B A D A N I A   O P E R A C Y J N E   I   D E C Y Z J E

Nr 4 2009

František TURNOVEC*

FAIRNESS AND SQUARENESS: FAIR DECISION MAKING RULES IN THE EU COUNCIL?

In this paper we analyze the following problem: assuming that the distribution of voting weights in a simple voting committee is fair (whatever this means), how should we set up a voting rule to guarantee that the distribution of influence (relative voting power) is as close as possible to relative voting weights.

Keywords: *fairness, indirect voting power, Penrose–Banzhaf power index, optimal quota, Shapley–Shubik power index, simple voting committee, square root rule*

1. Introduction

In the late spring of 2004 the following draft of an open letter of European scientists to the governments of the EU member states was distributed among the community of European academics:

In the last few years there has been an intense discussion on the voting procedure in the Council of Ministers of the European Union. With 25 member states (and two more in the near future) it is not a simple task to make reliable judgements on the implications of the various voting systems that have been suggested.

We the undersigned wish to draw the attention of EU Governments to the fact that scientific methods can be used and need to be used to analyse, understand and design complex voting systems. In particular:

1) From a scientific point of view there are obvious drawbacks to the systems of voting in the European Council discussed so far. The experts on voting theory agree that the Treaty of Nice gives too much power to a number of countries while others obtain less power than appropriate. On the other hand, the draft European

* IES Faculty of Social Sciences, Charles University in Prague, Opletalova 26, 110 00 Prague 1, Czech Republic. e-mail: turnovec@mbox.fsv.cuni.cz

Constitution assigns too much power to the biggest and the smallest states in a systematic way, while the middle size countries do not get their due share of influence (see the tables attached). Moreover, the Nice system will be extremely ineffective due to its high quotas.

2) The 'compromises' proposed recently to change the quota in the draft Constitution either to 65% of the population and 55% of the states or to 55% of the population and 55% of the states make the situation for several countries even worse than in the draft Constitution. As can be shown by mathematical analysis, it is not the quotas that are mainly at fault, but rather the system of proposed weights.

3) The basic democratic principle that the vote of any citizen of a Member State ought to be worth as much as for any other Member State is strongly violated both in the voting system of the Treaty of Nice and in the rules given in the draft Constitution. It can be proved rigorously that this principle is fulfilled if the influence of each country in the Council is proportional to the square root of its population. This is known as 'Penrose's Square Root Law'. Such a system may be complemented by a simple majority of states.

4) A voting system that obeys the Square Root Law, i.e., which gives equal power to all citizens, is easily implemented. It is representative, objective, transparent, and effective. Such a system was proposed by Swedish diplomats already in 2000, and recently endorsed in a number of scientific articles.

We urge our politicians to take into consideration the contribution of the scientific community to this issue. We are highly concerned that any system implemented without due regard to the scientific analysis of voting power may become a major drawback to a democratic development in the European Union.

This open letter was originally signed by a group of nine distinguished scientists from six EU countries calling themselves "Scientists for a democratic Europe", later cosigned by 38 other colleagues and submitted to the governments of the member states and to the Commission[1]. In this paper we want to explore these statements.

The basic idea of the proposal supported by the open letter is the following concept of "fairness": *If the European Union is a union of citizens, then it is fair when each citizen (independently of her national affiliation) exercises the same influence over union issues. This is achieved when the voting weight of each national representation in the Council of Ministers is proportional to the square root of population*.

The so-called square-root rule is attributed to the British statistician Lionel Penrose [11] and is closely related to indirect voting power measured by Penrose–Banzhaf power index. Different aspects of the square root rule are analysed in FELSENTHAL and MACHOVER [3], [4], [5], LARUELLE and WIDGRÉN [8], BALDWIN and WIDGRÉN

[1] The letter (including added tables) and list of its signatories see e.g. at the following web address: http://www.esi2.us.es/~mbilbao/pdffiles/letter.pdf

[1], PLECHANOVOVÁ [12], SŁOMCZYŃSKI and ŻYCZKOWSKI [14], [15], HOSLI [7] and LEECH and AZIZ [9].

The concept of indirect voting power is based on the following rather artificial construction: Assume n districts (e.g. regions) with different population sizes (numbers of voters) are represented in a super-regional committee that decides different agendas relevant to the whole entity. Each district representative in the committee has some voting weight (number of votes). The decision making process is performed by a series of referenda in each district and districts' representatives in the committee vote according to the results of these referenda. In each district an individual citizen has the same voting weight (one vote) that provides him with a voting power (all the citizens from one district have the same voting power). Also, each district representative has some voting power in the committee based on the district's voting weight. The indirect voting power of a citizen from a particular district is given by the product of her voting power in local referenda and the voting power of her representative in the committee. Representation of districts in the committee is considered fair, if each citizen has the same indirect voting power independently of the district he belongs to.

