conditioning on minimum portfolio variance model solution

Let $\mathbf{R} = [R_{ij}]$ $i = 1, \ldots, N, j = 1, \ldots, T$ be the matrix of N time series of the rates of return on stocks. The matrix $\mathbf{R}$ is then standardized – $\mathbf{R_s}$ is the standardized matrix of the rates of return $\mathbf{R_s} = [R_{\mathbf{s}ij}]$.

We can represent a sample linear correlation matrix $\mathbf{P}$ as follows

$$\mathbf{P} = \frac{1}{T}\mathbf{R}_\mathbf{s}^\mathbf{T}\mathbf{R}_\mathbf{s}\,. \tag{1}$$

Let $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ be the diagonal matrix, with eigenvalues of $\mathbf{P}$ on the main diagonal and $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_N]$ be the matrix of normalized eigenvectors related to $\lambda_1, \lambda_2, \ldots, \lambda_N$, $\mathbf{V}^\mathbf{T}\mathbf{V} = \mathbf{V}\mathbf{V}^\mathbf{T} = \mathbf{I}_N$ (see Ralston [9] and Wilkinson, Reinsch [11] for analytical and numerical procedures).

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Then

$$\mathbf{\Lambda} = \mathbf{V}^T\mathbf{P}\mathbf{V} = \frac{1}{T}\mathbf{V}^T\mathbf{R}_s^T\mathbf{R}_s\mathbf{V} \tag{2}$$

and j-th eigenvalue can be represented in a following way

$$\lambda_j = \frac{1}{T}(\mathbf{R}_s\mathbf{v}_j)^T(\mathbf{R}_s\mathbf{v}_j) = \frac{1}{T}\sum_{i=1}^{T}\left(\sum_{l=1}^{N} R_{sil}v_{lj}\right)^2, j = 1, ..., N\,. \tag{3}$$

If any eigenvalue of **P**, for instance λ_p, equals zero it means that

$$\lambda_p = 0 \Rightarrow \sum_{i=1}^{T}\left(\sum_{l=1}^{N} R_{sil}v_{lp}\right)^2 = 0\,,$$

and furthermore

$$\begin{aligned} &\sum_{l=1}^{\mathrm{N}} R_{sil}v_{lp} = 0\,, i = 1, ..., T, \\ &\mathbf{R}_s\mathbf{v}_p = \mathbf{0}. \end{aligned}$$

The above identity defines a precise linear relationship between rates of return series. The parameters of this relationship are the elements of an eigenvector corresponding to the zero eigenvalue.

Each eigenvalue equal to zero defines one linear relationship among analyzed series. In practice, we rarely observe precise multicollinearity. Much more often we are faced with numerical difficulties connected with ill-conditioning of data due to occurrence of “close to zero” eigenvalues. One of the proposed measures of the strength of the interdependence of time series is the number of “close to zero” (or less than one) eigenvalues of the **P** matrix. Condition index of the correlation matrix can be defined as a ratio of its largest to smallest eigenvalue. It was shown in Belsley, Kuh, Welsch [2] by simulation experiments in which progression 3, 10, 30, 100, 300, 1000... of condition indices corresponds roughly to the progression 0.9, 0.99, 0.999, 0.9999, ... of multiple correlation coefficients for a set of explanatory variables in linear regression model. In this work we try to look for consequences of linear interdependencies among time series of rates of return for the optimal portfolio model solution. First results of similar analysis helpful in optimal investment portfolio construction, where optimality criterion is defined as variance minimization, were presented in Konarzewska [7].

Portfolio variance can be presented as follows:

$$\sigma_P^2 = \mathbf{X}^T\mathbf{\Sigma}\mathbf{X}\,, \tag{4}$$

where x_i, $i = 1, ..., N$; $\mathbf{x} = [x_1, ..., x_N]$, weight (fraction of wealth invested) of i-th security in the portfolio, $\sum_{i=1}^{N} x_i = 1$, $x_i \geq 0$,

N – number of securities in the portfolio,

σ_i^2 – variance of i-th security rate of return,

σ_{ij} – covariance of i-th and j-th security rates of return,

$\mathbf{\Sigma} = [\sigma_{ij}]$ – $N \times N$ covariance matrix of rates of return.

Applying singular value decomposition of covariance matrix we obtain:

$$\sigma_P^2 = \sum_{j=1}^{N} \tilde{\lambda}_j \omega_j^2 , \tag{5}$$

where

$\omega = \mathbf{x}^T \operatorname{diag}(\sigma_i)\tilde{\mathbf{V}}$, $\quad \omega_j = \sum_{l=1}^{N} x_l \sigma_l \tilde{v}_{lj}$,

$\tilde{\lambda}_j$, $j = 1, \ldots, N$ are ordered in descending order of the covariance matrix eigenvalues,

$\tilde{\mathbf{v}}_j = [v_{lj}]$, $l, j = 1, \ldots, N$ – eigenvector of covariance matrix corresponding to $\tilde{\lambda}_j$,

$\tilde{\mathbf{V}} = [\tilde{\mathbf{v}}_1, ..., \tilde{\mathbf{v}}_N]$ – matrix of eigenvectors corresponding to $\tilde{\lambda}_j$.

Similar decomposition of portfolio variance as in (5) can be made based on correlation matrix $\mathbf{P}$ instead of covariance matrix $\mathbf{\Sigma}$. From the above formula it is easy to conclude that the identification of strong linear interdependencies ($\tilde{\lambda}_j = 0$ or $\lambda_j = 0$) is very important from the point of view of investment risk minimization. Our opinion is that securities which have the property of forming linear relationships are much desired in investment portfolio. The other consequence of the relation (5) is that portfolio variance minimization algorithm will choose portfolio weights to set the impact of the largest eigenvalue close to zero, if possible, that is $\omega_1^2 = 0$.

