O P E R A T I O N S   R E S E A R C H   A N D   D E C I S I O N S

No. 1 2021

DOI: 10.37190/ord210107

DISCUSSION ON THE TRANSIENT BEHAVIOUR OF SINGLE SERVER MARKOVIAN MULTIPLE VARIANT VACATION QUEUES

MANICKAM VADIVUKARASI, KALIAPPAN KALIDASS*

Department of Mathematics, Karpagam Academy of Higher Education, Coimbatore 641 021, Tamil Nadu, India

We consider an $M/M/1$ queue where beneficiary visits occur singly. Once the beneficiary level in the system becomes zero, the server takes a vacation at once. If the server finds no beneficiaries in the system, then the server can take another vacation after the return from the vacation. This process continues until the server has exhaustively taken all the J vacations. The closed form transient solution of the considered model and some important time-dependent performance measures are obtained. Further, the steady state system size distribution is obtained from the time-dependent solution. A stochastic decomposition structure of waiting time distribution and expression for the additional waiting time due to the presence of server vacations are studied. Numerical assessments are presented.

Keywords: *Markovian queue, multiple variant vacations, transient solution, waiting time distribution*

1. Introduction

From the works of Levy and Yechiali [12], new kind of queueing models, namely vacation queueing models, are started. Due to the plenty of applications of these models in the field of communication, computer networks, and production systems, etc., the models have drawn notice of many authors. The prime surveys on these vacation queueing models are Doshi [5] and Teghem [13]. Takagi [23] and Tian and Zhang [24] published books devoted to this topic. Readers may see the two surveys by Ke et al. [10] and Upadhyaya [25] on vacation queueing models and the references therein.

Two types of vacation schemes are discussed in the literature, namely a single vacation scheme, and a multiple vacation scheme. In the above schemes, the server halts the service process when the server is on vacation. Servi and Finn [16] introduce the

*Corresponding author, email address: dassmaths@gmail.com

Received 13 July 2020, accepted 29 March 2021

working vacation scheme in which the server provides service to the beneficiaries even when the server is on vacation but at a different rate which is slower than the normal service rate. This vacation scheme has attracted a lot of authors and thus has led to numerous research works. We recommend the survey paper [4] to view the various queues extended with this type of vacation studied by various researchers up to 2016.

Zhang and Tian [28] propose a variant of multiple vacation policy. In the model, the server is allowed to take a maximum number of K vacations consecutively, and then the server remains idle in the system. Works with this kind of policy with additional impressions of bulk entry, balking, N-policy, and beneficiaries' impatience can be found, in Banik [3], Ke [9], Ke et al. [11], and Yue et al. [27]. Pikkala and Pilla [15] study the transient behaviour of $M/M/1$ variant working vacation queues with balking customers.

Despite that various authors discussed the transient solution of queueing systems in the literature of vacation queueing systems, they rather see steady-state discussions with various policies than the time-dependent solution. Kalidass and Ramanath [8] analyze the time-dependent solution of queues with server vacations and a waiting server. Kalidass et al. [7] study the transient analysis of a Markovian single server queue with a repairable server and multiple vacations. Ammar [1] discusses the transient solution of single server multiple vacation queues with impatient customers. Ammar [2] extends the work of [1] by adding the possibility of beneficiaries impatience. Sudhesh and Francis Raj [19] discuss the transient distribution of a single server queue with working vacations. With impatient beneficiaries, Sudhesh et al. [18] extend the work of [19]. Sudhesh and Azhagappan [17] attain the transient solution of single server Markovian working vacation queueing models with variant impatient behaviour. Suranga Sampath and Kalidass [21] extend the results of [19] and incorporate them with server failures.

Ibe and Isijola [6] discuss a new kind of multiple vacation queueing model where the server is permitted to take two different vacations in which the duration of the second vacation is lesser than the first one. The transient solution of the model discussed in [6] is carried out by Vijayashree and Janani [26]. Later, Sampath and Liu [20] analyse the impact of beneficiary's impatience in [24]. The transient solution of $M/M/1$ queue with differentiated vacations and impatient beneficiaries can be found in [22] in which the server takes a vacation after a random waiting time after every end of the busy periods.

Anesthesiologists are physicians who play an important role in clinical practice of surgery, and they provide continuous medical care before, during, and after surgery, which is their primary job, and there many supplementary tasks, to mention a few, to maintain medical records, to prepare medical history of admitted patients, etc. Planned surgeries in many specialist hospitals are performed in succession. When all scheduled surgeries are completed and if there are no primary tasks, they decide to complete the additional task of preparing a record of the patient's medical histories. After completing this additional task, they start to do the primary job, if any awaiting, or decide to complete another additional task of placing order procedures to buy laboratory equipment, X-rays, and other types of medical equipment. Upon completing this additional task, they continue with the primary work (if any), otherwise they decide to complete another task that teaches students and staff the types and methods of anesthesia administration, the symptoms of complications, and the emergency procedures to deal with them. All of these additional tasks listed above complete the above process. When all additional tasks are completed, he/she is waiting for the primary work.

Consider the situation in the post office counter clerk. The clerk's main responsibility is to liaise with beneficiaries by booking posts, selling stamps, and answering, if beneficiaries need. Once he/she completes service to all the beneficiaries in front of the desk, he/she decides to complete one of the secondary tasks, such as preparing and completing bills, completing the daily, weekly, and monthly balance of the accounts spreadsheet and sorting. After completion one of the secondary tasks listed above, if beneficiaries awaiting the desk, the clerk starts to provide the primary service. Otherwise, the clerk decides to complete one of the remaining secondary tasks. The clerk continues the above process until completing all his/her secondary tasks. When all the secondary tasks are finished, the clerk simply waits in the service desk for new arrivals.

With the motivation of the above real-life examples, Markovian single server queuing models with multiple variant vacations are considered here. The paper is systemised as follows. Our queueing model is described in Section 2. The time-dependent solution and some prime performance measures in the time-dependent scenario are presented in Sections 3 and 4, respectively. Steady-state solution derived from the solution discussed in Section 3 is given in Section 5 with the stochastic decomposition structure. Distribution of waiting time of beneficiaries in the system is analysed in Section 6. The subsequent Sections 7 and 8 include the numerical discussion on the findings and concluding remarks, respectively.

2. The system description

Single server Markovian queueing systems with Poisson arrival beneficiaries J and multiple vacation policies are considered. Assumptions are as follows:

- Beneficiaries join singly in the waiting line. The interarrival times are independently, identically, and exponentially distributed with the parameter λ.
- The service takes place singly, and is provided by the single server to the first in the queue. The service times are assumed to be distributed according to an exponential distribution with mean $1/\mu$.
- After the end of every busy period, the server goes on vacation when the number of waiting beneficiaries is zero. If there is at least one beneficiary at vacation completion time, the server provides service to the beneficiary immediately. Otherwise, the situation lasted a maximum number of J vacations, and then the server remains in the service

facility to serve beneficiaries. The vacation times are assumed to be exponentially distributed with an intensity of γ.

• The inter-arrival times, service times, and vacation times are assumed to be independent of each other.

• Let $K(t)$ denote the number of beneficiaries in the system at time t.

• Let $S(t)$ be the status of the server at time t, which is defined as follows:

$S(t) = v_i$ if the server is in ith vacation and $i = 0, 1, ..., j-1$.

$S(t) = B$ if the server is busy or idle.

Then the bivariate process $\{(K(t), S(t)), t \geq 0\}$ confines a continuous-time Markov process with state space

$$\Delta = \{(m, i)\}: m \geq 0, i = B, v_0, v_1, ..., v_{j-1}$$

Let $q_{m,i}(t)$ be the probability to m beneficiaries in the system and the service provider is in ith state. We assume that the server is in the first vacation and zero beneficiaries in the system at time $t = 0$.

