O P E R A T I O N S   R E S E A R C H   A N D   D E C I S I O N S

No. 4 2019

DOI: 10.37190/ord190404

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# SPECIAL VERSUS GENERAL PROTECTION AND ATTACK OF TWO ASSETS

Two independent assets are analysed, being subject to special and general protection and attack, supplementing earlier research on individual and overarching protection and attack. Sixteen analytical solutions are developed to examine how a defender and attacker choose either two special efforts, one general effort, or one special effort and one general effort. The latter occurs when the special unit effort cost for one asset is lower than that of the other asset and the general unit effort cost. The article provides a tool for each player to realise which of these three options it should choose when facing an opponent who also chooses between these three options. The solutions are explained and illustrated with examples. The article focuses on specialization versus generalization of effort which is of paramount importance.

**Keywords:** *special effort, general effort, protection, defence, attack, reliability, vulnerability, independent assets, safety, security, terrorism*

## 1. Introduction

### 1.1. Background and contribution

This article analyses special and general protection and attack of two independent assets, supplementing earlier research by Hausken [14] on two parallel and series assets. An asset can be anything of value, e.g., the 22 target types listed in the Global Terrorism Database[2]. Each asset can be subject to one special effort (protection[3] or attack[4]) designed particularly for that asset. Additionally, each asset can be subject to one general

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[3]Protection refers to hardening the assets so that they remain operational, and are not destroyed, stolen, or compromised.

[4]Attack refers to attempts to break through the protection so that the assets do not remain operational, and furthermore, to destroy, steal, or compromise the assets.

effort (protection or attack) aimed at both assets. This contrasts with individual and overarching protection and attack, which is abundantly analysed in the literature, in a manner that may not be immediately obvious. The difference was discovered mathematically by the authors and can be illustrated by a plethora of examples.

The mathematical difference is that special and general protection and attack require only one contest for each asset. That one contest expresses the damage probability or vulnerability for the asset, depending on protection and attack. More specifically, the special and general efforts operate additively in the numerator and denominator of the contest success function which expresses the damage probability or vulnerability. In contrast, individual and overarching protection and attack for $n$ protection levels require one contest for each asset, plus $n - 1$ which is the number of levels above the individual level. For example, two independent assets and $n = 2$ protection levels require two contests at the individual level for the two independent assets, and one contest at the collective or aggregate level for the two independent assets considered as an aggregate unit, i.e., three contests[5].

Distinguishing between the individual and overarching protection and attack means that an attacker first has to break through the overarching protection. If that is impossible, due to overwhelming overarching protection or because the attacker somehow lacks the attack expertise, protection of each asset individually is irrelevant, and thus the attacker's potential preparation for an attack on each asset individually is also irrelevant. Only if the attacker breaks through the overarching protection, does individual protection and attack become relevant. Examples of overarching protection are border security, general intelligence, public health measures such as immunisation, and general methods to survive chemical, biological, and explosive terrorism.

In contrast, special and general protection and attack assumes only one level, which is realistic when individual assets are not aggregated in any way, and which is frequently the case in practice. Separating the different levels is often unrealistic and may sometimes be arbitrary. Letting special and general efforts operate additively in the numerator and denominator of the contest success function is mathematically complicated. We thus consider the simplest situation which gives strategic insight, i.e., the 16 possible combinations of how to allocate two special efforts and one general effort by a defender and an attacker.

[5]With $n - 3$ protection levels, regardless how the individual assets are aggregated first from level 1 to level 2, and later from level 2 to level 3, two more contests are required beyond what is required for special and general protection and attack analysed in this article.

## 1.2. Examples

### 1.2.1. Military example

Consider the military usually divided into the army, the navy, and the air force. Within each of these three branches, personnel and equipment differ across regions of the world (cities, mountains, deserts, oceans), seasons (summer, autumn, winter, spring), time of day (day, night), temperatures, humidity, etc. Examples of equipment types are small arms, artillery, vehicles, aircraft, vessels, attire, field equipment. These divisions illustrate how special versus general protection and attack can be designed in a plethora of different manners.