2. Model

Let N be the set of members of a committee and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a nonnegative vector of the weights (e.g. votes or shares) of the committee members. A subset $S \in N$ of committee members voting uniformly (YES or NO) is called a voting configuration. Let us denote by $w(S) = \sum_{i \in S} w_i$, the weight of configuration S. A voting rule is defined by a quota q, satisfying

$$0 \le q \le \sum_{i \in N} w_i$$

(the quota q represents the minimal total weight necessary to approve the proposal). Usually we require a majority quota, i.e.

$$\frac{1}{2}\sum_{i \in N} w_i < q \le \sum_{i \in N} w_i \, .$$

The triple $[N, q, \mathbf{w}]$ is called a simple voting committee. Voting configuration S is called *winning* if $w(S) \ge q$ and *losing* in the opposite case. By $W[N, q, \mathbf{w}]$ we denote the set of all winning configurations in $[N, q, \mathbf{w}]$.

Voting power analysis seeks an answer to the following question: Given a simple weighted committee, what is the influence of its members over the outcome of voting?

The voting power of a member i is the probability that i will be decisive, in the sense that she would be able to reverse the outcome of voting by reversing her vote. To define a particular power measure means identifying some qualitative property (decisiveness), whose presence or absence in a voting process can be established and quantified (e.g. NURMI [10]). Generally, there are two such properties related to committee members' positions in voting that are used as a starting point for the quantification of voting power: swing position and pivotal position of committee members.

Let S be a winning configuration in a simple weighted committee $[N, q, \mathbf{w}]$. A member $i \in S$ has a swing in configuration S if $w(S) \geq q$ and $w(S \backslash \{i\}) < q$. Assuming all configurations are equally likely, it makes sense to evaluate the a priori voting power of each member of the committee as the probability of having a swing given all the possible configurations that contain i, irrespectively of whether they be winning or losing. This probability is measured by the absolute Penrose–Banzhaf (PB) power index (see [2]]:

$$\Phi_i^{PB}(N, q, \mathbf{w}) = \frac{s_i}{2^{n-1}}$$

where s_i is the number of swings of member i and 2^{n-1} is the number of coalitions with i as a member.

To compare the relative power of different members of the committee, the relative form of the Penrose–Banzhaf power index is used:

$$\phi_i^{PB}(N, q, \mathbf{w}) = \frac{s_i}{\sum_{k \in N} s_k},$$

Let the numbers 1, 2, ..., n be the fixed names of committee members and $(i_1, i_2, ..., i_n)$ be a permutation of these members. Let us assume that member k is in position r in this permutation, i.e. $k = i_r$. Member k of the committee is in a pivotal situation (has a pivot) with respect to permutation $(i_1, i_2, ..., i_n)$, if $w(\{i_1, i_2, ..., i_{r-1}\}) < q$ and $w(\{i_1, ..., i_{r-1}, i_r\}) \geq q$, which implies $w(\{i_{r+1}, i_{r+2}, ..., i_n\}) < 0$ and $w(\{i_r, i_{r+1}, ..., i_n\}) \geq 1$. Hence, the outcome of voting will be in this case identical to the vote of member $k = i_r$, "yes" if she votes "yes" and "no" if she votes "no". Assume that the strict ordering of members in a given permutation expresses an intensity of their support (preferences) for a particular issue in the sense that, if member i_s precedes i_t in this permutation, then support by i_s for a particular proposal is stronger than support by i_t.

One can expect that the group supporting the proposal will be formed in the order of the positions of members in the given permutation. If this is so, then member k will be in the situation that the group composed of the preceding members in the given permutation still does not have enough votes to pass the proposal, but the group of members placed behind her in the permutation does not have enough votes to block the proposal. The group that obtains her support will win. A member in a pivotal situation has a decisive influence on the final outcome. Assuming there are many voting acts and all the possible preference orderings are equally likely, under a full veil of ignorance about the preferences of individual members, it makes sense to evaluate the a priori voting power of each committee member as the probability of being in a pivotal situation. This probability is measured by the Shapley–Shubik (SS) power index (see [13]):

$$\Phi_i^{SS}(N, q, \mathbf{w}) = \frac{p_i}{n!}$$

(p_i is the number of pivotal positions of committee member i, and $n!$ is the number of permutations of the committee members, i.e., the number of different orderings of n elements). From

$$\sum_{i=1}^{n} p_i = n!$$

it follows, that

$$\phi_i^{SS}(N, q, \mathbf{w}) = \frac{p_i}{\sum_{k \in N} p_k} = \Phi_i^{SS}(N, q, \mathbf{w})$$

(i.e. the relative SS-power index is equal to the absolute one).