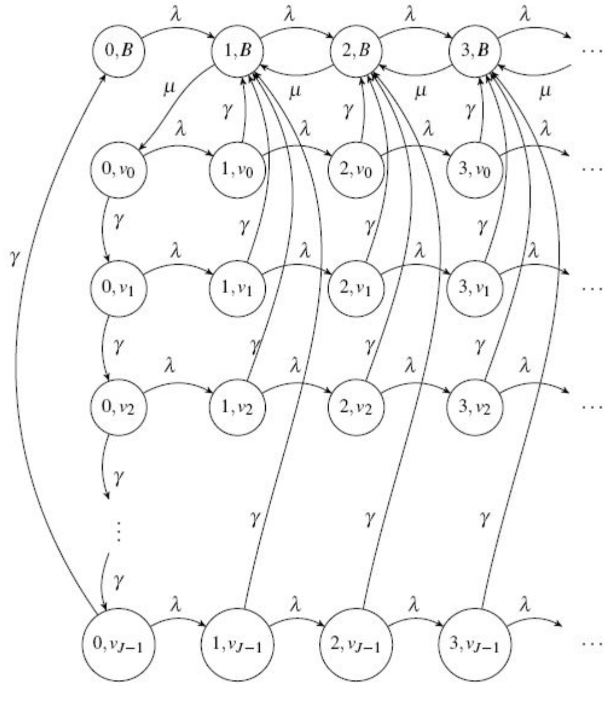

Fig. 1. State transition diagram

The system of Kolmogorov differential difference equations is given by

$$q'_{0,v_0}(t) = -(\lambda+\gamma)\, q_{0,v_0}(t) + \mu q_{1,B}(t) \tag{1}$$

$$q'_{n,v_0}(t) = -(\lambda+\gamma)\, q_{n,v_0}(t) + \lambda q_{n-1,v_0}(t), \ n \geq 1 \tag{2}$$

$$q'_{0,v_i}(t) = -(\lambda+\gamma)\, q_{0,v_i}(t) + \gamma q_{0,v_{i-1}}(t) \tag{3}$$

$$q'_{n,v_i}(t) = -(\lambda+\gamma)\, q_{n,v_i}(t) + \lambda q_{n-1,v_i}(t), \quad n \geq 1,\ i = 1,\ 2,\ ...,\ J-1 \tag{4}$$

$$q'_{0,B}(t) = -\lambda q_{0,B}(t) + \gamma q_{0,v_{J-1}}(t) \tag{5}$$

$$q'_{n,B}(t) = -(\lambda+\mu)\, q_{n,B}(t) + \lambda q_{n-1,B}(t) + \mu q_{n+1,B}(t) + \gamma \sum_{i=0}^{J-1} q_{n,V_i}(t), \quad n \geq 1 \tag{6}$$

with the initial conditions $q_{0,v_0}(0) = 1,\ q_{n,v_i}(0) = 0, \forall n \geq 0, i = 1,\ 2,\ ...,\ J-1,\ q_{0,v_n}(0) = 0,$ $\forall n \geq 1,\ q_{n,B}(0) = 0,\ n \geq 0.$

3. The transient solution

Let $q^*_{n,i}(s)$ be the Laplace transform of $q_{n,i}(t),\ n \geq 0$ and $i = 0,\ 1,\ ...,\ J-1.$ Taking the Laplace transform of (1)–(6) gives

$$sq^*_{0,v_0}(s) - q_{0,v_0}(0) = -(\lambda+\gamma)\, q^*_{0,v_0}(s) + \mu q^*_{1,B}(s) \tag{7}$$

$$sq^*_{n,v_0}(s) - q_{n,v_0}(0) = -(\lambda+\gamma)\, q^*_{n,v_0}(s) + \lambda q^*_{n-1,v_0}(s), \quad n \geq 1 \tag{8}$$

$$sq^*_{0,v_i}(s) - q_{0,v_i}(0) = -(\lambda+\gamma)\, q^*_{0,v_i}(s) + \gamma q^*_{0,v_{i-1}}(s), i = 1,\ 2,\ ...,\ J-1 \tag{9}$$

$$sq^*_{n,v_i}(s) - q_{n,v_i}(0) = -(\lambda+\gamma)\, q^*_{n,v_i}(s) + \lambda q^*_{n-1,v_i}(s), \quad i = 1,\ 2,\ ...,\ J-1,\ n \geq 1 \tag{10}$$

$$sq^*_{0,B}(s) - q_{0,B}(0) = -\lambda q^*_{0,B}(s) + \gamma q^*_{0,v_{J-1}}(s) \tag{11}$$

$$sq^*_{n,B}(s) - q_{n,B}(0) = -(\lambda + \mu)\, q^*_{n,B}(s) + \lambda q^*_{n-1,B}(s) + \mu q^*_{n+1,B}(s)$$
$$+ \gamma \sum_{i=0}^{J-1} q^*_{n,v_i}(s), \quad n \geq 1 \tag{12}$$

Equation (7) gives

$$q^*_{0,v_0}(s) = \frac{1}{s+\lambda+\gamma} + \frac{\mu}{s+\lambda+\gamma}\, q^*_{1,B}(s) \tag{13}$$

Similarly, from (8), we get

$$q^*_{n,v_0}(s) = \frac{\lambda}{s+\lambda+\gamma}\, q^*_{n-1,v_0}(s), \quad n \geq 1 \tag{14}$$

which recursively gives

$$q^*_{n,v_0}(s) = \left(\frac{\lambda}{s+\lambda+\gamma}\right)^n q^*_{0,v_0}(s), \quad n \geq 1 \tag{15}$$

Substituting (13) into (15), we get

$$q^*_{n,v_0}(s) = \frac{\lambda^n}{(s+\lambda+\gamma)^{n+1}} + \frac{\lambda^n \mu}{(s+\lambda+\gamma)^{n+1}}\, q^*_{1,B}(s), \quad n \geq 1 \tag{16}$$

Also, from (9), we get

$$q^*_{0,v_i}(s) = \frac{\gamma}{s+\lambda+\gamma}\, q^*_{0,v_{i-1}}(s), \quad i = 1,\ 2,\ \ldots,\ J-1 \tag{17}$$

From (10), we get

$$q^*_{n,v_i}(s) = \frac{\lambda}{s+\lambda+\gamma}\, q^*_{n-1,v_i}(s), \quad n \geq 1 \tag{18}$$

which recursively gives

$$q^*_{n,v_i}(s) = \frac{\lambda^n \gamma}{(s+\lambda+\gamma)^{n+1}}\, q^*_{0,v_{i-1}}(s), \quad n \geq 1,\ i = 1,\ 2,\ \ldots,\ J-1 \tag{19}$$

After some algebra,

$$q_{n,v_i}^*(s) = \frac{\lambda^n \gamma^i}{\left(s+\lambda+\gamma\right)^{n+i+1}} + \frac{\lambda^n \gamma^i \mu}{\left(s+\lambda+\gamma\right)^{n+i+1}}\, q_{1,B}^*(s), \quad i = 1,\ 2,\ \dots,\ J-1 \tag{20}$$

(11) is modified as

$$q_{0,B}^*(s) = \frac{\gamma}{s+\lambda}\, q_{0,v_{J-1}}^*(s)$$

From (13) and (17), we get

$$q_{0.B}^*(s) = \frac{\gamma^J}{(s+\lambda)\left(s+\lambda+\gamma\right)^J}\left(1+\mu q_{1,B}^*(s)\right) \tag{21}$$

Inverse Laplace transform on both sides of equations (17) and (20), we get

$$q_{n,v_i}(t) = \lambda^n \gamma^i \frac{t^{n+i}}{(n+i)!}\,\mathrm{e}^{-(\lambda+\gamma)t} + \lambda^n \gamma^i \mu \left(\frac{t^{n+i}}{(n+i)!}\,\mathrm{e}^{-(\lambda+\gamma)t} \star q_{1,B}(t) \right), \quad n \ge 0,\ i = 0, 1,\ \dots,\ J-1 \tag{22}$$

where $\star$ represents convolution.

