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ON SOME INFLATION MODEL BASED ON THE PAST PRICE DYNAMICS

The paper describes a model of a price movement path formed upon the past dynamics, following from a couple of assumptions. A system of delayed differential equations has been built and the properties of solutions analyzed. At the end, the main conclusions drawn from the model are presented.

Keywords: *inflation, economic system, a delay vector, a price movement path*

1. Assumptions and model formulation

Let us consider a closed system of two time dependent variables, price level $p(t)$ and time delay $\tau(t)$ describing the price dynamics. Furthermore, we claim that $p, \tau \in C^1(0, \infty)$.

1.1. The inflation equation

In order to define the model we make several assumptions on the economy and agents. Let us first assume the following.

Assumption 1. *The change of current price level is a function of the past values of an index*.

The delay used to estimate current index level is determined by the function $\tau(t)$. Hence, we may rewrite assumption 1 in a mathematical way as follows

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$$\dot{p} = f(p(t - \tau(t))) . \tag{1}$$

As regards the function f we assume that it is regular enough and convenient to analyze the model. Moreover, the dependence between the change and past values is governed by:

Assumption 2. *Not given any information which could influence the price expectations* (*change of* $\tau(t)$), *agents await the price trend to sustain, i.e., facing a high inflation rate agents await its further rise and conversely in the case of a low inflation level agents await its further decrease*.

In terms of function f this means that its first derivative is positive.

More interesting from the agent's point of view seems to be the problem of delay, the real value indicating the time interval from which the agent takes the index values to estimate current price level.

1.2. The delay equation

Market agent uses all available (but still imperfect) information about the economy, hence he considers all market events to estimate current price trend. Taking all this into account we make the following assumption of market participants.

Assumption 3. *Changes of price level cause agents to variate the delay used to forecast current price index, depending on the rapidity of index movement*.

The value of the function $\tau(t)$ reflects mood on the markets. Having full information about past price levels, market participants take decision about the delay. According to assumption 2, agents are guided by the price trend. The change of $\tau(t)$ value is therefore price trend function denoted in our model by $g(\cdot)$. It is also very important for the market agents to know how impetuous the shifts of the observed inflation level are. The rapidity of shifts in the period $\tau(t)$ can be measured with the directional coefficient of the secant through points $(t - \tau(t), p(t - \tau(t)))$ and $(t, p(t))$. Additionally, as time goes by, agents have still better knowledge, hence the delay elongates in a constant measure $\gamma > 0$. We may rewrite the above considerations in a formal way as follows

$$\dot{\tau} = g\left(\frac{p(t) - p(t - \tau(t))}{\tau(t)}\right) + \gamma . \tag{2}$$

The existence of the constant γ has an important interpretation. The situation when the increase of price is rapid enough (the absolute value of the coefficient

$g\left(\frac{p(t)-p(t-\tau(t))}{\tau(t)}\right)$ exceeds γ) agents start to think of the market events as an economic crisis.

Assumption 4. *In case of falling prices, when the decrease accelerates, participants tend to lengthen the delay needed to estimate current price level.*

An average agent is aware of the danger caused by sudden price raise as well as by instability resulting from the growing rate of index decrease. That is why in our model market agent makes use of past price level but the delay depends on the momentum of price change which was observed. When price level is falling, and the decrease rate grows, market participants become more suspicious. They rather do not believe in price stability and even expecting price increase, elongate the historical horizon from which the data used to make forecasts are collected. The above assumptions let us claim on the function $g(\cdot)$ that its first derivative be negative, conversely to as the first derivative of function $f(\cdot)$. In case of accelerating price increase, directional coefficient of the secant grows, $\tau(t)$ becomes smaller, which means the shortening of the delay. It is easy to notice that assumption 4 is also fulfilled.