3. Penrose–Banzhaf indirect power

Let N be a set of districts (regions) and P_i the set of citizens (population) of district i, card $(N) = n$, card $(P_i) = p_i$. Consider a randomly selected "yes–no" issue and suppose that the people in each district decide their approval or disapproval by a referendum (each citizen has one vote). For simplicity, assume that the number of voters participating in a referendum in district i is equal to the respective population size, then $[P_i, m_i, \mathbf{e}_{pi}]$ is a referendum committee of the citizens of district i (by $\mathbf{e}_{pi}$ we denote a p_i-dimensional vector of 1's, by m_i a majority quota, i.e. int $\left(\frac{1}{2}p_i\right) < m_i \leq p_i$). Let $[N, q, \mathbf{w}]$ be a committee of districts (w_i is the weight of district i in the committee, q is a majority quota). Let S be a winning configuration in $[P_i, m_i, \mathbf{e}_{pi}]$. A citizen of district i has a swing in a winning voting configuration if and only if card $(S) = m_i$. Such a winning configuration in $[P_i, m_i, \mathbf{e}_{pi}]$ will be called a swing winning configuration.

The number of all swing winning configurations in $[P_i, m_i, \mathbf{e}_{pi}]$ is given by:

$$\binom{p_i}{m_i} = \frac{p_i!}{(p_i - m_i)!m_i!}.$$

The number of all swing winning configurations without the participation of a particular citizen is given by:

$$\binom{p_i - 1}{m_i} = \frac{(p_i - 1)!}{(p_i - 1 - m_i)!m_i!} = \frac{p_i!(p_i - m_i)}{(p_i - m_i)!m_i!\,p_i}.$$

The number of all swing winning configurations a particular citizen is a member of is given by:

$$\binom{p_i}{m_i} - \binom{p_i - 1}{m_i} = \binom{p_i}{m_i}\left[1 - \frac{(p_i - m_i)}{p_i}\right] = \binom{p_i}{m_i}\frac{m_i}{p_i}.$$

This gives the number of swings of a single citizen in a district referendum (by changing her vote, she will change the outcome). Using the probability of having a swing as a measure of voting power and assuming that all voting configurations are equally probable, we obtain the Penrose–Banzhaf absolute power index of a single citizen in district i as

$$\Pi_i^{PB}(p_i, m_i) = \frac{1}{2^{p_i - 1}}\binom{p_i}{m_i}\frac{m_i}{p_i}.$$

The voting power of each citizen is a decreasing function of the size of the population.

If we assume a simple majority quota (a proposal must be supported by more than half of voters plus one) equal to (for large p_i)

$$m_i = \mathrm{int}\left(\frac{1}{2}p_i + 1\right) \approx \frac{1}{2}p_i$$

(the smallest integer greater than $p_i/2$), then the number of cases in which an average citizen of district i has a swing (the outcome of the district referendum will be identical to her vote) is equal to

$$\frac{1}{2}\binom{p_i}{\left(\frac{p_i}{2}\right)} = \frac{1}{2}\frac{p_i!}{\left(\frac{p_i}{2}\right)!\left(\frac{p_i}{2}\right)!} = \frac{1}{2}\frac{p_i!}{\left(\left(\frac{p_i}{2}\right)!\right)^2}$$

and the probability of having a swing is

$$\Pi_i^{PB}\left(P_i, \mathrm{int}\left(\frac{1}{2}p_i + 1\right)\right) \approx \frac{1}{2}p_i\frac{p_i!}{\left(\left(\frac{p_i}{2}\right)!\right)^2}$$

(the power of a citizen of i, the absolute Penrose–Banzhaf index). Using Stirling's formula

$$n! \approx \frac{n^n}{e^n}\sqrt{2\pi\, n}.$$

(see FELSENTHAL and MACHOVER (1998)), for sufficiently large p_i we obtain the approximation

$$\Pi_i^{PB}\left(P_i, \mathrm{int}\left(\frac{1}{2}p_i + 1\right)\right) \approx \sqrt{\frac{2}{\pi\, p_i}}$$

(for a proof, see LARUELLE and WIDGREN (1998)). The larger the size of the population of district i, the smaller is an individual citizen's Penrose–Banzhaf power in referendum-type district voting.

If the district representatives in the committee of districts vote on each issue according to the results of district referenda and $\Phi_i^{PB}(N, q, \mathbf{w})$ is the absolute Penrose–Banzhaf power of the district i representative in the districts' committee, then

$$\Phi_i^{PB}(N, q, \mathbf{w})\Pi_i^{PB}\left(P_i, \mathrm{int}\left(\frac{1}{2}p_i + 1\right), \mathbf{e}_{p_i}\right) \approx \Phi_i^{PB}(N, q, w)\sqrt{\frac{2}{\pi\, p_i}}$$

is the indirect absolute voting power of an individual citizen of district i in the districts' committee (the probability that the result of voting in the districts' committee will be the same as the referendum vote of an average individual citizen from district i).