Define the partial probability generating function

$$Q_B(z,\ t) = \sum_{n=1}^{\infty} q_{n,B}(t)\, z^n$$

then

$$\frac{\partial Q_B(z,\ t)}{\partial t} = \sum_{n=1}^{\infty} q'_{n,B}(t)\, z^n$$

Multiplying and summing (12) by appropriate z^n gives

$$(s+\lambda+\mu)\, Q_B^*(z, s) = \lambda \sum_{n=1}^{\infty} q_{n-1,B}^*(s) z^n + \mu \sum_{n=1}^{\infty} q_{n+1,B}^*(s) z^n + \gamma \sum_{n=1}^{\infty} \sum_{i=0}^{J-1} q_{n,v_i}^*(s) z^n$$

$$Q_B^*(z,s)=\frac{z}{-\lambda z^2+(s+\lambda+\mu)z-\mu}\left(\lambda z q_{0,B}^*(s)-\mu q_{1,B}^*(s)+\gamma\sum_{n=1}^{\infty}\sum_{i=0}^{J-1} q_{n,v_i}^*(s)z^n\right)$$

where $Q_B^*(z,s)$ is the Laplace transform of $Q_B(z,t)$.

Since the denominator of the above equation vanishes at

$$z_1=\frac{\omega-\sqrt{\omega^2-4\lambda\mu}}{2\lambda}$$

where $\omega = s+\lambda+\mu$, we have

$$\lambda z_1 q_{0,B}^*(s)-\mu q_{1,B}^*(s)+\gamma\sum_{n=1}^{\infty}\sum_{i=0}^{J-1} q_{n,v_i}^*(s)z_1^n=0$$

From this,

$$\begin{aligned}0=\mu q_{1,B}^*(s)&-\frac{\lambda z_1}{s+\lambda}\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J-\frac{\lambda z_1}{s+\lambda}\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J\mu q_{1,B}^*(s)\\&-\frac{\lambda z_1}{s+\lambda}\left(\frac{\gamma}{s+\lambda(1-z_1)+\gamma}\right)+\frac{\lambda z_1}{s+\lambda}\left(\frac{\gamma}{s+\lambda(1-z_1)+\gamma}\right)\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J\\&-\frac{\lambda z_1}{s+\lambda}\left(\frac{\gamma}{s+\lambda(1-z_1)+\gamma}\right)\left(1-\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J\right)\left(1+\mu q_{1,B}^*(s)\right)\end{aligned}\tag{23}$$

After some algebra,

$$q_{1,B}^*(s)=\frac{1}{\mu}\times\left(\frac{\frac{\lambda z_1}{s+\lambda}\left(\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J+\frac{\gamma}{s+\lambda(1-z_1)+\gamma}-\frac{\gamma}{s+\lambda(1-z_1)+\gamma}\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J\right)}{1-\frac{\lambda z_1}{s+\lambda}\left(\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J+\frac{\gamma}{s+\lambda(1-z_1)+\gamma}-\frac{\gamma}{s+\lambda(1-z_1)+\gamma}\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J\right)}\right)\tag{24}$$

$$q_{1,B}^*(s) = \frac{1}{\mu}\left(\frac{g^*(s)}{1-g^*(s)}\right) \tag{25}$$

where

$$g^*(s) = \frac{\lambda z_1}{s+\lambda}\left(\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J + \frac{\gamma}{s+\lambda(1-z_1)+\gamma} - \frac{\gamma}{s+\lambda(1-z_1)+\gamma}\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J\right) \tag{26}$$

Then $q_{1,B}^*(s)$ takes the form of

$$q_{1,B}^*(s) = \frac{1}{\mu} g^*(s)\sum_{d=0}^{\infty}\left(g^*(s)\right)^d = \frac{1}{\mu}\sum_{d=1}^{\infty}\left(g^*(s)\right)^d \tag{27}$$

By inverting the above, we get

$$q_{1,B}(t) = \frac{1}{\mu}\sum_{d=1}^{\infty}\left(g(t)\right)^{\star d} \tag{28}$$

where $\left(g(t)\right)^{\star d}$ represents the *d*-fold convolution of $g(t)$,

$$\begin{aligned} g(t) = \lambda\Bigg(& \mathrm{e}^{-\lambda t} \star \frac{1}{\beta t}\mathrm{e}^{-(\lambda+\mu)t} I_1(\alpha t)\Bigg) \star \Bigg(\gamma^J \frac{t^{J-1}}{(J-1)!}\mathrm{e}^{-(\lambda+\gamma)t} \\ & + \frac{2\gamma}{\alpha}\sum_{m=0}^{\infty}\left(\frac{\alpha}{2\lambda} - \frac{2\gamma}{\alpha}\right)^m \left(\frac{m+1}{t}\right) I_{m+1}(\alpha t)\,\mathrm{e}^{-(\lambda+\mu)t} \\ & - \gamma^J \mathrm{e}^{-(\lambda+\gamma)t}\frac{t^{J-1}}{(J-1)!} \star \frac{2\gamma}{\alpha}\sum_{m=0}^{\infty}\left(\frac{\alpha}{2\lambda} - \frac{2\gamma}{\alpha}\right)^m \left(\frac{m+1}{t}\right) I_{m+1}(\alpha t)\mathrm{e}^{-(\lambda+\mu)t}\Bigg) \end{aligned} \tag{29}$$

Derivation of $q_{n,B}(t)$: Multiplying the appropriate z^n with (12), we get

$$\frac{\partial Q_B(z,t)}{\partial t}-\left(-(\lambda+\mu)+\frac{\mu}{z}+\lambda z\right)Q_B(z,t)=\lambda z q_{0,B}(t)-\mu q_{1,B}(t)$$

$$+\ \gamma\sum_{n=1}^{\infty}\sum_{i=0}^{J-1}q_{n,v_i}(t)z^n \tag{30}$$

Integrating the above linear differential equation with respect to t yields

$$Q_B(z,t)=\lambda\int_0^t zq_{0,B}(y)\,\mathrm{e}^{-(\lambda+\mu)(t-y)}\mathrm{e}^{((\mu/z)+\lambda z)(t-y)}dy$$

$$-\mu\int_0^t q_{1,B}(y)\mathrm{e}^{-(\lambda+\mu)(t-y)}\mathrm{e}^{((\mu/z)+\lambda z)(t-y)}dy$$

$$+\gamma\int_0^t\sum_{i=0}^{J-1}\sum_{n=1}^{\infty}q_{n,v_i}(y)z^n\mathrm{e}^{-(\lambda+\mu)(t-y)}\mathrm{e}^{((\mu/z)+\lambda z)(t-y)}dy \tag{31}$$

We use the Bessel function identity, if $\alpha=2\sqrt{\lambda\mu}$ and $\beta=\sqrt{\lambda/\mu}$, then

$$\exp\left(\frac{\mu}{z}+\lambda z\right)t=\sum_{n=-\infty}^{\infty}(\beta z)^n\, I_n(\alpha t)$$

where $I_n(t)$ is the modified Bessel function of the first kind.