We present two examples of correlation matrices showing different strength of interdependence of the data set to illustrate the portfolio variance function shape as we change weights of portfolio components. Data in the first example were generated as univariate random variables. Correlation matrix and optimization results are summarized in Table 1. Condition index equal to the ratio of the largest to the smallest eigenvalue of the correlation matrix is about 1.55 – so data are almost orthogonal – minimal strength of interdependence.

Table 1. Example 1. Correlation matrix for almost orthogonal data and optimal portfolio weights

Portfolio component	1	2	3	Eigenvalues	Eigenvectors			Optimal weights
1	1.0000	0.0671	0.0133	1.22	–0.2538	0.9498	0.1829	0.2362
2	0.0671	1.0000	0.2051	0.9922	–0.6998	–0.0498	–0.7126	0.2321
3	0.0133	0.2051	1.0000	0.7878	–0.6677	–0.3088	0.6773	0.5317

Corresponding shape of portfolio variance function presented in Figure 1 is almost conical. The results of portfolio optimization – optimal weights are consequently unique.

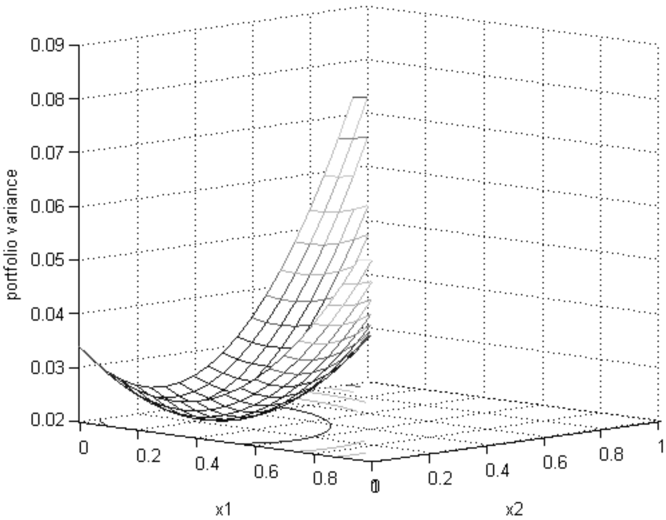

Fig. 1. Conical shape of portfolio variance – 3-components portfolio

Next example represents strong interdependence among data series. Correlation matrix spectral decomposition and optimization results are summarized in Table 2.

Table 2. Example 2. Correlation matrix for strongly interdependent data and optimal portfolio weights

Portfolio component	1	2	3	Component standard deviation	Eigen-values	Eigenvectors			Optimal weights
1	1.0000	0.9875	–0.9855	3.0277	2.9743	–0.5771	0.7636	–0.2896	0.3077
2	0.9875	1.0000	–0.9885	4.0879	0.0146	–0.5777	–0.1310	0.8057	0.2228
3	–0.9855	–0.9885	1.0000	3.9101	0.0111	0.5773	0.6322	0.5167	0.4695

The plot in Figure 2 shows completely different conditioning of portfolio variance minimization problem. Condition index of correlation matrix generated in Example 2 is about 268.7 which indicates strong interdependence of data series. The resulting shape shows that although precise numerical algorithms can have no problem to find the optimum the result is almost not unique – there are numerous different portfolio structures with similar portfolio variance as the minimal one. For the analyzed in Example 2 three-component portfolio the weights for all "near" optimal portfolios lie on a straight line. The analytical formula for this line can be approximately found applying formula (5) – we can check that minimization of portfolio variance corresponds to such component weights that the ω_1^2 is close to zero. It means that in this case, which represents an extremely strong linear relationship among data series, the impact of the largest eigenvalue for the portfolio variance is extremely reduced.

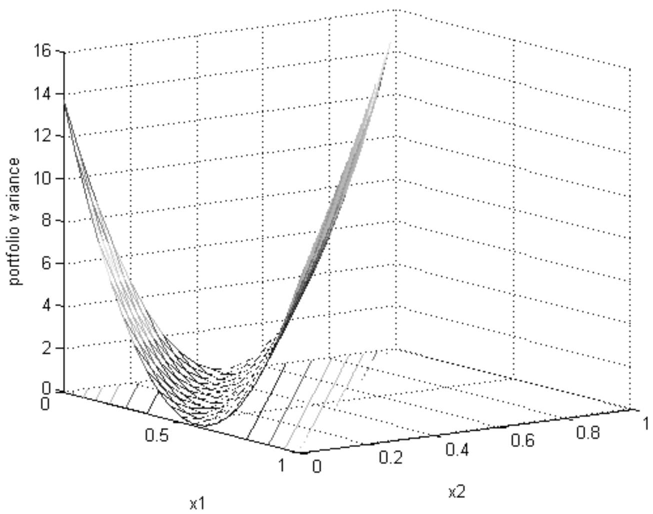

Fig. 2. Tunnel shape of portfolio variance function for different weights of 3-component portfolio – strong interdependence among series of data

We found the following expression for the fitted regression line between x_1 and x_2 in the bottom of the tunnel in Figure 2 using least squares estimation method:

$$\hat{x}_2 = -1.0892x_1 + 0.5532 .$$

The determination coefficient for the above regression line was equal to 0.9922 so the relation between weights is very strong. Portfolio variances for the weights corresponding to this line are very close to each other and to the optimal solution presented in Table 2. Thus the minimum variance portfolio model solution is in this sense, taking into account also that portfolio model parameters (covariances) are estimated with statistical errors, not unique.

The problem of alternate optimal portfolio structures for minimum variance portfolio model when data series are linearly interdependent was an inspiration to check to what extent it was present on real stock exchange market and if it could be also observed when another risk measure, such as CVaR, was taken into account. We analyzed Warsaw Stock Exchange data during the period of 2003 to the end of April 2006.