To guarantee equal indirect power for citizens from all the different districts in the districts' committee, the following must hold:

$$\Phi_i^{PB}(N, q, \mathbf{w})\sqrt{\frac{2}{\pi\, p_i}} = \alpha$$

for all i, where α is a positive constant. This holds if

$$\Phi_i^{PB}(N, q, \mathbf{w}) = \alpha\sqrt{\frac{\pi\, p_i}{2}}$$

i.e. if the absolute voting power of member states is proportional to the square root of their population. This happens if and only if the relative voting power of districts in the districts' committee is equal to

$$\phi_i^{PB}(N, q, \mathbf{w}) = \frac{\sqrt{p_i}}{\sum_{i \in N}\sqrt{p_i}}.$$

Substituting European Union member states for districts and the Council of Ministers for the districts' committee, we can apply the concept of "fair representation" to the distribution of the voting weights of member states in the Council.

To illustrate the concepts discussed above, let us consider the simple example of a hypothetical union of four member states A, B, C, and D (see Table 1). The data provided in Table 1 are not based on a calculation of the member states' PB-power indices with some particular voting quota in the hypothetical union Council; they indicate citizens' absolute PB-power in the member states.

Table 1. Citizens' national absolute PB-power in a hypothetical union

Country	Population	Square root of population	Relative square root of population	National power of voter
A	100×10^6	10×10^3	0.416667	0.000079788
B	49×10^6	7×10^3	0.291667	0.000113984
C	25×10^6	5×10^3	0.208333	0.000159577
D	4×10^6	2×10^3	0.083333	0.000398942
total	178×10^6	24×10^3	1	

Source: Author's own calculations.

4. Voting weights, quota and voting power

It was rigorously proved what a fair distribution of voting power in the Council should look like in order to guarantee equal indirect voting power of all European citizens (providing a system of referenda is the mechanism of decision making)[2]. But there is still one problem to be solved: what allocation of voting weights among member states leads to proportionality of Penrose–Banzhaf relative voting power to the square root of the population? Supporters of the square root rule propose allocating the weights in the Council in proportion to the square root of population, assuming that in committees with a large number of members the distribution of weights is a good proxy of voting power. But a priori voting power seldom reflects the exact distribution of voting weights. If $[N, q, \mathbf{w}]$ is a simple weighted committee and $\phi[N, q, \mathbf{w}]$ is a vector of the relative power indices of its members, then usually $\phi[N, q, \mathbf{w}] \neq \alpha\mathbf{w}$.

Originally, it was assumed that voting weights proportional to the square root of the population, together with a simple majority quota, would provide a solution of the problem. But it appears that this is generally not the case (or is only the case in the limit $n \to \infty$). Being aware of this, SŁOMCZYŃSKI and ŻYCZKOWSKI [14] formulated the following minimization problem.

[2] It can be proved rigorously that this principle is fulfilled if the influence of each country in the Council is proportional to the square root of its population. This is known as 'Penrose's Square Root Law' (see Open letter).

Find a majority quota q that minimizes the sum of square residuals between the relative Penrose–Banzhaf power indices and relative voting weights defined to be proportional to the square root of the population.

$$\sigma^2(q) = \sum_{i \in N} \left(\phi_i(N, q, \sqrt{p}) - \frac{\sqrt{p_i}}{\sum_{k \in N} \sqrt{p_k}} \right)^2 .$$

We shall refer to (21) as the *SZ distance* and to the quota q^* minimizing (21) as the *optimal quota*.

Using a heuristic, Słomczyński and Życzkowski found the following approximation of the optimal quota: $q^* \approx 61.4\%$ for the EU composed of 27 countries. Hence, the final proposal, known as the "Jagiellonian Compromise", reads as follows: "*The voting weight of each member state is allocated proportionally to the square root of its population, the decision of the Council being taken if the sum of weights exceeds a (certain) quota*" (SŁOMCZYŃSKI and ŻYCZKOWSKI ([14], p. 3136). Later they gave a general formula for approximating the optimal quota

$$\underline{q} = \frac{1}{2} \sum_{i \in N} \sqrt{p_i} \left(1 + \frac{1}{\sqrt{n}} \right) \le q \le \frac{1}{2} \left(\sum_{i \in N} \sqrt{p_i} + \sqrt{\sum_{i \in N} p_i} \right) = \bar{q}$$

minimizing the distance between the vector of relative weights and the vector of relative PB power indices under the assumption that n = card (N) is sufficiently large, i.e. $n \to \infty$ (see SŁOMCZYŃSKI and ŻYCZKOWSKI [15]). Let us call this approximation the *SZ optimal quota*.