Comparing the coefficients of z^n in (31), we get

$$q_{n,B}(t)=\lambda\int_0^t zq_{0,B}(y)\,\mathrm{e}^{-(\lambda+\mu)(t-y)}\beta^n I_n\left(\alpha(t-y)\right)dy$$

$$-\mu\int_0^t q_{1,B}(t)\,\mathrm{e}^{-(\lambda+\mu)(t-y)}\beta^n I_n\left(\alpha(t-y)\right)dy$$

$$+\gamma\int_0^t\sum_{i=0}^{J-1}\sum_{r=1}^{\infty}q_{r,v_i}(y)\,\mathrm{e}^{-(\lambda+\mu)(t-y)}\beta^{n-r}I_{n-r}\left(\alpha(t-y)\right)dy \tag{32}$$

Comparing the coefficients of z^{-n} in (31), we get

$$0 = \lambda \int_0^t z q_{0,B}(y)\, \mathrm{e}^{-(\lambda+\mu)(t-y)} \beta^n I_n\left(\alpha\left(t-y\right)\right) dy$$

$$- \mu \int_0^t q_{1,B}(t)\, \mathrm{e}^{-(\lambda+\mu)(t-y)} \beta^n I_n\left(\alpha\left(t-y\right)\right) dy$$

$$+ \gamma \int_0^t \sum_{i=0}^{J-1} \sum_{r=1}^{\infty} q_{r,V_i}(y)\, \mathrm{e}^{-(\lambda+\mu)(t-y)} \beta^{n-r} I_{n+r}\left(\alpha\left(t-y\right)\right) dy \tag{33}$$

Subtracting (33) from (32), we get

$$q_{n,B}(t) = \gamma \int_0^t \sum_{i=0}^{J-1} \sum_{r=1}^{\infty} q_{r,V_i}(y)\, \mathrm{e}^{-(\lambda+\mu)(t-y)} \beta^{n-r}$$

$$\times \left(I_{n-r}\left(\alpha\left(t-y\right)\right) - I_{n+r}\left(\alpha\left(t-y\right)\right)\right) dy, \quad n \geq 1 \tag{34}$$

To evaluate $q_{\bullet,v_0}(t)$, $q_{\bullet,v_i}(t)$, and $q_{\bullet,B}(t)$, let $q^*_{\bullet,v_0}(s)$ denote the Laplace transform of $q_{\bullet,v_0}(t)$

$$q^*_{\bullet,v_0}(s) = \sum_{n=0}^{\infty} q^*_{n,v_0}(s) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{s+\lambda+\gamma} \right)^n q^*_{0,v_0}(s) = \frac{1}{s+\gamma}\left(1+\mu q^*_{1,B}(s)\right) \tag{35}$$

Inversion of the above equation yields

$$q_{\bullet,v_0}(t) = \exp\left(-\gamma t\right) + \mu \exp\left(-\gamma t\right) \star q_{1,B}(t)$$

Let $q^*_{\bullet,v_i}(s)$ denote the Laplace transform of $q_{\bullet,v_i}(t)$, $i = 1, 2, ..., J-1$

$$q^*_{\bullet,v_i}(s) = \sum_{n=0}^{\infty} q^*_{n,v_i}(s) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{s+\lambda+\gamma} \right)^n \left(\frac{\gamma}{s+\lambda+\gamma} \right)^i q^*_{0,v_0}(s)$$

$$= \frac{\gamma^i}{\left(s+\lambda+\gamma\right)^i \left(s+\gamma\right)} \left(1+\mu q^*_{1,B}(s)\right) = h(s)\left(1+\mu q^*_{1,B}(s)\right) \tag{36}$$

where

$$h(s)=\frac{\gamma^i}{(s+\lambda+\gamma)^i\,(s+\gamma)}$$

Inversion of the above equation yields

$$q_{\bullet,v_i}(t)=h(t)+\mu h(t)\star q_{1,B}(t)$$

where

$$h(t)=\gamma^i\left(\mathrm{e}^{-\gamma t}\star\frac{t^{i-1}}{(i-1)!}\mathrm{e}^{-(\lambda+\gamma)t}\right)$$

and $q_{1,B}(t)$ has already been found.

$$q^*_{\bullet,B}(s)=\sum_{n=0}^{\infty}q^*_{n,B}(s)=q^*_{0,B}(s)+\sum_{n=1}^{\infty}q^*_{n,B}(s)$$

$$=\frac{\gamma^J}{(s+\lambda)(s+\lambda+\gamma)^J}\left(1+\mu q^*_{1,B}(s)\right)+f^*(s)+\mu q^*_{1,B}(s)f^*(s)-\frac{\mu}{s}q^*_{1,B}(s)$$

where

$$f^*(s)=\frac{\lambda\gamma^J}{s(s+\lambda)(s+\lambda+\gamma)^J}+\frac{\gamma\lambda}{s(s+\gamma)(s+\lambda)}-\frac{\gamma\lambda}{s(s+\gamma)(s+\lambda)}\left(\frac{\gamma}{s+\lambda+\gamma}\right)^J$$

On inversion, we get

$$\begin{aligned}q_{\bullet,B}(t)&=\mathrm{e}^{-\lambda t}\star\frac{\gamma^J t^{J-1}}{(J-1)!}\mathrm{e}^{-(\lambda+\gamma)t}+\mu\mathrm{e}^{-\lambda t}\star\frac{\gamma^J t^{J-1}}{(J-1)!}\mathrm{e}^{-(\lambda+\gamma)t}\star q_{1,B}(t)\\&\quad+f(t)+\mu q_{1,B}(t)\star f(t)-\mu_{1,B}(t)\end{aligned}\tag{37}$$

where

$$f(t)=a_0+b_0\mathrm{e}^{-\lambda t}+c_0\mathrm{e}^{-\gamma t}+c_1\mathrm{e}^{-(\lambda+\gamma)t}+c_2t\mathrm{e}^{-(\lambda+\gamma)t}+\ldots+c_J\frac{t^{J-1}}{(J-1)!}\mathrm{e}^{-(\lambda+\gamma)t}$$

Here a_0, b_0, c_0, c_1, ..., c_J are constants.

4. System performance measures

4.1. Time dependent mean

The mean number of beneficiaries in the system at a time t is given by

$$m(t) = E(X(t)) = \sum_{n=1}^{\infty} n\left(q_{n,B}(t) + \sum_{i=0}^{J-1} q_{n,v_i}(t)\right)$$

Differentiating with respect to t, we get

$$m'(t) = \sum_{n=1}^{\infty} n\left(q'_{n,B}(t) + \sum_{i=0}^{J-1} q'_{n,v_i}(t)\right) = \lambda\left(\sum_{n=0}^{\infty} q_{n,B}(t) + \sum_{n=0}^{\infty}\sum_{i=0}^{J-1} q_{n,v_i}(t)\right)$$

$$-\mu\sum_{n=1}^{\infty} q_{n,B}(t) = \lambda - \mu\sum_{n=1}^{\infty} q_{n,B}(t)$$

$$m(t) = \lambda t - \mu\sum_{n=1}^{\infty}\int_0^t q_{n,B}(y)dy \tag{38}$$

where $q_{n,B}(t)$ has already been found.