2. Value-at-Risk and Conditional Value-at-Risk

Let $\mathbf{r} = [r_1, r_2, ..., r_N]$ denotes a vector of portfolio components rates of return and $R_p = \mathbf{x}^{\mathrm{T}}\mathbf{r}$ denotes portfolio rate of return with cumulative distribution function $F(\mathbf{x}, u) = P\{R_p \le u\}$, and $(1 - \alpha) \times 100\%$ is the chosen confidence level.

We accept the following definition of the Value-at-Risk: an α-quantile of the portfolio rate of return distribution with changed sign (or $(1 - \alpha)$-quantile of portfolio losses), that is:

$$\mathrm{VaR}(\mathbf{x},\alpha) = -\min\{u \in \mathcal{R} : F(\mathbf{x},u) \ge \alpha\} = -F^{-1}(\mathbf{x},\alpha)\,. \tag{6}$$

Conditional Value-at-Risk CVaR is in this work defined as the conditional expected loss for losses greater or equal to VaR:

$$\mathrm{CVaR}(\mathbf{x},\alpha) = -E[R_p(\mathbf{x}) \,|\, R_p(\mathbf{x}) \le -\mathrm{VaR}(\mathbf{x},\alpha)]\,. \tag{7}$$

Statistical problems of VaR and consequently CVaR estimation arise when we have difficulty in process generating rates of return identification. The reason is usually a lack of data from the left tail of portfolio rate of return distribution, non-constant mean and/or variance, etc. In this research we applied historical simulation for evaluating portfolio VaR. This means that we assumed that future changes in component rates of return will look like the past ones. Another approach is to estimate parameters of chosen time series model like GARCH(1,1) for portfolio components and calculate appropriate forecasts of the rates of return. Other methods apply multivariate normality assumption, application of copulae, statistical verification of probability distribution including extreme statistics, interpolation, Monte Carlo simulation, resampling methods. Jorion [6] gives some prescriptions for calculations. It has to be mentioned here that VaR is not the ideal measure of investment risk – it lacks subadditivity property (see Pflug [8]) for non-normal empirical distributions which are quite often asymmetric. Moreover, it is not a convex smooth function of **x** so it is hard to apply classical methods of optimization based on gradient function. In the literature we can find examples of data-driven optimization heuristics like threshold acceptance algorithm applied and presented in Gilli et al. [4] where we can find also references to other algorithms. CVaR is a measure which satisfies the properties of coherent measure of risk – see Artzner et al. [1]. As we can check in theory and practice, CVaR is much easier to optimize – even linear programming techniques can be applied (see Rockafellar and Uryasev [10]).

In empirical research we give report about it in §5, the chosen confidence level was equal to $1 - \alpha = 0.95$. Optimal portfolio allocation minimizing CVaR was performed with Excel Solver.

3. Measuring sensitivity of portfolio structures

Sensitivity analysis of risk measured by VaR with respect to portfolio allocation was elaborated by Gaivoronski and Pflug [3], Gourieroux et al. [5]. Optimal portfolio structure is the main result we are looking for in solving various optimization problems. The question is how much the structures differ when we change the assumptions about model parameter values (standard deviations of the rates of return, covariances, skewness of the distribution instead of symmetry etc.). To compare structures, we propose two measures of distance between vectors of portfolio weights:

- Angular distance (based on sine function), equal to one for orthogonal vectors

$$d_A = \sqrt{1 - \frac{\left(\sum_{j=1}^{N} x_{1j} x_{2j} \right)^2}{\sum_{j=1}^{N} x_{1j}^2 \sum_{j=1}^{N} x_{2j}^2}} .$$

- Scaled Euclidean distance (notice that maximal distance equal to one is in the case of orthogonal unit weight vectors)

$$d_E = \sqrt{\frac{\sum_{j=1}^{N} (x_{1j} - x_{2j})^2}{2}} .$$

We applied these measures in empirical research to compare optimal portfolios calculated for different approaches called:

• Optimistic, when we assume the security was bought at the session at minimal price and sold at maximal noticed price.

• Neutral, when we consider final prices for sessions when the security was bought and sold.

• Pessimistic, when we assume buying at maximal and selling at minimal registered price.

Optimal allocations for two different models with non-negative weights and no restrictions at minimal portfolio rate of return were compared:

• Minimization of portfolio variance,

• Minimization of portfolio CVaR.

At the same time we kept control of conditioning of correlation matrices.

4. Selected results from empirical research

The empirical analysis took into account the period between January 2003–April 2006. The objects were 10-days rates of return on 13 securities – the largest firms on the Warsaw Stock Exchange. We distinguished 7 subsamples which were analyzed separately. Table 3 summarizes data conditioning including the number of greater than zero correlation matrix eigenvalues, maximal eigenvalues and condition indices.

Table 3. Conditioning of data

Time period	Optimistic approach			Neutral approach			Pessimistic approach		
	No. > 0	Max. eigenvalue	Cond. index	No. > 0	Max. eigevalue	Cond. index	No. > 0	Max. eigenvalue	Cond. index
2003 1st half	4	4.9474	82.40	4	4.9905	71.24	4	5.2448	72.28
2003 2nd half	2	8.7003	171.61	2	8.7405	190.36	2	8.9064	199.56
2004 1st half	3	6.3743	85.69	3	6.4504	91.88	3	6.4338	89.11
2004 2nd half	5	3.8628	45.66	4	3.7367	35.52	4	3.7450	40.88
2005 1st half	5	5.9485	66.85	4	5.9507	60.20	4	5.9695	64.70
2005 2nd half	5	3.9622	30.81	4	4.2576	32.74	5	4.4885	33.45
2006 I–IV	5	4.7090	65.38	5	5.0482	70.83	5	5.1184	65.58

As we can see the condition indices for correlation matrices for the analyzed series were observed between 30.81 to 199.56. It means that series of rates of return observed are in general interdependent for all three approaches.