Let us check the relation between the weights and quota for our hypothetical union from Table 1. Assume that the weights in the Council are equal to the square roots of the population $\sqrt{p_i}$ and the quota q is from the Słomczyński and Życzkowski interval:

$$\underline{q} = \frac{1}{2} \sum_{i \in N} \sqrt{p_i} \left(1 + \frac{1}{\sqrt{n}} \right) = 18000 \le q \le 18670.83 = \frac{1}{2} \left(\sum_{i \in N} \sqrt{p_i} + \sqrt{\sum_{i \in N} p_i} \right) = \bar{q} .$$

In Table 2 we present the absolute and relative Penrose–Banzhaf power indices of the national representatives using the quota q = 18335.415 from the above interval and the resulting indirect absolute PB voting power of citizens of each member state in the union (in fact, in this case we obtain the same results for both the lower and upper values of q from the SZ interval). The SZ distance is used to compare the calculated relative voting power of member states with relative square root shares. The resulting indirect voting power of citizens from different member states is far from being equal. The "rigorously proved" square root rule remains problematic, even if we accept the "national referenda" mechanism.

Table 2. Indirect voting power of citizens from different member states under the SZ optimal quota

SZ quota	[18000, 18670]		relative SR	Swings	PB abs	PB rel.	Citizens' national power	Citizens' indirect power	SZ distance
Country	Population	SR							
A	100×10^6	10×10^3	0.416667	3.00	0.3750	0.3750	0.000079788	0.000029921	0.001736111
B	49×10^6	7×10^3	0.291667	3.00	0.3750	0.3750	0.000113984	0.000042744	0.006944444
C	25×10^6	5×10^3	0.208333	1.00	0.1250	0.1250	0.000159577	0.000019947	0.006944444
D	4×10^6	2×10^3	0.083333	1.00	0.1250	0.1250	0.000398942	0.000049868	0.001736111
Total	178×10^6	24×10^3	1	8	1.0000	1			0.017361111

Source: Author's own calculations.

In our case, the square root rule does not lead to the equalization of indirect power: either the SZ optimal quota is not correct (possibly due to the small number of countries), or the square root rule is not as good an approximation as it is declared to be (or both).

5. Exact solution of the optimal quota problem

The optimal quota problem has an exact solution which is independent of asymptotic properties (limiting behavior as $n \to \infty$). TURNOVEC ([18], pp. 335–348) showed that for any simple weighted committee there exists a finite number of different intervals $(\gamma_0, \gamma_1]$, ..., $(\gamma_{m-1}, \gamma_m]$ such that for any quota from a particular interval $(\gamma_{k-1}, \gamma_k]$ the sets of winning voting configurations are the same and for quotas from different intervals the sets of winning configurations are different. These intervals are called *quota intervals of stable power*; the power index of any voter is constant for all quotas from the same interval of stable power. Let $(\gamma_{k-1}, \gamma_k]$ be a quota interval of stable power, then γ_k is called a marginal quota and to obtain the power indices for all $q \in (\gamma_{k-1}, \gamma_k]$ it is enough to calculate them for the marginal quota γ_k only. Quota intervals of stable power can be calculated as follows.

Define a partition of the set of all voting configurations (the power set 2^N) into equal weight classes $\Omega_0, \Omega_1, ..., \Omega_r$ (such that the weights of different configurations from the same class are the same and the weights of configurations from different classes are different). For completeness, set $w(\emptyset) = 0$. Consider the increasing ordering, according to weight, of equal weight classes $\Omega^{(0)}, \Omega^{(1)}, ..., \Omega^{(r)}$ such that for any $t < k$ and $S \in \Omega^{(t)}$, $R \in \Omega^{(k)}$ we have $w(S) < w(R)$. Denote $q_t = w(S)$ for any $S \in \Omega^{(t)}$, $t = 1, 2, ..., r$. There is a finite number $r \leq 2^n - 1$ of different quota intervals of stable power $(q_{t-1}, q_t]$ such that $W[N, q_t, \mathbf{w}] \subset W[N, q_{t-1}, \mathbf{w}]$.

In Table 3 we present all the quota intervals of stable power in the Council of the hypothetical Union from Table 1, where the voting weights of the various member states in the council are equal to the square roots of their population. All the voting configurations with the same value are listed in the first column, the values of these configurations in the second column and quota intervals of stable power in the third column. In our case, we have 12 quota intervals of stable power with 6 majority quota intervals.

Table 3. Quota intervals of stable power for the hypothetical union

$\Omega^{(t)}$	$W(S)$	$(\gamma_{t-1}, \gamma_t]$
∅	0	
{4}	2000	(0,2000]
{3}	5000	(2000,5000]
{2}, {3,4}	7000	(5000,7000]
{2, 4}	9000	(7000,9000]
{1}	10000	(9000,10000]
{1, 4}, {2, 3}	12000	(10000,12000]
{2, 3, 4}	14000	(12000,14000]
{1, 3}	15000	(14000,15000]
{1, 2}, {1, 3, 4}	17000	(15000,17000]
{1, 2, 4}	19000	(17000,19000]
{1, 2, 3}	22000	(19000,22000]
{1, 2, 3, 4}	24000	(22000,24000]