2. Mathematical analysis of the model

According to assumptions and equations described in section 1, we will consider the following system of equations

$$\dot{p} = f(p(t-\tau(t))), \tag{3}$$

$$\dot{\tau} = g\left(\frac{p(t)-p(t-\tau(t))}{\tau(t)}\right) + \gamma. \tag{4}$$

In the further part of the paper, taking into account claims made on functions $f(\cdot)$ and $g(\cdot)$ we analyze the system mainly for small time values. It is then possible to consider a linearized system

$$\dot{p} = \alpha\, p(t-\tau(t)), \tag{5}$$

$$\tau\dot{\tau} = \beta\, p(t) - \beta\, p(t-\tau(t)) + \gamma\tau(t), \tag{6}$$

where $\beta < 0 < \alpha$, with Cauchy type initial conditions $p(0) = p_0$, $\tau(0) = 0$. For the sake of our considerations let us claim $\gamma + \beta\alpha p_0 > 0$.

The basic question regarding the above system of equations concerns existence of its solutions. Although the two equations form a closed system, we will treat them rather separately paying grater attention to (5). The paper presents a standard existence theorem and a form of Gronwall's inequality.

Theorem 1. *Let* $K = [0, a] \times \{|p - p_0| < b\}$ *and* $M = \sup_p |p - p_0 < b\}$. *Then for each continuous function* $\tau(t)$ *fulfilling the bound* $\forall_{t\in[0,\,T]}\tau(t) \in [0, t)$ *and a constant* $\alpha > 0$ *there exists locally a solution such that* $\forall_{t\in[0,\,a']}(t, p(t)) \in K$

$$\dot{p} = \alpha\, p(t - \tau(t)),$$

$$p(0) = p_0,$$

for $t \in [0, a']$, *where* $a' = \min\left\{a, \dfrac{b}{\alpha M}, \dfrac{c}{2b}\right\}$ *for an arbitrary constant* $0 < c < 1$.

Proof. Let us consider an operator

$$\mathcal{L}(p)(t) = p_0 + \int_0^t \alpha p(s - \tau(s))ds.$$

Notice that $\mathcal{L}$ transforms $C([0, a'], \{|p - p_0| \le b\})$ into itself for

$$|\mathcal{L}p(t) - \mathrm{p}_0| \le \int_0^t \alpha\, |\, p(s - \tau(s))\,|\, ds \le \alpha \frac{b}{\alpha} = b.$$

Moreover operator $\mathcal{L}$ is a contraction. Taking $p_1(t), p_2(t) \in C$, for each $0 \le t \le a$ we have

$$\begin{aligned}|\mathcal{L}(p_1)(t) - \mathcal{L}(p_2)(t)| &\le \int_0^t |p_1(s - \tau(s)) - p_2(s - \tau(s))\,|\, ds \\ &\le c \sup_{s\in[0,t]} |\, p_1(s) - p_2(s)\,|.\end{aligned}$$

The space $C([0, a'], \{|p - p_0| \le b\})$ on which operator $\mathcal{L}$ acts is in supremum metrics complete. Hence following the Banach fixed point theorem there exists exactly one fixed point $p^* = \mathcal{L}(p^*)$. The function p^* is the solution to the problem. ■

According to theorem 1 we proved locally the existence of a solution to (5) having an arbitrary chosen function $\tau(t)$ which fulfills the assumptions of theorem 1.

Lemma 1. (Gronwall's inequality) *For all* $t \in \mathbb{R}^+$ *and an arbitrary function* $\tau(t)$ *fulfilling the bound* $\forall_{t\in[0,\,T]}\tau(t) \in [0, t)$ the following *holds*

$$|p(t)| \le |p_0|e^{\alpha t}.$$

Proof. Assume first $p_0 = 0$. Denote

$$P(t) = |p_0| + \int_0^t \alpha\, |p(s - \tau(s))|\, ds\,.$$

We have $P(t) > 0$. There is also $P'(t) = \alpha|p(t - \tau(t))|$. From (5) we have

$$|p(t - \tau(t))| \le |p_0| + \int_0^{t-\tau(t)} \alpha\, |p(s - \tau(s))|\, ds \le |p_0| + \int_0^t \alpha\, |p(s - \tau(s))|\, ds\,,$$

hence $|p(t - \tau(t))| < P(t)$. Therefore, we get $(\log P(t))' = \dfrac{P'(t)}{P(t)} \le \alpha$, which means

$$\log P(t) - \log P(0) = \int_0^t (\log P(s))' ds \le \alpha t\,.$$

Integrating (5) from 0 to t

$$|p(t)| \le |p_0| + \int_0^t \alpha\, |p(s - \tau(s))|\, ds = P(t)\,,$$

from which we obtain

$$|p(t)| \le P(t) \le |p_0|\, e^{\alpha t}\,. \tag{7}$$

If $p_0 = 0$ we may substitute it with any arbitrary $p_0' > 0$. Then inequality (7) holds for p_0' but because p_0' was chosen arbitrarily, we get $|p(t)| = 0$. ■