The first analyzed model was the minimum variance portfolio model. The results of sensitivity are presented in Table 4. For each sub-period we estimated correlations, standard deviations.

Looking at the values of distances between portfolios calculated for extremely different approaches in Table 4 we can see that the optimal solutions lie not far from each other – distances are rather small as well as standard deviations. At the same time mean rates of return differ much which is not surprising. Concluding – minimum variance portfolios were not sensitive on even not so small changes in data matrices. The period of time when we observed the strongest sensitivity was 1st half of 2004. The weakest sensitivity of the results wase for the 2nd half of 2003. For both mentioned periods we observed strong interdependence of data series.

Table 4. Results for minimum variance optimal portfolios

Time period	Approach	Portfolio standard deviation	Portfolio mean rate of return	Angular distance OPT vs PES	Euclidean distance OPT vs PES
2003 1st half	OPT	2.53%	2.57%	0.2810	0.1039
	NEUTRAL	2.47%	0.81%		
	PES	2.40%	–1.00%		
2003 2nd half	OPT	5.00%	4.82%	0.0402	0.0298
	NEUTRAL	5.13%	2.24%		
	PES	5.23%	–0.70%		
2004 1st half	OPT	2.88%	2.68%	0.3388	0.1115
	NEUTRAL	2.96%	0.17%		
	PES	3.00%	–2.29%		
2004 2nd half	OPT	1.51%	3.17%	0.1650	0.0506
	NEUTRAL	1.60%	1.22%		
	PES	1.52%	–0.80%		
2005 1st half	OPT	2.21%	1.85%	0.1310	0.0556
	NEUTRAL	2.19%	–0.25%		
	PES	2.23%	–2.25%		
2005 2nd half	OPT	1.80%	4.57%	0.2466	0.0745
	NEUTRAL	1.87%	2.11%		
	PES	1.96%	–0.36%		
2006 I–IV	OPT	2.18%	4.27%	0.2167	0.0734
	NEUTRAL	2.27%	1.16%		
	PES	2.13%	–1.56%		

Minimum CVaR portfolios are much more sensitive. The results are given in Table 5. The angular distances show almost orthogonal portfolio structures obtained for optimistic and pessimistic approach for 2nd half of 2003. Negative values of VaR and CVaR were sometimes observed for optimistic approach. It is worth mentioning that minimum CVaR portfolios, beside their more visible sensitivity on changes in data series, demonstrated positive skewness in almost all cases.

Table 6 presents distances between optimal portfolio weight vectors calculated with the minimum variance and minimum CVaR criteria. They are quite far from each other – for 2nd half 2003 almost orthogonal.

Table 5. Results for minimum CVaR portfolios

Time period	Approach	**VaR**	**CVaR**	**rr**	**Std**	**Skewness**	**Angular distance**	**Euclidean distance**
2003 1st half	OPT	0.1791%	0.2975%	4.37%	3.36%	0.4228	0.3531	0.1478
	NEUTRAL	1.5663%	2.3326%	2.75%	3.37%	0.5105		
	PES	3.7184%	4.4651%	0.64%	3.17%	0.4767		
2003 2nd half	OPT	2.8904%	3.7336%	7.96%	7.74%	1.0128	0.9737	0.5541
	NEUTRAL	5.5590%	6.7531%	4.32%	6.74%	0.3690		
	PES	9.3567%	10.7497%	–1.08%	5.47%	0.0684		
2004 1st half	OPT	1.4913%	1.7941%	5.14%	4.68%	0.7845	0.2768	0.0975
	NEUTRAL	4.4824%	4.9826%	2.63%	4.95%	0.7455		
	PES	6.6717%	7.5803%	–0.67%	4.01%	0.3950		
2004 2nd half	OPT	–0.9307%	–0.7460%	3.31%	1.54%	0.5084	0.4990	0.2147
	NEUTRAL	1.1210%	1.3903%	1.49%	1.72%	0.3237		
	PES	3.3679%	3.5960%	–0.54%	1.68%	0.3690		
2005 1st half	OPT	0.9301%	1.1250%	2.84%	3.01%	0.5646	0.5994	0.2084
	NEUTRAL	3.1406%	3.7725%	0.55%	2.83%	0.4824		
	PES	5.6767%	5.8544%	–1.85%	2.82%	0.4067		
2005 2nd half	OPT	–1.6819%	–0.8287%	5.42%	2.44%	0.1924	0.6981	0.3131
	NEUTRAL	1.3775%	1.8099%	2.49%	2.25%	-0.1077		
	PES	4.5375%	4.8344%	0.11%	2.33%	-0.5029		
2006 I–IV	OPT	–2.4879%	–1.9588%	7.66%	3.62%	0.2788	0.3287	0.1379
	NEUTRAL	1.1433%	1.5665%	3.23%	2.99%	0.5419		
	PES	3.6853%	4.1488%	–0.56%	2.56%	0.4894		

To give a deeper insight into the problem of the impact of data ill-conditioning on portfolio optimization with different criteria, we have chosen two sample periods and neutral rates of return for only three series and composed optimal portfolios. Values of statistical characteristics were calculated changing contribution of assets from 0 to 1 with 0.05 step. The first period was 2nd half 2003 – the highest interdependence observed. For selected data condition index was equal to 21.54 with maximal eigenvalue equal to 2.414 (for three components portfolio sum of eigenvalues is equal to three). Figures 3–6 present the following relations:

Fig. 3 mean portfolio rate of return vs portfolio standard deviation,

Fig. 4 mean portfolio rate of return vs portfolio VaR,

Fig. 5 mean portfolio rate of return vs portfolio CVaR,

Fig. 6 portfolio CVaR vs portfolio standard deviation.