From the final number of quota intervals of stable power, it follows that there exists an exact solution to the optimal quota problem of Słomczyński and Życzkowski:

$$q^* = \arg\min_j \sum_{i \in N} \left(\phi_i(N, q_j, \sqrt{\mathbf{p}}) - \frac{\sqrt{p_i}}{\sum_{k \in N} \sqrt{p_k}} \right)^2$$

where $j = 1, 2, ..., m$, m is the number of majority quota intervals of stable power and $q_j \in (\gamma_{j-1}, \gamma_j]$. It is enough to consider only marginal quotas. The quota q^* provides only the best approximation of square root fairness, it does not guarantee the exact proportionality of power indices and weights. Moreover, this property is not related specifically to the square root rule and holds for any reasonable power index (e.g. Shapley–Shubik) as well. If $(\gamma_{t-1}, \gamma_t]$ is the quota interval of stable power minimizing the distance between the vector of relative power indices and the vector of relative weights and the approximation of the SZ optimal quota is correct, then the following must hold:

$$\gamma_{t-1} < \underline{q} \le \overline{q} \le \gamma_t .$$

Table 4. All possible PB power indices for majority quota intervals of stable power

Country	Population	SR	Relative SR	Swings	PB abs. power	PB rel. power	Citizens' national power	Citizens' indirect power	SZ distance
QISP	(12000, 14000]								
A	100×10^6	10×10^3	0.416667	5	0.6250	0.4167	0.000079788	0.000049868	0
B	49×10^6	7×10^3	0.291667	3	0.3750	0.2500	0.000113984	0.000042744	0.001736111
C	25×10^6	5×10^3	0.208333	3	0.3750	0.2500	0.000159577	0.000059841	0,001736111
D	4×10^6	2×10^3	0.083333	1	0.1250	0.0833	0.000398942	0.000049868	0
Total	178×10^6	24×10^3	1	12		1			0.003472222
QISP	(14000, 15000]								
A	100×10^6	10×10^3	0.416667	6	0.7500	0.6000	0.000079788	0.000059841	0.03361111
B	49×10^6	7×10^3	0.291667	2	0.2500	0.2000	0.000113984	0.000028496	0.00840278
C	25×10^6	5×10^3	0.208333	2	0.2500	0.2000	0.000159577	0.000039894	0.00006944
D	4×10^6	2×10^3	0.083333	0	0.0000	0.0000	0.000398942	0.000000000	0.00694444
Total	178×10^6	24×10^3	1	10		1	0.000059804		0.04902778
QISP	(1500, 17000]								
A	100×10^6	10×10^3	0.416667	5	0.6250	0.5000	0.000079788	0.000049868	0.00694444
B	49×10^6	7×10^3	0.291667	3	0.3750	0.3000	0.000113984	0.000042744	0.00006944
C	25×10^6	5×10^3	0.208333	1	0.1250	0.1000	0.000159577	0.000019947	0.01173611
D	4x10^6	2×10^3	0.083333	1	0.1250	0.1000	0.000398942	0.000049868	0.00027778
Total	178×10^6	24×10^3	1	10		1	0.000059804		0.01902778
QISP	(17000, 19000]								
A	100×10^6	10×10^3	0.416667	3	0.3750	0.3750	0.000079788	0.000029921	0.00173611
B	49×10^6	7×10^3	0.291667	3	0.3750	0.3750	0.000113984	0.000042744	0.00694444
C	25×10^6	5×10^3	0.208333	1	0.1250	0.1250	0.000159577	0.000019947	0.00694444
D	4×10^6	2×10^3	0.083333	1	0.1250	0.1250	0.000398942	0.000049868	0.00173611
Total	178×10^6	24×10^3	1	8		1	0.000059804		0.01736111
QISP	(19000, 22000]								
A	100×10^6	10×10^3	0.416667	2	0.2500	0.3333	0.000079788	0.000019947	0.00694444
B	49×10^6	7×10^3	0.291667	2	0.2500	0.3333	0.000113984	0.000028496	0.00173611
C	25×10^6	5×10^3	0.208333	2	0.2500	0.3333	0.000159577	0.000039894	0.01562500
D	4×10^6	2×10^3	0.083333	0	0.0000	0.0000	0.000398942	0.000000000	0.00694444
Total	178×10^6	24×10^3	1	6		1	0.000059804		0.03125000
QISP	(22000, 24000]								
A	100×10^6	10×10^3	0.416667	1	0.1250	0.2500	0.000079788	0.000009974	0.02777778
B	49×10^6	7×10^3	0.291667	1	0.1250	0.2500	0.000113984	0.000014248	0.00173611
C	25×10^6	5×10^3	0.208333	1	0.1250	0.2500	0.000159577	0.000019947	0.00173611
D	4×10^6	2×10^3	0.083333	1	0.1250	0.2500	0.000398942	0.000049868	0.02777778
Total	178×10^6	24×10^3	1	4		1.0000			0.05902778

Source: Author's own calculations.