### 1.2.2. Bank example

Consider a bank equipped with physical assets (money, gold, silver), digital assets (computers with storage of digital currencies, securities, bonds, etc.), human assets (employees, customers, visitors, etc.), and office equipment, electricity, and building structure. The bank may design one special protection for each of these four kinds of assets, i.e. four specially designed protections. Thus, physical assets may be protected by storage in vaults with a security guard opening the vault when needed. The digital assets may be stored on specialised or general storage computers, and be protected by cryptography, firewalls, antivirus, etc. designed in a special or general manner dependent on their nature. The human assets may be protected by security personnel regulating who enters the bank, emergency buttons which personnel can push when threatened, armed patrolling guards, etc. The office equipment, electricity, and building structure may be protected by personnel trained in maintenance, replacement, and theft prevention. Alternatively, or additionally, the bank may design general protection against all the four kinds of assets, e.g., the trained general guards who know the nature of the four kinds of assets and which kinds of threats are likely.

Similarly, a criminal organisation specialising in bank robberies may hire attackers (thieves, robbers, etc.) specialising in attacking each of the four kinds of assets, or general ones attacking two or three or all four kinds of assets. For example, poisonous gas, food poisoning, or taking hostages are special attacks which work only against the human assets, and not against the other three kinds of assets. Special protections are gas masks or training people to somehow avoid the gas or avoid being impacted by it, food purification or methods that prevent or alleviate food poisoning, and possibly methods to prevent taking hostages. Compromising the power supply is another special attack, against which back-up power supply constitutes protection. Examples of general attack are dynamite, a bulldozer, and a helicopter which may be successful against several of the four kinds of assets. Examples of general protection are explosive detection dogs, antiballistic missiles, and surveillance of the air space above the bank, and the ground below it.

#### 1.2.3. Museum example

Consider a museum with items that vary in cost, age, size, weight, fame, rarity, fragility, sophistication, appearance, and a plethora of other characteristics. The museum can hire one special guard trained to have intimate knowledge into how to protect some of the items, say the costly fragile sophisticated items, and one second special guard trained to have intimate knowledge into how to protect the remaining items, which are not costly, fragile, and sophisticated. Alternatively, or additionally, the museum can hire one general guard trained to protect all items regardless of their characteristics.

Similarly, a criminal cartel may hire an attacker specializing in attacking (stealing, destroying, compromising, etc.) one set of items, e.g., paintings which may be secured on walls in particular ways known to the attacker who may additionally know how to transport, store, and sell paintings through its acquaintances on the black market. The cartel may hire a second attacker specializing in attacking the museum jewellery, which is usually protected and secured differently, and that requires different methods, mode of transport, storage, and network for further sale or distribution. Alternatively, the cartel may hire a general attacker trained to attack all kinds of items, including paintings and jewellery. Cross-fertilisation of knowledge of how to attack different items may enable the general attacker to be more successful.

#### 1.2.4. A general view of the examples

The examples illustrate how special protection and attack utilise division of labour and the application of different methods and equipment suitable for multiple assets. Examples of specialised labour are particular skill sets such as language competence, negotiation experience, cryptography knowledge, weapons training, combat experience. General protection and attack may comprise multiple skill sets. The examples are straight-forwardly extended or supplemented. Today’s literature does not enable analysing such examples which do not involve multiple protection levels. Instead, the examples illustrate one level where special and general efforts operate additively, as analysed in this article.

## 1.3. Literature

#### 1.3.1. One protection level with non-additive efforts

The common approach in the literature is to analyse one protection level. This multifarious literature is reviewed by Hausken and Levitin [18]. Let us mention a few relevant works. For infrastructures and series and parallel systems, see [2, 4, 6, 10–12]. For border security and control weapons of mass destruction, see [1, 5, 9, 30]. For element separation and protection in multi-state systems, see [23, 17]. For protection based on differing measures of target attractiveness, see [3]. For multiple targets, see [38, 39, 35, 8]. Further examples are protection against terrorism and natural disasters [40], resource allocation between target hardening and border security [34], intrusion detection in heterogeneous networks [29], ranking the elements of water-supply networks depending on their value to the network's owner [31], and assessing the importance measures for ranking the system elements in complex systems [32].