Table 6. Distances between minimum variance and minimum CVaR portfolios

Time period	Approach	Angular distance	Euclidean distance
2003 1st half	OPT	0.8644	0.3691
	NEUTRAL	0.8010	0.3557
	PES	0.7563	0.3286
2003 2nd half	OPT	0.9992	0.6767
	NEUTRAL	0.9834	0.6218
	PES	0.3216	0.1925
2004 1st half	OPT	0.9762	0.4203
	NEUTRAL	0.9850	0.4537
	PES	0.9192	0.3702
2004 2nd half	OPT	0.2525	0.0789
	NEUTRAL	0.4141	0.1386
	PES	0.5501	0.1795
2005 1st half	OPT	0.7472	0.3003
	NEUTRAL	0.6309	0.2622
	PES	0.5490	0.2241
2005 2nd half	OPT	0.7494	0.2824
	NEUTRAL	0.5518	0.1986
	PES	0.6823	0.2166
2006 I–IV	OPT	0.7005	0.2958
	NEUTRAL	0.6822	0.2642
	PES	0.5162	0.1935

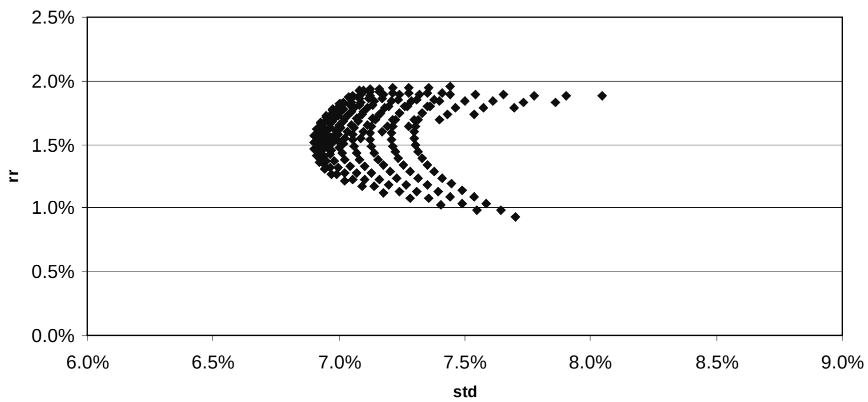

Fig. 3. Mean portfolio rate of return vs. portfolio standard deviation – 3-component portfolios for 2nd half 2003

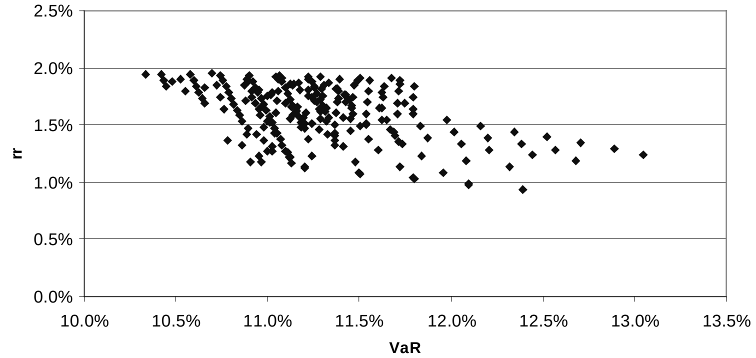

Fig. 4. Mean portfolio rate of return vs portfolio VaR – 3-component portfolios for 2nd half 2003

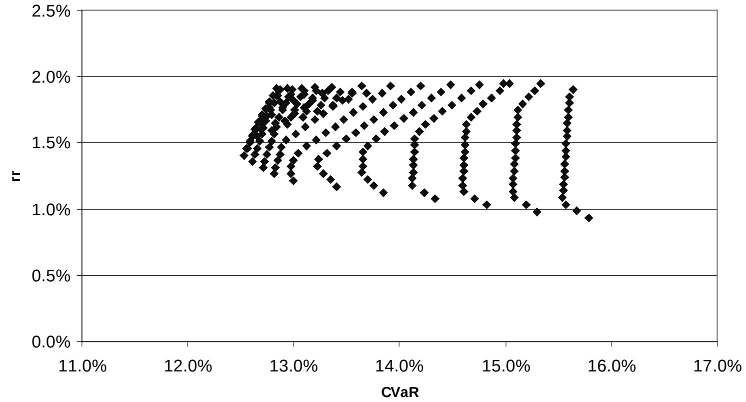

Fig. 5. Mean portfolio rate of return vs portfolio CVaR – 3-component portfolios for 2nd half 2003

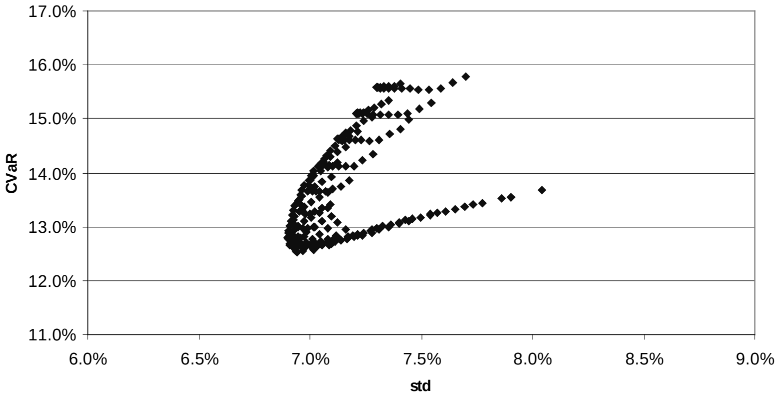

Fig. 6. Portfolio CVaR vs portfolio standard deviation – 3-component portfolios for 2nd half 2003