Gronwall's inequality let us elongate any local solution obtained from 1, that is, enables us to obtain a global solution to the equation describing function $p(t)$ for any given continuous delay function fulfilling the bound conditions described above.

Gronwall's inequality provides us with one more important result. Under conditions of $\tau(t) \in [0, t)$ the price level cannot increase too rapidly (the maximal growth rate is exponentially bounded with the exponent proportional to constant α).

Assume now that the solution of (5) is regular enough. For t small enough we may expand $p(t)$ in Taylor series around $t = 0$. Taking into account equation $\dot{p} = \alpha p(t - \tau(t))$ and considering initial conditions, equation (6) becomes

$$\tau\dot{\tau} = \beta(p_0 + \alpha p_0 t) - \beta(p_0 + \alpha p_0(t - \tau)) + \gamma\tau\,.$$

After transformations we get

$$\tau\dot{\tau} = (\gamma + \beta\alpha p_0)\tau\,. \tag{8}$$

Substituting $\frac{1}{2}\tau^2 = h$, we obtain

$$\dot{h} - (\gamma + \beta\alpha\, p_0)\sqrt{2}h^{\frac{1}{2}} = 0\,.$$

Now, we have got a Bernoulli equation. Hence, given an inflation level function $p(t)$, we are able to find locally also a suitable delay function.

As equation (8) does not depend on the choice of unknown function $p(t)$, we may point the delay function $\tau_1(t)$ for small values of t. Then, making use of the $\tau_1(t)$ we are able to solve (5) for $p(t)$ and returning to (6) we can consider new initial conditions shifted by this small t and once again solve (8). This way we are able to construct an approximate local solution to the system where the approximation is $0(t^2)$.

3. Conclusions

In section 3 we described basic facts about the existence of solutions to the system of equations formed on the basis of assumptions from section 2. In the following part we consider phenomena that may occur in the economy described with the above model.

Corollary 1. *Given a delay function $\tau(t)$, the inflation path formed mainly by this factor, is unique.*

If inflation is determined by past values, price dynamics is unique within the delay horizon.

It is necessary for economic stability to avoid unnaturally large changes in the price level within too short period of time, i.e., inflation should move only moderately. Taking into account the results from section 3 we may formulate the following corollary

Corollary 2. *In the case of the economy described by our model, price growth is bounded at most exponentially and depends on initial value.*

In our model we consider inflation problem analyzing consumers' decisions. However, we may treat equation (5) as a tool for monetary policy institutions controlling price movements by different economic instruments. The problem stated in that way enables us to treat the $\tau(t)$ function as an independent value describing not only the delay of the impact of the above tools on the inflation level but also as an aggregate measure containing impact of factors as well as the delay itself. Then, analyzing our problem we may restrict ourselves to consider only the first equation

$$\dot{p} = \alpha\, p(t - \tau(t)) \tag{9}$$

with a given function $\tau(t)$. Figure 1 presents and exemplary price dynamics generated by a discretization of equation (9) and a randomly chosen vector $\tau(t)$.