#### 1.3.2. Individual and overarching protection and attack

Two protection levels are analysed by Haphuriwat and Bier [9], Hausken [20, 13], Levitin and Hausken [26] Levitin et al. [27], Peng et al. [33], and Levitin and Hausken [25, 24]. Many protection levels are analyzed by Levitin [22] Korczak and Levitin [21], Levitin et al. [28], and Golalikhani and Zhuang [7].

#### 1.3.3. Three rare examples of additivity in the contest success function

First, Zhuang and Bier [40] and Hausken and Zhuang [19] consider a contest where the defender chooses protection added to an exogenously given inherent protection level. Second, Hausken et al. [15] assess two contests where a defender chooses one general effort against both natural disaster and terrorism. In the first contest, the defender adds special protection against natural disaster only. In the second contest the defender adds special protection against terrorism only. Third, Hausken [14] analyses special versus general protection and attack of parallel and series assets. Section 2 formulates the problem. Section 3 analyses the system. Section 4 provides sensitivity analysis. Section 5 concludes the line of reasoning.

## 2. Nomenclature and problem formulation

### 2.1. Nomenclature

#### Parameters

$r_i$ – defender's valuation for asset $i$, $i = 1, 2$, $r_i \geq 0$
$R_i$ – attacker's valuation for asset $i$, $i = 1, 2$, $R_i \geq 0$
$c_i$ – defender's special unit cost of protecting asset $i$, $i = 1, 2$
$C_i$ – attacker's special unit cost of attacking asset $i$, $i = 1, 2$
$c$ – defender's general unit cost of protecting both assets
$C$ – attacker's general unit cost of attacking both assets
$m_i$ – contest intensity for asset $i$, $i = 1, 2$
$m$ – contest intensity when $m_1 = m_2 = m$

**Strategic choice variables**

$t_i$ – defender's special protection effort for asset $i$, $i = 1, 2$
$T_i$ – attacker's special attack effort for asset $i$, $i = 1, 2$
$t$ – defender's general protection effort for both assets
$T$ – attacker's general attack effort for both assets

**Dependent variables**

$V_i$ – vulnerability of asset $i$, $i = 1, 2$, due to special and general protection and attack
$u$ – defender's expected utility
$U$ – attacker's expected utility

### 2.2. Game formulation

We consider two independent assets which the defender values as $r_i \geq 0$ and the attacker values as $R_i \geq 0$, $i = 1, 2$. Figure 1 shows how the defender applies special effort $t_i$ at unit cost $c_i$ to protect asset $i$, $i = 1, 2$. The defender applies general effort $t$ at unit cost $c$ to protect both assets. Analogously, the attacker applies special effort $T_i$ at unit cost $C_i$ to attack asset $i$, and general effort $T$ at unit cost $C$ to attack both assets.

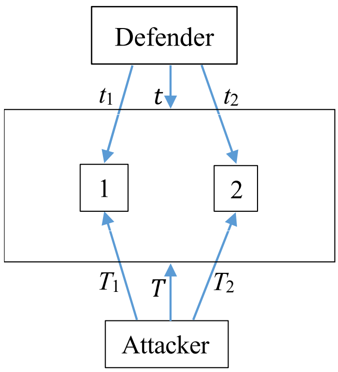



Fig. 1. Special protection $t_1$ and $t_2$, special attack $T_1$ and $T_2$, general protection $t$, and general attack $T$ for two independent assets 1 and 2

Asset $i$'s vulnerability is its damage probability. Asset $i$ is protected with effort $t_i + t$, and attacked with effort $T_i + T$. Utilising the ratio form contest success function [36, 37] asset $i$'s vulnerability is

$$V_i = \frac{(T_i + T)^{m_i}}{(t_i + t)^{m_i} + (T_i + T)^{m_i}} \tag{1}$$

where $\partial V_i/\partial t_i < 0$ and $\partial V_i/\partial T_i > 0$, which follows from differentiating (1) with respect to $t_i$ and $T_i$, and $m_i \geq 0$ is asset $i$'s contest intensity. When $0 \leq m_i \leq 1$, exerting less effort than the other $m_i$ player has disproportional advantage. When $m_i = 1$, each effort has proportional advantage. When $m_i > 1$, exerting more effort than the other player has disproportional advantage. When $m_i = \infty$, exerting slightly more effort than the other player gives a 'winner-takes-all' situation. For further illustration of the intensity parameter $m_i$, see Hausken and Levitin [16]. The expected damage summed over the two assets is