Trying to comment on the results on Figure 3 we have to notice that for ill-conditioned data minimization of portfolio variance, quadratic programming problem, meets many different solutions with similar values of the objective function. Trying to find portfolio minimizing VaR, as it can be noticed on Figure 4, one is faced with numerical problems. We cannot find something similar to the efficient frontier – non-dominated portfolios as in the former case. Unique portfolio minimizing VaR at the graph is at the same time maximizing mean rate of return. Figure 5 illustrates linear steep shape of the efficient frontier when we minimize CVaR with constraint on minimal satisfying value of mean rate of return. Although there exists further possibility of increasing rate of return from 1.90% to about 1.95% it involves considerable increase of CVaR from 12.84% to 14.99%. Figure 6 relates portfolio CVaR to portfolio standard deviation. We can observe that minimal values of both risk measures focus in a small region. The structures of the two optimal portfolios: first minimizing CVaR and second minimizing variance, and distances between them are as follows:

$$\mathbf{x}_{\text{CVaR}} = [0 \quad 0.5058 \quad 0.4942],$$

$$\mathbf{x}_{\text{variance}} = [0.2031 \quad 0.4067 \quad 0.3902],$$

$$d_A = 0.3392,$$

$$d_E = 0.1759.$$

As we can observe optimal portfolios are quite neighbouring. Figure 6 illustrates, however, that there does not exist a relationship between this two measures – the same value of standard deviation is observed for portfolios with very distant values of CVaR and vice versa.

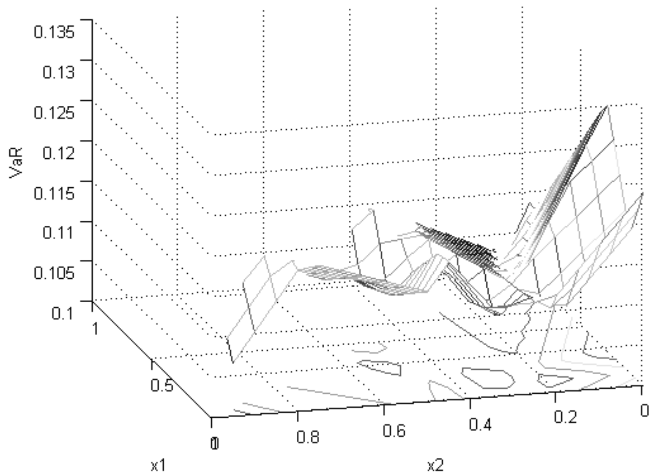

Fig. 7. Portfolio VaR for 2nd half 2003 – example of 3-component portfolio

Figures 7 and 8 illustrate portfolio VaR and CVaR surface for different allocations. In the case of VaR we can observe non convex, multimodal shape. Surface drawn by CVaR is smooth, easy to look for optimal allocation. However, the bottom of the surface is flat which suggests the possibility of different structures with similar small CVaR characteristics.

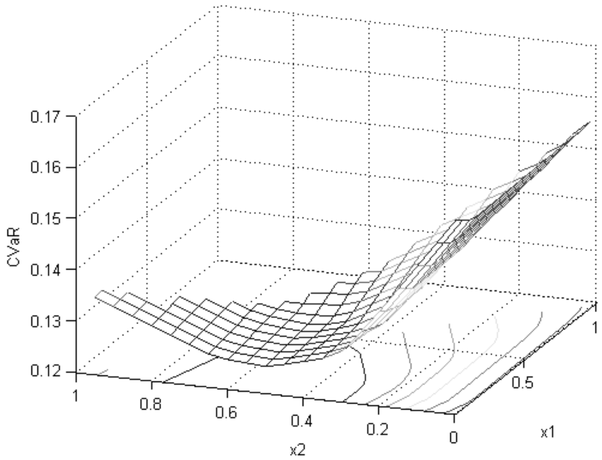

Fig. 8. Portfolio CVaR for 2nd half 2003 – example of 3-component portfolio

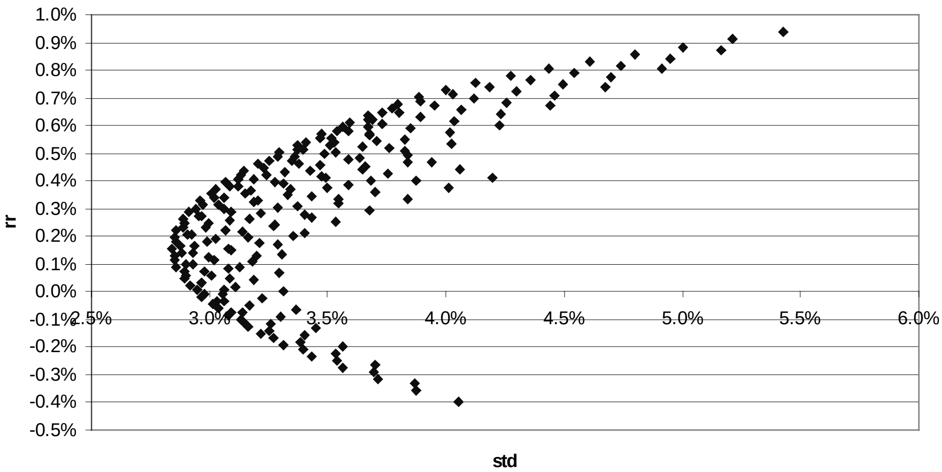

Fig. 9. Portfolio mean rates of return vs standard deviation – 3-component portfolio for 2nd half 2004

Second chosen period was 2nd half 2004. Figures 9–12 correspond to Figures 3–6 for the former example. Conditioning of 3-component portfolio data matrix is much better – condition index was equal to 1.67 with maximal eigenvalue equal to 1.26.