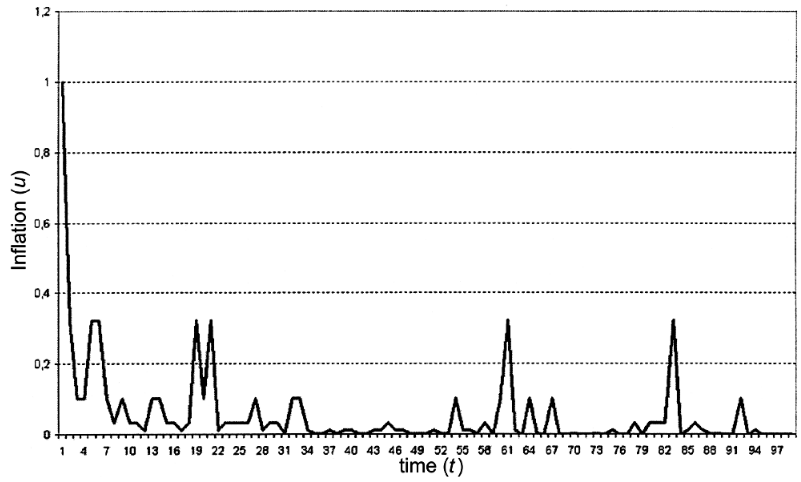

Fig. 1. An exemplary price dynamics formed by a monetary policy institution modeled using equation (9) for 90 periods

The vector of variable $\tau(t)$ contains all information about market conditions and some predictable events. We may modify equation (9) making constant α time-dependent

$$\dot{p} = \alpha(t) p(t - \tau(t)). \tag{10}$$

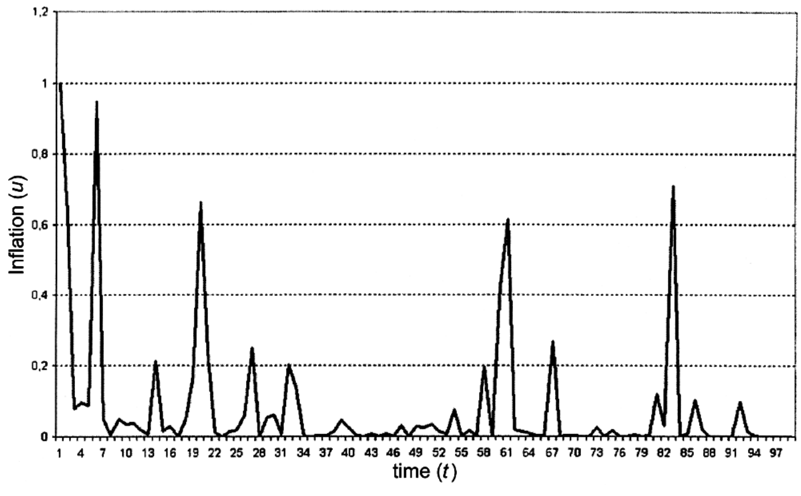

Fig. 2. An exemplary price dynamics formed by a monetary policy institution modeled using equation (10) for 90 periods

This means that the impact of monetary policy institutions on inflation varies in different time periods. The influence of central bank decisions is not constant over time. Figure 2 presents an exemplary discrete price dynamics generated upon equation (10) using a random vector $\alpha(t)$ and a random vector of delay $\tau(t)$. The analysis of function $\tau(t)$ as a tool possessed by monetary policy institutions may be a good starting point for further considerations. The central bank using these tools is able to influence expectations and the behaviour of market agents and this way it may indirectly control inflation level in the economy. Omitting the coupling between functions $\tau(t)$ and $p(t)$ in our model enables us to take advantage of the results pertaining to the theory of equation (9), especially in analyzing oscillatory solutions.

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O pewnym modelu inflacji opartym na dynamice cen w przeszłości

W artykule opisano model inflacji oparty na dynamice cen w przeszłości. Zachowania rynku, które są kluczowym elementem rzutującym na wynikową ścieżkę inflacji, są włączone do modelu w postaci wektora opóźnień $\tau(t)$. Rozważany jest również wariant modelu z losowym parametrem α (wektor $\alpha(t)$), uwzględniający inercję systemu gospodarczego. Przedstawiono analizę matematyczną modelu, a jego działanie zilustrowano rysunkami dla losowego parametru $\tau(t)$ i losowego oraz nielosowego parametru α.

Opisany model może stanowić pomocne narzędzie dla instytucji finansowych, kontrolujących zmiany cen za pomocą różnych instrumentów ekonomicznych.

Słowa kluczowe: *inflacja, system gospodarczy, wektor opóźnień, ścieżka zmiany ceny*