4.2. Time dependent variance

The variance of number of beneficiaries at time t is given by

$$V(t) = w(t) - \left(m(t)\right)^2$$

where $w(t)$ be the second moment of number of beneficiaries at time t

$$w(t) = E\left(X^2(t)\right) = \sum_{n=1}^{\infty} n^2\left(q_{n,B}(t) + \sum_{i=0}^{J-1} q_{n,v_i}(t)\right)$$

Differentiating with respect to t, we get

$$w'(t) = \sum_{n=1}^{\infty} n^2\left(q'_{n,B}(t) + \sum_{i=0}^{J-1} q'_{n,v_i}(t)\right)\lambda\left(\sum_{n=0}^{\infty}(2n+1)\,q_{n,B}(t)\right.$$

$$\left. + \sum_{i=0}^{J-1}\sum_{n=0}^{\infty}(2n+1)q_{n,v_i}(t)\right) - \mu\sum_{n=1}^{\infty}(2n-1)\,q_{n,B}(t) \tag{39}$$

Integrating (39) with respect to t, we get

$$w(t)=\lambda\left(\sum_{n=0}^{\infty}(2n+1)\int_0^t q_{n,B}(y)dy\right)-\mu\sum_{n=1}^{\infty}(2n-1)\int_0^t q_{n,B}(y)dy$$

$$+\lambda\sum_{i=0}^{J-1}\sum_{n=0}^{\infty}(2n+1)\int_0^t q_{n,v_i}(y)dy \tag{40}$$

Substituting (38) and (40) in $V(t)$, we get

$$V(t)=\lambda\left(\sum_{n=0}^{\infty}(2n+1)\int_0^t q_{n,B}(y)dy+\sum_{i=0}^{J-1}\sum_{n=0}^{\infty}(2n+1)\int_0^t q_{n,v_i}(y)dy\right)$$

$$-\mu\sum_{n=1}^{\infty}(2n-1)\int_0^t q_{n,B}(y)dy-\left(\lambda t-\mu\sum_{n=1}^{\infty}\int_0^t q_{n,B}(y)dy\right)^2 \tag{41}$$

where $q_{n,B}(t),\ q_{n,v_i}(t)$ have already been found.

5. The steady state solution

Let $\pi_{n,j}$ denote the steady state probability for the system to be in state j with n beneficiaries. Mathematically,

$$\pi_{n,j}=\lim_{t\to\infty}q_{n,j}(t)$$

Using the initial value theorem of the Laplace transforms which states

$$\lim_{t\to\infty}q_{n,j}(t)=\lim_{s\to 0}sq_{n,j}^{*}(s)$$

it is observed that

$$\pi_{n,j}=\lim_{s\to 0}sq_{n,j}^{*}(s)$$

From (24),

$$sq_{1,B}^{*}(s)=\frac{1}{\mu}\left(\frac{\lambda\left((\mu-\lambda)-s+o\left(s^2\right)\right)\Theta}{s\mu-s\lambda(1-\Theta)+\lambda(\mu-\lambda)(1-\Theta)+o\left(s^2\right)\Theta}\right)$$

where

$$\Theta = \left(\left(\frac{\gamma}{s+\lambda+\gamma} \right)^J + \frac{\gamma}{s+\lambda(1-z_1)+\gamma} - \frac{\gamma}{s+\lambda(1-z_1)+\gamma} \left(\frac{\gamma}{s+\lambda+\gamma} \right)^J \right) \quad (42)$$

Taking limit as $s \to 0$ on both sides and using the Tauberian theorem, we obtain

$$\pi_{1,B} = \lim_{s \to 0} s q_{1,B}^*(s) = \frac{\dfrac{\lambda\gamma(\mu-\lambda)}{\mu^2}}{\gamma + \lambda \left(1 - \left(\dfrac{\gamma}{\lambda+\gamma} \right)^J \right)} \quad (43)$$

From (21),

$$\pi_{0,B} = \lim_{s \to 0} s q_{0,B}^*(s) = \frac{\mu\gamma^J}{\lambda(\lambda+\gamma)^J} \pi_{1,B}$$

Taking the Laplace transform on both sides of (34), we get

$$q_{n,B}^*(s) = \gamma \sum_{r=1}^{\infty} \sum_{i=0}^{J-1} q_{r,v_i}^*(s) \beta^{n-r} \left(\frac{\left((s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - \alpha^2} \right)^{n-r}}{\alpha^{n-r} \sqrt{(s+\lambda+\mu)^2 - \alpha^2}} - \frac{\left((s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - \alpha^2} \right)^{n+r}}{\alpha^{n+r} \sqrt{(s+\lambda+\mu)^2 - \alpha^2}} \right)$$

$$\pi_{n,B} = \lim_{s \to 0} s q_{n,B}^*(s) = \lim_{s \to 0} s\gamma \sum_{r=1}^{\infty} \sum_{i=0}^{J-1} \pi_{r,v_i}^*(s) \beta^{n-r} \left(\frac{\left((s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - \alpha^2} \right)^{n-r}}{\alpha^{n-r} \sqrt{(s+\lambda+\mu)^2 - \alpha^2}} - \frac{\left((s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - \alpha^2} \right)^{n+r}}{\alpha^{n+r} \sqrt{(s+\lambda+\mu)^2 - \alpha^2}} \right)$$

$$=\gamma\sum_{r=1}^{\infty}\sum_{i=0}^{J-1}\pi_{r,v_i}\beta^{n-r}\left(\frac{\left((\lambda+\mu)-\sqrt{(\lambda+\mu)^2-\alpha^2}\right)^{n-r}}{\alpha^{n-r}\sqrt{(\lambda+\mu)^2-\alpha^2}}-\frac{\left((\lambda+\mu)-\sqrt{(\lambda+\mu)^2-\alpha^2}\right)^{n+r}}{\alpha^{n+r}\sqrt{(\lambda+\mu)^2-\alpha^2}}\right)$$

In the same way,

$$\pi_{n,v_i}=\lim_{s\to 0}sq^*_{n,v_i}(s)=\frac{\lambda^n\gamma^i\mu}{(\lambda+\gamma)^{n+i+1}}\pi_{1,B},\quad n\geq 0,\ i=0,\ 1,\ ...,\ J-1$$

Remark 1. When $J\to 0$, the steady state probabilities converted into standard $M/M/1$ queue.

$$\pi_{1,B}=\frac{\lambda(\mu-\lambda)}{\mu^2}=\rho(1-\rho),\quad \pi_{0,B}=\frac{\mu}{\lambda}\pi_{1,B}=1-\rho$$

Theorem 1. If $\rho<1$, the number of beneficiaries present in the system can be decomposed into the sum of two independent random variables: $Q=Q_1+Q_{\text{mav}}$ where Q_1 is the number of beneficiaries present in the standard $M/M/1$ queue, and Q_{mav} the additional system length due to the corresponding J vacations with its probability generating function (PGF) is

$$PGF=\left(\frac{z\left(\rho-\alpha_1\alpha_2^J\right)}{\rho-\alpha_1}-\frac{z\alpha_1\left(1-\alpha_2^J\right)(1-\rho z)}{(\rho-\alpha_1)(1-\alpha_1 z)}\right.$$
$$\left.+\frac{1-\rho z}{\rho(1-\alpha_1 z)}-\frac{\alpha_1\alpha_2^J z(1-\rho z)}{\rho(1-\alpha_1 z)}\right)\left(\frac{\rho\gamma}{\gamma+\lambda\left(1-\alpha_2^J\right)}\right)$$

where $\alpha_1+\alpha_2=1$ and $\alpha_1=\lambda/(\lambda+\gamma)$.