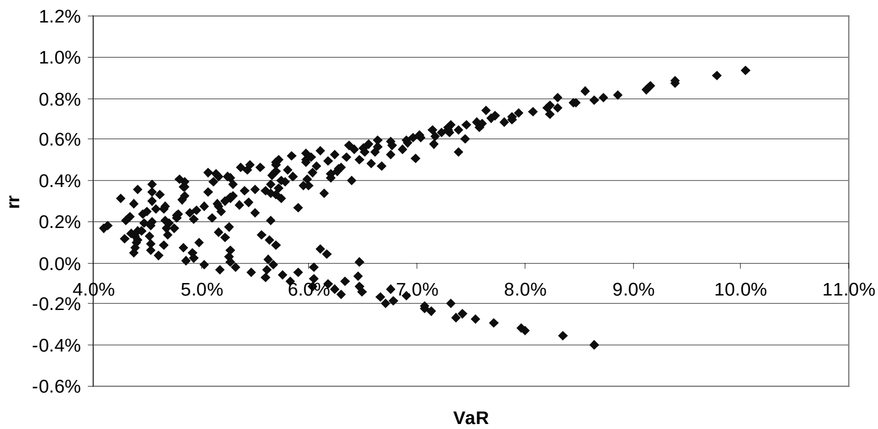

Fig. 10. Portfolio mean rates of return vs. VaR – 3-component portfolio for 2nd half 2004

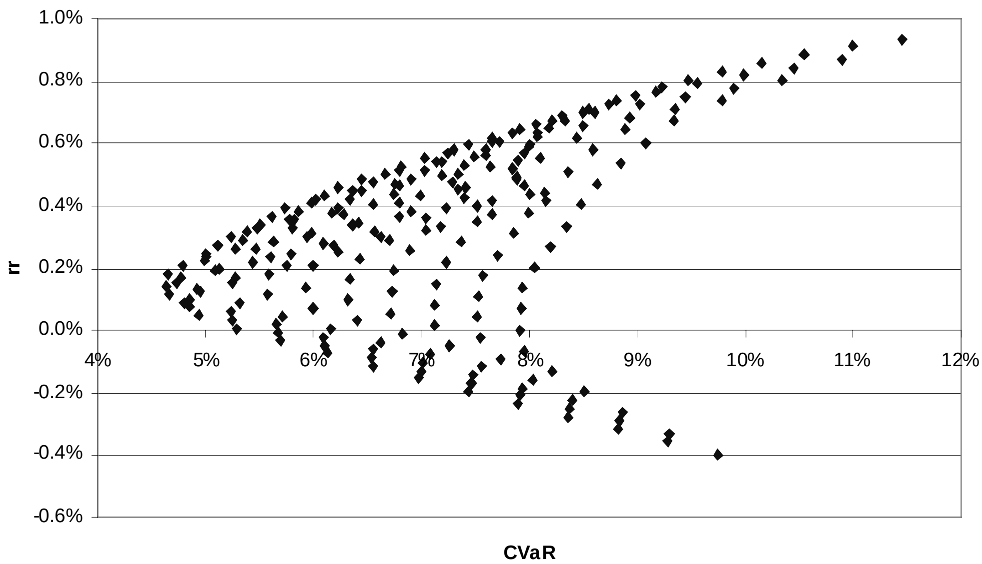

Fig. 11. Portfolio mean rate of return vs. CVaR – 3-component portfolio for 2nd half 2004

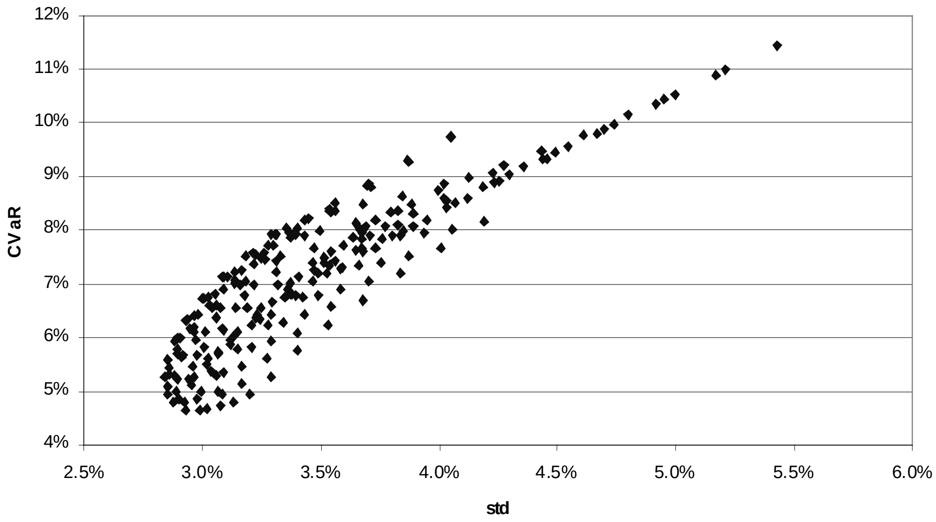

Fig. 12 Portfolio CVaR vs standard deviation – 3-component portfolio for 2nd half 2004

We can easily see the differences between the relations illustrated on corresponding graphs for the two chosen periods:

• we do not observe grouping of portfolios near minimal value of standard deviation in Figure 9 comparing it with Figure 3,

• VaR for portfolios presents much smoother behaviour than in the former example,

• the shape of the efficient frontier for minimization of CVaR problem in Figure 11 does not allow increasing the mean rate of return without significant increase in CVaR as presented in Figure 5,

• we observe in Figure 12 almost linear relationship between portfolio CVaR and standard deviation; no functional relation in Figure 6.

The optimal structures of portfolios and distances were as follows:

$$\mathbf{x}_{\text{CVaR}} = [0.0936 \quad 0.3948 \quad 0.5115],$$

$$\mathbf{x}_{\text{variance}} = [0.2000 \quad 0.4461 \quad 0.3539],$$

$$d_A = 0.3002,$$

$$d_E = 0.1397.$$

Comparing the distances between the two structures for nearly orthogonal data with corresponding results for strongly interdependent series of observations we can see only a slight difference – for nearly independent series the distances are a little bit smaller.

Figure 13 presents VaR surface for portfolios – although it is also rough as for former example it shows very different shape – not so many local minima exist and they represent quite similar values of VaR – different portfolio allocations but focused in close region.