$$\begin{aligned} d &= r_1 \frac{(T_1+T)^{m_1}}{(t_1+t)^{m_1}+(T_1+T)^{m_1}} + r_2 \frac{(T_2+T)^{m_2}}{(t_2+t)^{m_2}+(T_2+T)^{m_2}} \\ D &= R_1 \frac{(T_1+T)^{m_1}}{(t_1+t)^{m_1}+(T_1+T)^{m_1}} + R_2 \frac{(T_2+T)^{m_2}}{(t_2+t)^{m_2}+(T_2+T)^{m_2}} \end{aligned} \tag{2}$$

as experienced by the defender and the attacker, respectively. The defender maximises the sum $r_1 + r_2$ of the asset values minus the expected damage $d$ minus the effort expenditures. The attacker maximises the expected damage $D$ minus the effort expenditures. The players' expected utilities are

$$\begin{aligned} u &= r_1 + r_2 - r_1 \frac{(T_1+T)^{m_1}}{(t_1+t)^{m_1}+(T_1+T)^{m_1}} - r_2 \frac{(T_2+T)^{m_2}}{(t_2+t)^{m_2}+(T_2+T)^{m_2}} - c_1 t_1 - c_2 t_2 - ct \\ U &= R_1 \frac{(T_1+T)^{m_1}}{(t_1+t)^{m_1}+(T_1+T)^{m_1}} + R_2 \frac{(T_2+T)^{m_2}}{(t_2+t)^{m_2}+(T_2+T)^{m_2}} - C_1 T_1 - C_2 T_2 - CT \end{aligned} \tag{3}$$

Equation (3) implies that if $c \leq \mathrm{Min}\{c_1, c_2\}$ or $C \leq \mathrm{Min}\{C_1, C_2\}$, then no player applies special effort, and if $c \geq c_1 + c_2$ and $C \geq C_1 + C_2$, then no player applies general effort. The 12 parameters in the model, known to both players, are the four asset valuations $r_1, r_2, R_1, R_2$, the six unit effort costs $c_i, C_i, c, C$, and the two contest intensities $m_i$, $i = 1, 2$. Assuming complete information about parameters is a good first approximation. The defender's three strategic choice variables are $t_i$ and $t$, $i = 1, 2$. The attacker's three strategic choice variables are $T_i$ and $T$. Both players choose their strategies independently and simultaneously. Simultaneous moves are chosen since they are believed to be descriptive, and also to make the analysis more tractable. One common alternative, where the defender moves first, ignores the fact that the defender in practice often chooses protection based on observations, analysis, and reasoning of what the attackers usually may have done earlier or in similar situations. Another alternative, where the attacker moves first, is often used to analyse a defender's emergency response; see Hausken and Levitin [18] for a review of these alternatives, which may be examined in future research.

## 3. Solving the model

The first-order conditions, from differentiating (3), are

$$\frac{\partial u}{\partial t_i}=\frac{r_i m_i (t_i+t)^{m_i-1}(T_i+T)^{m_i}}{\left((t_i+t)^{m_i}+(T_i+T)^{m_i}\right)^2}-c_i=0,\quad \frac{\partial U}{\partial T_i}=\frac{R_i m_i (T_i+T)^{m_i-1}(t_i+t)^{m_i}}{\left((t_i+t)^{m_i}+(T_i+T)^{m_i}\right)^2}-C_i=0,$$
$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial t_1}+\frac{\partial u}{\partial t_2}+c_1+c_2-c=0,\quad \frac{\partial U}{\partial T}=\frac{\partial U}{\partial T_1}+\frac{\partial U}{\partial T_2}+C_1+C_2-C=0 \tag{4}$$

The six first-order conditions in (4) have four unknown, $t_1+t$, $t_2+t$, $T_1+T$, and $T_2+T$. An interior solution, which is a Nash equilibrium in pure strategies, occurs only when $c_1+c_2=c$ and $C_1+C_2=C$. This causes multiple optima. Hence, general protection and attack, on the one hand, and special protection of and attack against each asset individually, on the other hand, are equally effective. A minuscule perturbation of any single unit cost violates one of the equalities. Thus, the equalities rarely hold in practice. We consequently do not consider this unstable interior solution. Mixed strategies are not considered. Corner solutions, which are also Nash equilibria, are analysed. These arise when one or both players do not apply one or several efforts, because they are too costly, or to avoid negative expected utilities.