Proof. Consider

$$Q(z)=\sum_{n=0}^{\infty}\pi_{n,B}z^n+\sum_{n=0}^{\infty}\sum_{i=0}^{J-1}\pi_{n,v_i}z^n=\left(\frac{\mu}{\lambda}\right)\left(\frac{\gamma}{\lambda+\gamma}\right)^J\pi_{1,B}$$
$$+\sum_{n=1}^{\infty}\left(\frac{(\rho z)^n}{\rho}+\frac{\alpha_1\left(1-\alpha_2^J\right)}{\rho-\alpha_1}\left(\frac{(\rho z)^n}{\rho}-\frac{(\alpha_1 z)^n}{\alpha_1}\right)\right)$$
$$+\sum_{n=0}^{\infty}\left(\frac{\lambda z}{\lambda+\gamma}\right)^n\sum_{i=0}^{J-1}\left(\frac{\gamma}{\lambda+\gamma}\right)^i\left(\frac{\mu}{\lambda+\gamma}\right)\pi_{1,B}$$

$$= \frac{\mu\gamma^J \pi_{1,B}}{\lambda(\lambda+\gamma)^J} + \frac{z\pi_{1,B}}{1-\rho z} + \left(\frac{\alpha_1\left(1-\alpha_2^J\right)}{\rho-\alpha_1}\right)\left(\frac{z}{1-\rho z} - \frac{z}{1-\alpha_1 z}\right)\pi_{1,B}$$

$$+ \frac{1}{1-\alpha_1 z}\left(\frac{\mu}{\lambda}\right)\left(1-\left(\frac{\gamma}{\lambda+\gamma}\right)^J\right)\pi_{1,B}$$

$$= \left(\frac{\pi_{1,B}}{1-\rho z}\right)\left(\frac{\rho z-\alpha_1\alpha_2^J z}{\rho-\alpha_1}\right) - \left(\frac{\alpha_1\left(1-\alpha_2^J\right)}{\rho-\alpha_1}\right)\left(\frac{\pi_{1,B} z}{1-\alpha_1 z}\right) + \frac{\pi_{1,B}}{\rho\left(1-\alpha_1 z\right)}$$

$$- \left(\frac{\gamma^J \pi_{1,B}}{\rho(\lambda+\gamma)^J}\right)\left(\frac{\alpha_1 z}{1-\alpha_1 z}\right) = \left(\frac{1-\rho}{1-\rho z}\right) Q_{\text{mav}}(z) \tag{44}$$

where

$$Q_{\text{mav}}(z) = \left(\frac{z\left(\rho-\alpha_1\alpha_2^J\right)}{\rho-\alpha_1} - \frac{z\alpha_1\left(1-\alpha_2^J\right)(1-\rho z)}{(\rho-\alpha_1)(1-\alpha_1 z)}\right.$$

$$\left. + \frac{1-\rho z}{\rho(1-\alpha_1 z)} - \frac{\alpha_1\alpha_2^J z(1-\rho z)}{\rho(1-\alpha_1 z)}\right)\left(\frac{\rho\gamma}{\gamma+\lambda\left(1-\alpha_2^J\right)}\right)$$

It is easy to prove that $Q_{\text{mav}}(1)=1$ and therefore $Q_{\text{mav}}(z)$ is the *PGF* of additional system length due to the effect of multiple adoptive vacations.

6. Waiting time distribution

Let W denote the waiting time in the system of an arbitrary beneficiary and let $W^*(s)$ be the Laplace–Steiltjes transform (LST) of the distribution of waiting time in the system.

Theorem 2. If $\rho > 1$, the steady state waiting time W can be decomposed into the sum of independent random variables: $W = W_q + W_{\text{mav}}$, where W_q is the waiting time of beneficiaries in the system without adaptive vacation, and W_{mav} is the additional stationary waiting time due to the effect of multiple adaptive vacations and has a distribution with its Laplace transform

$$W^*_{\text{mav}}(s)=\left(\frac{\rho\gamma}{\gamma+\lambda\left(1-\alpha_2^J\right)}\right)\left(\frac{\rho-\alpha_1\alpha_2^J}{\rho-\alpha_1}-\frac{\left(\rho-\alpha_1\alpha_2^J\right)s}{\lambda\left(\rho-\alpha_1\right)}\right.$$
$$-\frac{\lambda\left(1-\alpha_2^J\right)(1-\rho)}{\rho-\alpha_1}\left(\frac{1}{s+\gamma_1}\right)-\frac{\left(1-\alpha_2^J\right)(2\rho-1)}{\rho-\alpha_1}\left(\frac{s}{s+\gamma}\right)$$
$$\left.+\frac{\left(1-\alpha_2^J\right)\rho(1-\rho)}{\rho-\alpha_1}\left(\frac{s^2}{s+\gamma}\right)+\frac{(\lambda+\gamma)(1-\rho)}{\rho}\left(\frac{1}{s+\gamma}\right)+\frac{1}{\alpha_1}\left(\frac{s}{s+\gamma}\right)\right)$$

Proof. From [14], the relationship between *PGF* of system size distribution and the LST of the waiting time of a beneficiary is

$$Q(z)=W^*\left(\lambda(1-z)\right)$$

Assume $s=\lambda(1-z)$, and applying $z=1-\frac{s}{\lambda}$ in (44), we have

$$W^*(s)=\left(\frac{\mu(1-\rho)}{\mu(1-\rho)+s}\right)W^*_{\text{mav}}(s)$$

Expected average waiting time during the vacation period becomes

$$E\left(W_{\text{mav}}\right)=\left(\frac{\gamma\left(1-\alpha_2^J\right)(2\rho-1)}{\rho-\alpha_1}-\frac{\lambda\alpha_1\left(1-\alpha_2^J\right)(1-\rho)+\gamma\left(\rho-\alpha_1\right)}{\alpha_1\left(\rho-\alpha_1\right)}\right.$$
$$\left.+\frac{(\lambda+\gamma)(1-\rho)}{\rho}\right)\left(\frac{\rho}{\gamma\left(\gamma+\lambda\left(1-\alpha_2^J\right)\right)}\right)$$

and the expected waiting time is

$$E(W)=\frac{\rho}{1-\rho}+E\left(W_{\text{mav}}\right)$$

7. Numerical illustrations

In this section, we present some numerical illustrations to analyse the effect of the parameters of our model. We take $J=15$, that is the number of vacations permitted to the server take is 15.

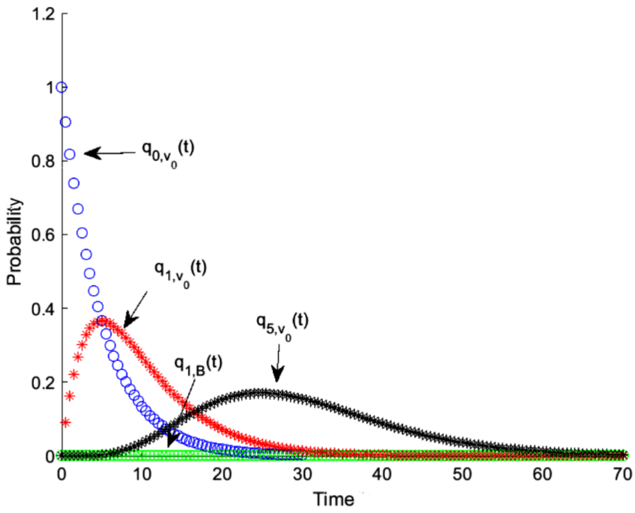

Fig. 2. Various probabilities against time

In Figure 2, various probabilities, such as $q_{0,v_0}(t)$, $q_{1,v_0}(t)$, $q_{5,v_0}(t)$, and $q_{1,B}(t)$, are shown for the parameters $\lambda = 0.2$, $\mu = 0.3$, $\gamma 0.001$. Note that the system initially started with zero beneficiaries when on the first vacation. We notice that the estimated values $q_{1,v_0}(t)$ and $q_{5,v_0}(t)$ increase gradually and decrease later until they arrive at the stable in contrast to the values of $q_{0,v_0}(t)$ and $q_{1,B}(t)$.