In Table 4 we present the distribution of voting power for all the majority quota intervals of stable power (QISP) in our hypothetical union and the corresponding indirect power of citizens in the union. We can see that while any quota from the interval [18000, 18670.83] is an SZ optimal quota, the exact solution to the optimal quota problem is given by any quota from the quota interval of stable power (12000, 14000] (or in relative terms, any quota between 50.01% and 59.33%). There is no quota granting equal indirect power to citizens, the best SZ distance (21) provided by the exact optimal quota is 0.00347222, while the SZ distance based on the SZ optimal quota approximation is 0.01736111.

5. Several remarks on the square root rule

The model of the equalization of indirect Penrose–Banzhaf power applied to the distribution of voting weights in the EU is legitimate and scientifically justified. But it is not the unique way of implementing the fairness principle and the statements from the open letter of European scientists are rather exaggerated. They are based on two different premises: a) on the concept of a referendum type mechanism for decision making, which is crucial to the principle of fairness used itself, b) on the implicit assumption that the Penrose–Banzhaf model is the only way to quantify voting power, which is crucial to implementation. However, exactly the same can be done with the Shapley–Shubik model of voting power.

Let us comment first on the model of a referendum based mechanism of decision making. It contradicts the intuition of a representative democracy and introduces in reality a process of direct democracy. National representation in the Council of the EU means government representation. With few exceptions, governments do not feel political responsibility to the majority of citizens, even in internal affairs. Citizens in multi-party systems do not elect the government, but they decide on the composition of the parliament. Government formation is based on trade-offs between political parties and individual members of parliament, minority governments exist etc.

Even if we accept the square root principle of fairness, the implementation has weak formal points (the approximation for the optimal quota).

It has not been sufficiently justified why fairness in voting is an exclusive attribute of the Penrose–Banzhaf power index[3]. If one selects Shapley–Shubik power as

[3] Supporters of the square root rule associate its justification exclusively with the Penrose–Banzhaf concept of power. Their objections to the Shapley–Shubik concept of power are based on a classification of power measures as either so called I-power (a voter's potential influence over the outcome of voting) and P power (expected relative share in a fixed prize available to the winning group of committee members, based on cooperative game theory), introduced by FELSENTHAL, MACHOVER and ZWICKER [6]. The Shapley–Shubik power index was declared to represent P-power and as such unsuitable for measuring influence in voting. In TURNOVEC [16] and TURNOVEC, MERCIK and MAZURKIEWICZ [17] we tried to show that the objections against the Shapley–Shubik power index, based on its interpretation as a concept of P-power, are not sufficiently justified. Both the Shapley–Shubik and Penrose–Banzhaf measures could be successfully derived as values of a cooperative game and, at the same time, both of them can be interpreted as probabilities of being in some decisive position (pivot, swing) without using cooperative game theory at all.

a measure of influence, then the same scheme of fairness can be implemented. A fair distribution of power among member states in council voting should be equal, in this case, to population size. Let p_i be the size of the population in member state i, then $1/p_i$ is the Shapley–Shubik voting power of a single citizen of country i in a national referendum. Then, if the SS-power of country i in the council is

$$\phi_i^{SS} = \frac{p_i}{\sum_{k=1}^{n} p_k},$$

then the SS indirect voting power

$$\frac{1}{p_i}\frac{p_i}{\sum_{k=1}^{n} p_k} = \frac{1}{\sum_{k=1}^{n} p_k}$$

is the same for every citizen of the EU. For voting weights proportional to population size we can find a quota minimizing the distance between a "fair" SS-power distribution and the power distribution generated by the population weights using

$$q^* = \arg\min_j \sum_{i \in N} \left(\phi_i^{SS}(N, q_j, \mathbf{p}) - \frac{p_i}{\sum_{k \in N} p_k} \right)^2 .$$

Such a concept of fairness would be more transparent and understandable to citizens and politicians than using Stirling's or other complicated combinatorial formulas.

The choice of a "fairness principle" is a problem of political consensus among member states and cannot be resolved by the "scientific community" using mathematical models. However, clarification, clear formulation and representation of the problem can be of help to political decision making. From this point of view, discussion of the square root rule is useful and legitimate. What is wrong is that it is presented as the only correct way of dealing with the problem and this creates the illusion that the "fairness" issue has been solved.

Acknowledgements

This research was supported by the Czech Government Research Target Program, project No. MSM0021620841 and by the Grant Agency of the Czech Republic, project No. 402/09/1066 "Political economy of voting behavior, rational voter theory and models of strategic voting". The author is grateful to Běla Plechanovová and to three anonymous referees for valuable comments and suggestions.

References

[1] Baldwin R., Widgrén M., *Winners and Losers under Various Dual Majority Rules in the EU's Council of Ministers*, [in:] M. Wiberg (ed.), *Reasoned Choices*, The Finnish Political Science Association, Turku, 2004, 42–89.