Consider the following eight solutions for the defender's strategies: ($t_1=0$, $t_2=0$, $t=0$), ($t_1=0$, $t_2=0$, $t\geq 0$), ($t_1=0$, $t_2\geq 0$, $t=0$), ($t_1=0$, $t_2\geq 0$, $t\geq 0$), ($t_1\geq 0$, $t_2=0$, $t=0$), ($t_1\geq 0$, $t_2=0$, $t\geq 0$), ($t_1\geq 0$, $t_2\geq 0$, $t=0$), ($t_1\geq 0$, $t_2\geq 0$, $t\geq 0$). We analyse four of these solutions where each asset receives either special or general protection, or both special and general protection. That is, ($t_1=0$, $t_2=0$, $t\geq 0$) means only general protection, ($t_1=0$, $t_2\geq 0$, $t\geq 0$) means only general protection of asset 1 and both special and general protection of asset 2, ($t_1\geq 0$, $t_2=0$, $t\geq 0$) means only general protection of asset 2 and both special and general protection of asset 1, and ($t_1\geq 0$, $t_2\geq 0$, $t=0$) means only special protection. We exclude the last solution, ($t_1\geq 0$, $t_2\geq 0$, $t\geq 0$), which is not a corner solution. We also exclude the first solution, ($t_1=0$, $t_2=0$, $t=0$), which means that no asset is protected. That exclusion follows from the insight that no player withdraws from applying effort into protecting or attacking when applying the ratio form contest success function in a simultaneous game. Regardless how costly protection or attack is, exerting some minimum effort is always optimal.

There are thus two solutions left among the eight. In (19) in Section 3.5, we analyse ($t_1 = 0, t_2 \geq 0, t = 0$) as a special corner solution of ($t_1 \geq 0, t_2 \geq 0, t = 0$), where $t_1$ decreases to $t_1 = 0$. Then, asset 1 is not protected due to $c_2 \leq c_1$ against an overwhelming general attack $T > 0$ against both assets. Analogously, in (18) in Section 3.5, we analyse ($t_1 \geq 0$, $t_2 = 0$, $t = 0$) as a special corner solution of ($t_1 \geq 0$, $t_2 \geq 0$, $t = 0$), where $t_2$ decreases to $t_2 = 0$. Then, asset 2 is not protected due to $c_1 \leq c_2$ against an overwhelming general attack $T > 0$ against both assets. Consequently, out of the defender's eight solutions, we exclude two solutions, we analyse four solutions, and we analyse the last two solutions as special corner solutions of the four solutions.