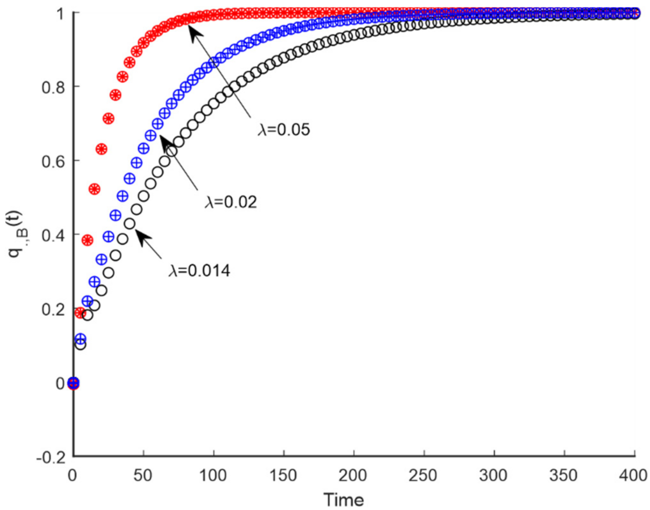

Fig. 3. Probability for the server to be in a busy state

Figure 3 shows the time dependences of $q_{\bullet,B}(t)$ for the values of λ equal to 0.014, 0.02, and 0.05. Here, we take μ = 0.2, and γ= 0.5. From this, we spot that when λ increases, $q_{\bullet,B}(t)$ also increases. Also, the curves reach stability after time points. Among the three values of λ, this happens earlier for the value of 0.05, as hoped.

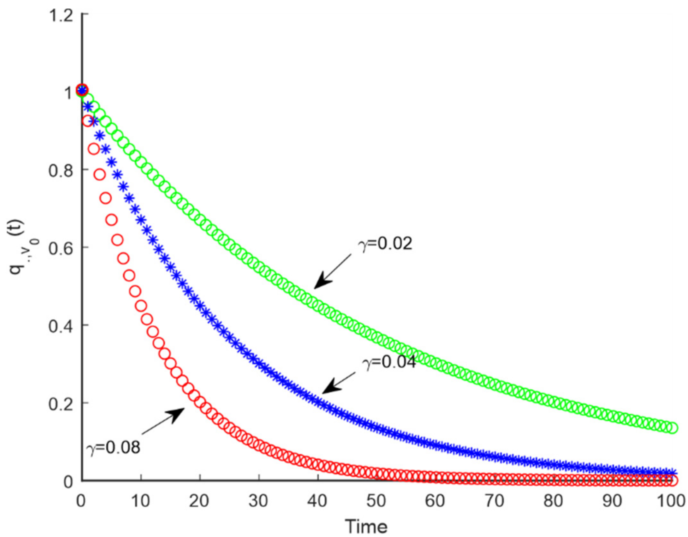

Fig. 4. Probability for the server to be in the first vacation

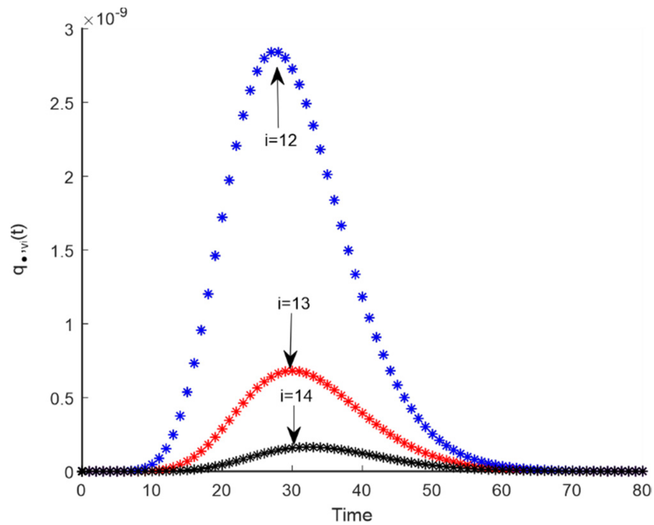

Fig. 5. Probability for the server to be in the vacations

From Figure 4 we observe that the probability for the server to be in the first vacation decreases with increasing time. We take $\lambda = 0.2$, $\mu = 0.3$. All three curves regarding $\gamma = 0.02, 0.04, 0.08$ reach stability after some time, but it happens earlier to the values of $\gamma = 0.02, 0.04$. It is also seen that the decrease is high for $\gamma = 0.08$ relatively comparing with $\gamma = 0.02$, and 0.04.

Figure 5 exhibits the time dependences of $q_{\bullet,v_i}(t)$, the probability to the server is on ith vacation. Here we take $\lambda = 0.2$, $\mu = 0.3$ and show three curves for $i = 12, 13$, and 14. All three curves behave as concave.

Figure 6 shows the time dependences of $E(W(t))$, the mean waiting time of beneficiaries in the system for three different values of γ, namely 0.001, 0.01, and 0.5. Initially, the mean waiting time increases as time increases, and later after some time, it reaches stability. For $\gamma = 0.5$, the server availability in the system is high, thus reducing the average waiting time of users.

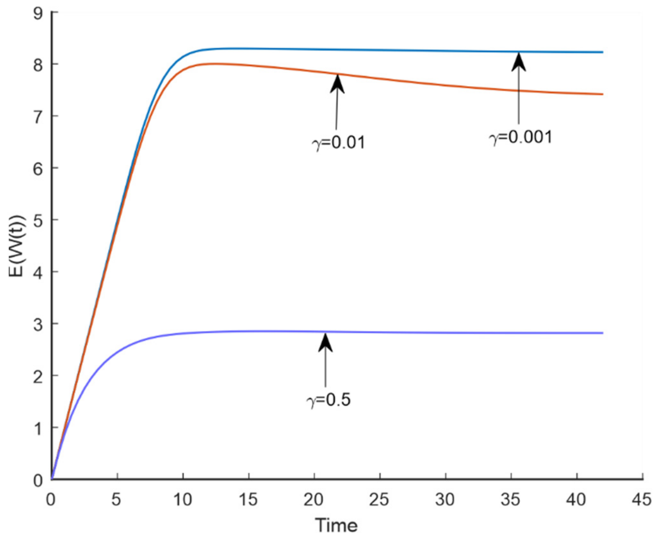

Fig. 6. Expected system size

Figures 7 and 8 display the dependences of the average waiting time of beneficiaries in the system against the service rate μ for different values of the arrival rate λ and the number of vacations permitted to the server, respectively. From Figure 7, we realised that there is a drastic decrease in the mean waiting time as the service rate increases, as intuitively expected. In this example, we take $\gamma = 0.1$ and $J = 5$. If the number of vacations permitted to the server is bigger, then the mean waiting time of beneficiaries in the system is also high, as expected, in Fig. 8. In this example, we take $\gamma = 0.1$, and $\lambda = 0.051$.