[2] Banzhaf J.F., *Weighted Voting doesn't Work: a Mathematical Analysis*, Rutgers Law Review, 1965, 19, 317–343.

[3] Felsenthal D.S., Machover M., *The Measurement of Voting Power, Theory and Practice*, Cheltenham, Edward Elgar, 1998.

[4] Felsenthal D.S., Machover M., *Analysis of QM Rule adopted by the EU Inter-Governmental Conference, Brussels, 18 June 2004*, Research report online, LSE, London 2004, [http://eprints.lse.ac.uk/2531]

[5] Felsenthal D.S., Machover M., *Analysis of QM Rule adopted by the Council of the European Union*, Brussels, 23 June 2007. Research report online, LSE, London, 2007, [http://eprints.lse.ac.uk/archive0000431]

[6] Felsenthal D.S., Machover M., Zwicker W., *The Bicameral Postulates and Indices of A Priori Voting Power*, Theory and Decision, 1998, 44, 83–116.

[7] Hosli M.O., *Council Decision Rules and European Union Constitutional Design*, AUCO Czech Economic Review, 2008, 2, 76–96.

[8] Laruelle A., Widgrén M., *Is the Allocation of Voting Power among EU States Fair?* Public Choice, 1998, 94, 317–339.

[9] Leech D., Aziz H., *The Double Majority Voting Rule of the EU Reform Treaty as a Democratic Ideal for an Enlarging Union: an Appraisal Using Voting Power Analysis*, Presented at the 2008 EPCS meeting in Jena, 2008.

[10] Nurmi H., *On Power Indices and Minimal Winning Coalitions*, Control and Cybernetics, 1997, 26, 609–611.

[11] Penrose L.S., *The Elementary Statistics of Majority Voting*, Journal of the Royal Statistical Society, 1946, 109, 53–57.

[12] Plechanovová B., *The Draft Constitution and the Decision-Making Rule for the Council of Ministers of the EU: Looking for an Alternative Solution*, European Integration Online Papers, 2004, 8 (12), [http://eiop.or.at/eiop/texte/2003-006a.htm].

[13] Shapley L.S., Shubik M., *A Method for Evaluation the Distribution of Power in a Committee System*, American Political Science Review, 1954, 48, 787–792.

[14] Słomczyński W., Życzkowski K., *Penrose Voting System and Optimal Quota*, Acta Physica Polonica, B, 2006, 37, No. 11, 3133–3143.

[15] Słomczyński W., Życzkowski K., *From a Toy Model to the Double Square Root System*, Homo Oeconomicus, 2007, 24, No. 3–4, 381–400.

[16] Turnovec F., *New Measure of Voting Power*, AUCO Czech Economic Review, 2007, 1, 4–14.

[17] Turnovec F., Mercik J.W., Mazurkiewicz M., *Power Indices Methodology: Decisiveness, Pivots and Swings*, [in:] M. Braham and F. Steffen (eds.), *Power, Freedom and Voting, Essays in Honor of Manfred J. Holler*, Springer Verlag, Berlin, Heidelberg, 2008, 23–37.

[18] Turnovec F., *The Efficiency of Fairness in Voting Systems*, Homo Oeconomicus, 2008, 25, No. 3–4, 335–348.

Sprawiedliwość i metoda pierwiastka kwadratowego: sprawiedliwe reguły podejmowania decyzji w Radzie Unii Europejskiej?

Przedstawiono analizę pojęcia sprawiedliwej reprezentacji wyborców w ciele reprezentującym różne ich grupy, takim jak np. parlament federalny w kraju będącym federacją lub unią. Pojęcie to, wprowadzone do dyskusji nad rozdziałem głosów i zasadami głosowania w Radzie Unii Europejskiej w roku 2004, sprowadzono do propozycji rozdziału wagi głosów między kraje członkowskie według pierwiastka kwadratowego z liczby mieszkańców poszczególnych krajów. Taki rozdział wag głosów gwarantowałby tę samą siłę pośrednią głosu wszystkim obywatelom UE mierzoną według indeksu siły głosu Penrose'a–Banzhafa. W pracy rozważa się jednak zagadnienie sprawiedliwej reprezentacji w głosowaniu w podobnych ciałach w sposób ogólniejszy, zakładając, że rozkład sił głosów w takim ciele jest sprawiedliwy, cokolwiek to by miało oznaczać: jak mianowicie ustalić reguły decydowania, aby zagwarantować, że rozkład wpływów – faktyczna względna siła głosu – będzie możliwie najbliższy względnym wagom głosów.

Słowa kluczowe: *sprawiedliwość, pośrednia siła głosu, indeks siły głosu Penrose'a–Banzhafa, optymalny próg większości, indeks siły głosu Shapleya–Shubika, proste głosowania, reguła pierwiastka kwadratowego*