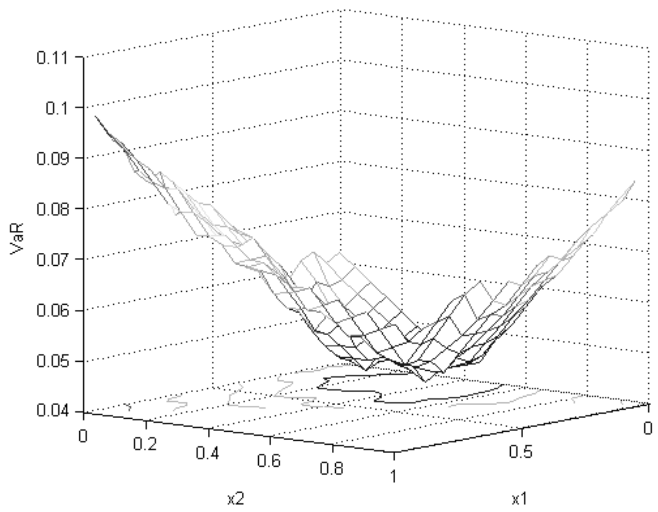

Fig. 13. Portfolio VaR for 2nd half 2004 – example of 3-component portfolio

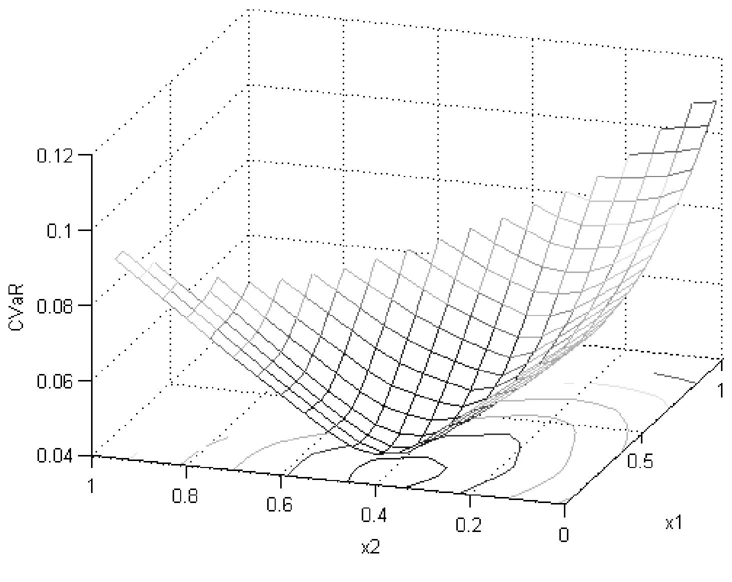

Fig. 14. Portfolio CVaR for 2nd half 2004 – example of 3-component portfolio

The surface of CVaR presented in Figure 14 for this well-conditioned set of data is not so flat – minimum is for sure unique, easy to find.

5. Conclusions

Analyzing data from the Warsaw Stock Exchange we have checked that strong interdependence among series of rates of return on the real stock exchange market is present. The aim of the work was to illustrate how strongly it influences the sensitivity of optimal portfolio structure when two different measures of risk were taken into account – portfolio variance and CVaR. The result for minimum variance model is obvious – quadratic programming problem becomes ill-conditioned and there is a problem of alternate optima which can be located far from each other looking at the optimal portfolio structure. Minimizing CVaR when the data are interdependent we can also identify many distant structures with similar values of minimal risk although global minimum exists. Strong interdependence among series of the rates of return enlarges problem of identifying portfolio structure minimizing VaR – the surface which is always rough with many local extrema becomes much more difficult to analyze with heuristic search methods.

The work gives only first insight into the problem of sensitivity of the optimal portfolio allocation for small changes in data. The next step will need improvements in VaR and CVaR estimation with the help of Monte Carlo simulation preserving covariances and empirical distributions. We plan to check how far optimal allocations are sensitive on wrong identification of data generating process.

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O pomiarze wrażliwości optymalnych struktur portfeli akcji

W artykule podjęto rozważania na temat wrażliwości rozwiązań problemu optymalizacji portfeli akcji, gdy brane są pod uwagę różne miary ryzyka: odchylenie standardowe stóp zwrotu, VaR portfela, CVaR portfela. Uwarunkowanie danych, mierzone za pomocą stopnia uwarunkowania macierzy korelacji stóp zwrotu z akcji, odgrywa, jak pokazano, zasadniczą rolę dla własności rozwiązań modelu minimalizacji wariancji portfela. Model, który jest modelem programowania kwadratowego, jest źle uwarunkowany i pojawia się problem niejednoznaczności rozwiązania optymalnego – wielu strukturom portfeli, nawet bardzo odległym, odpowiada podobna, bliska minimalnej wartości wariancji. Minimalizując CvaR w warunkach współzależności, napotykamy na podobny problem, choć nie występuje on z taką samą siłą. Silna współzależność szeregów stóp zwrotu zwiększa również problemy minimalizacji VaR – nasila się problem występowania wielu lokalnych ekstremów, co powoduje znaczne trudności w stosowaniu metod heurystycznych.

Przeprowadzono badanie empiryczne, biorące pod uwagę 13 największych spółek na GPW w Warszawie. Stwierdzono silną współzależność stóp zwrotu. Zbadano siłę wrażliwości rozwiązań modeli minimalizujących wariancję i CVaR z wykorzystaniem miary odległości kątowej oraz skalowanej odległości euklidesowej. W efekcie badań stwierdzono, że portfele znalezione przy kryterium minimalizacji wariancji nie wykazały dużej wrażliwości na zmiany w macierzy danych. Portfele minimalizujące CVaR okazały się bardziej wrażliwe.

Słowa kluczowe: *optymalizacja portfela akcji, wartość zagrożona ryzykiem VaR, warunkowa wartość zagrożona ryzykiem CVaR*