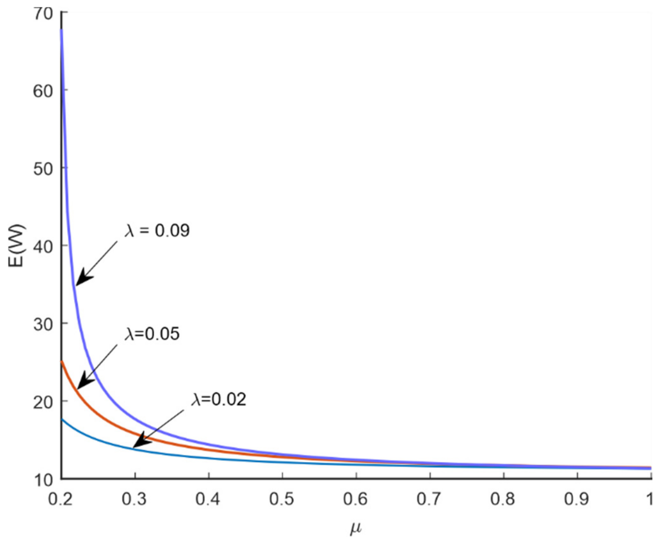

Fig. 7. Mean waiting time versus service rate for various values of λ

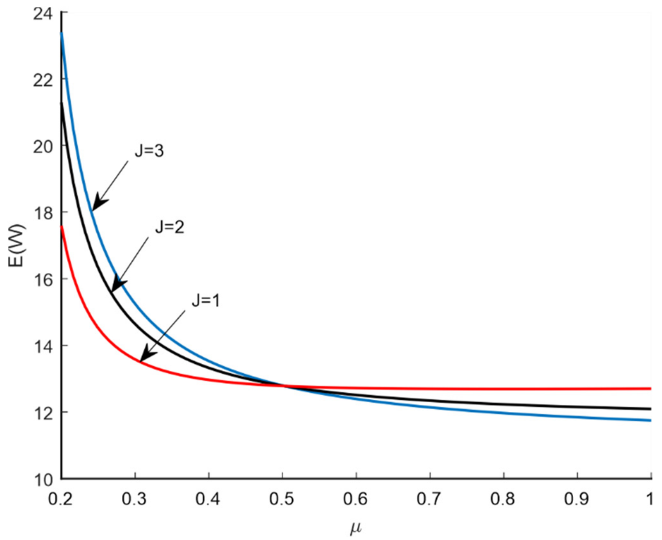

Fig. 8. Mean waiting time versus service rate for various values of J

8. Conclusion

In this paper, we derive a time-dependent solution of a single server Markovian queueing system with a variant of multiple vacation policy. By obtaining the transient solution of our model, we extend the work of Banik [3] where single arrival is possible. Also, we obtain numerically computable time-dependent solution and steady-state solution of Pikkala and Pilla [15] in which the server is not providing service while is in the vacations. We study the stochastic decomposition structures of steady-state system size distribution and waiting time distributions. The impacts of some variables on some crucial measures, like the expected waiting time of beneficiaries depicted numerically, shows that our closed-form solution is computable.

References

[1] AMMAR S.I., *Transient analysis of an M/M/1 queue with impatient behavior and multiple vacations*, Appl. Math. Comp., 2015, 260 (1), 97–105.

[2] AMMAR S.I., *Transient solution of an M/M/1 vacation queue with a waiting server and impatient customers*, J. Egypt. Math. Soc., 2017, 25 (3), 337–342.

[3] BANIK A.D., *The infinite-buffer single server queue with a variant of multiple vacation policy and batch Markovian arrival process*, Appl. Math. Model., 2009, 33 (7), 3025–3039.

[4] CHANDRASEKARAN V.M., INDHIRA K., SARAVANARAJAN M.C., RAJADURAI P., *A survey on working vacation queueing models*, Int. J. Pure Appl. Math., 2016, 106 (6), 33–41.

[5] DOSHI B.T., *Queueing systems with vacations. A survey*, Queueing Syst., 1986, 1, 29–66.

[6] IBE O.C., ISIJOLA O.A., *M/M/1 multiple vacation queueing systems with differentiated vacations*, Model. Sim. Eng., 2014, 3, 1–6.

[7] KALIDASS K., GNANARAJ J., GOPINATH S., KASTURI R., *Transient analysis of an M/M/1 queue with a repairable server and multiple vacations*, Int. J. Math. Oper. Res., 2014, 6 (2), 193–216.

[8] KALIDASS K., KASTURI R., *Time dependent analysis of M/M/1 queue with server vacations and a waiting server*, QTNA 2011, Proc. 6th International Conference on Queueing Theory and Network Applications, 2011, 77–83.

[9] KE J.C., *Operating characteristic analysis on the M/M/1 system with a variant vacation policy and balking*, Appl. Math. Model., 2007, 31 (7), 1321–1337.

[10] KE J.C., WU C.H., ZHANG G., *Recent developments in vacation queueing models: A short survey*, Int. J. Oper. Res., 2010, 7 (4), 3–8.

[11] KE J.C., HUANG H.I., CHU Y.K., *Batch arrival queue with N-policy and at most J vacations*, Appl. Math. Model., 2010, 34 (2), 451–466.

[12] LEVY Y., YECHIALI U., *An M/M/1 queues with servers' vacations*, INFOR., 1976, 14 (2), 153–163.

[13] LORIS-TEGHEM J., *Analysis of single server queueing systems with vacation periods*, Belg. J. Oper. Res. Stat. Comp. Sci., 1985, 25, 47–54.

[14] LIU W., XU X., TIAN N., *Some results on the M/M/1 queue with working vacations*, Oper. Res. Lett., 2002, 50, 41–52.

[15] PIKKALA V.L., PILLA R., *Transient Solution of M/M/1 variant working vacation queue with balking*, Int. J. Math. Model. Comp., 2018, 8, 17–27.

[16] SERVI L.D., FINN S.G., *M/M/1 queues with working vacations (M/M/1/WV)*, Perform. Eval., 2002, 50 (1), 41–52.

[17] SUDHESH R., AZHAGAPPAN A., *Transient analysis of an M/M/1 queue with variant impatient behavior and working vacations*, Opsearch, 2018, 55 (1), 787–806.

[18] SUDHESH R., AZHAGAPPAN A., DHARMARAJA S., *Transient analysis of M/M/1 queue with working vacation, heterogeneous service and customers' impatience*, RAIRO-Oper. Res., 2017, 51 (3), 591–606.

[19] SUDHESH R., FRANCIS RAJ L., *Computational analysis of stationary and transient distribution of single server queue with working vacation*, International Conference on Computing and Communication Systems, 2012, 480–489.

[20] SURANGA SAMPATH M.I.G., JICHENG LIU, *Impact of customers impatience on an M/M/1 queueing system subject to differentiated vacations with a waiting server*, Qual. Technol. Quant. Manage., 2018, 17 (2), 125–148.

[21] SURANGA SAMPATH M.I.G., KALIDASS K., *Transient analysis of a repairable single server queue with working vacations and system disasters*, Springer, 2019, 258–272.

[22] SURANGA SAMPATH M.I.G., KALIDASS K., JICHENG LIU., *Transient analysis of an M/M/1 queueing system subjected to multiple differentiated vacations, impatient customers and a waiting server with application to IEEE 802.16E power saving mechanism*, Indian145-146 J. Pure Appl. Math., 2020, 51 (1), 297–320.

[23] TAKAGI H., *Queueing analysis. A foundation of performance evaluation*, Vol. I. *Vacation and priority systems*, North-Holland, Amsterdam 1991.

[24] TIAN N., ZHANG Z.G., *Vacation queueing models*, Springer, 2006.

[25] UPADHYAYA S., *Queueing systems with vacation. An overview*, Int. J. Math. Oper. Res., 2016, 9 (2), 167–213.

[26] VIJAYASHREE K.V., JANANI B., *Transient analysis of an M/M/1 queueing system subject to differentiated vacations*, Qual. Technol. Quant. Manage., 2017, 15 (6), 730–748.

[27] YUE D., YUE W., SAFFER Z., CHEN X., *Analysis of an M/M/1 queueing system with impatient customers and a variant of multiple vacation policy*, J. Ind. Manage. Optim., 2014, 10 (1), 89–112.

[28] ZHANG Z.G., TIAN N., *Discrete time Geo/G/1 queue with multiple adaptive vacations*, Queueing Syst., 2001, 38, 419–429.