Table 1. Sixteen solutions arising from four corner solutions for each player

| R | Defender | Attacker | S |
|---|---|---|---|
| 1 | Ose, $c \geq c_1 + C_2 \Rightarrow t = 0, t_1 \geq 0, t_2 \geq 0$ | Ose, $C \geq C_1 + C_2 \Rightarrow T = 0, T_1 \geq 0, T_2 \geq 0$ | 3.1 |
| 2 | Ose, $c \geq c_1 + C_2 \Rightarrow t = 0, t_1 \geq 0, t_2 \geq 0$ | Oge, $C \leq \text{Min}\{C_1, C_2\} \Rightarrow T_1 = T_2 = 0, T \geq 0$ | 3.5 |
| 3 | Ose, $c \geq c_1 + C_2 \Rightarrow t = 0, t_1 \geq 0, t_2 \geq 0$ | Sge, $C_1 \leq C_2, C_1 < C < C_1 + C_2 \Rightarrow T_2 = 0, T_1 \geq 0, T \geq 0$ | 3.9 |
| 4 | Ose, $c \geq c_1 + C_2 \Rightarrow t = 0, t_1 \geq 0, t_2 \geq 0$ | Sge, $C_1 > C_2, C_2 < C < C_1 + C_2 \Rightarrow T_1 = 0, T_2 \geq 0, T \geq 0$ | 3.9 |
| 5 | Oge, $c \leq \text{Min}\{c_1, c_2\} \Rightarrow t_1 = t_2 = 0, t \geq 0$ | Ose, $C \geq C_1 + C_2 \Rightarrow T = 0, T_1 \geq 0, T_2 \geq 0$ | 3.4 |
| 6 | Oge, $c \leq \text{Min}\{c_1, c_2\} \Rightarrow t_1 = t_2 = 0, t \geq 0$ | Oge, $C \leq \text{Min}\{C_1, C_2\} \Rightarrow T_1 = T_2 = 0, T \geq 0$ | 3.2 |
| 7 | Oge, $c \leq \text{Min}\{c_1, c_2\} \Rightarrow t_1 = t_2 = 0, t \geq 0$ | Sge, $C_1 \leq C_2, C_1 < C < C_1 + C_2 \Rightarrow T_2 = 0, T_1 \geq 0, T \geq 0$ | 3.7 |
| 8 | Oge, $c \leq \text{Min}\{c_1, c_2\} \Rightarrow t_1 = t_2 = 0, t \geq 0$ | Sge, $C_1 > C_2, C_2 < C < C_1 + C_2 \Rightarrow T_1 = 0, T_2 \geq 0, T \geq 0$ | 3.7 |
| 9 | Sge, $c_1 \leq c_2, c_1 < c < c_1 + C_2 \Rightarrow t_2 = 0, t_1 \geq 0, t \geq 0$ | Ose, $C \geq C_1 + C_2 \Rightarrow T = 0, T_1 \geq 0, T_2 \geq 0$ | 3.6 |
| 10 | Sge, $c_1 \leq c_2, c_1 < c < c_1 + C_2 \Rightarrow t_2 = 0, t_1 \geq 0, t \geq 0$ | Oge, $C \leq \text{Min}\{C_1, C_2\} \Rightarrow T_1 = T_2 = 0, T \geq 0$ | 3.8 |
| 11 | Sge, $c_1 \leq c_2, c_1 < c < c_1 + C_2 \Rightarrow t_2 = 0, t_1 \geq 0, t \geq 0$ | Sge, $C_1 \leq C_2, C_1 < C < C_1 + C_2 \Rightarrow T_2 = 0, T_1 \geq 0, T \geq 0$ | 3.3 |
| 12 | Sge, $c_1 \leq c_2, c_1 < c < c_1 + C_2 \Rightarrow t_2 = 0, t_1 \geq 0, t \geq 0$ | Sge, $C_1 > C_2, C_2 < C < C_1 + C_2 \Rightarrow T_1 = 0, T_2 \geq 0, T \geq 0$ | 3.3 |
| 13 | Sge, $c_1 > c_2, c_2 < c < c_1 + C_2 \Rightarrow t_1 = 0, t_2 \geq 0, t \geq 0$ | Ose, $C \geq C_1 + C_2 \Rightarrow T = 0, T_1 \geq 0, T_2 \geq 0$ | 3.6 |
| 14 | Sge, $c_1 > c_2, c_2 < c < c_1 + C_2 \Rightarrow t_1 = 0, t_2 \geq 0, t \geq 0$ | Oge, $C \leq \text{Min}\{C_1, C_2\} \Rightarrow T_1 = T_2 = 0, T \geq 0$ | 3.8 |
| 15 | Sge, $c_1 > c_2, c_2 < c < c_1 + C_2 \Rightarrow t_1 = 0, t_2 \geq 0, t \geq 0$ | Sge, $C_1 \leq C_2, C_1 < C < C_1 + C_2 \Rightarrow T_2 = 0, T_1 \geq 0, T \geq 0$ | 3.3 |
| 16 | Sge, $c_1 > c_2, c_2 < c < c_1 + C_2 \Rightarrow t_1 = 0, t_2 \geq 0, t \geq 0$ | Sge, $C_1 > C_2, C_2 < C < C_1 + C_2 \Rightarrow T_1 = 0, T_2 \geq 0, T \geq 0$ | 3.3 |

Ose – only special efforts, Oge – only general efforts, Sge – one special effort and general effort. R – row, S – section.

The reasoning is analogous for the attacker's eight solutions. More specifically, we exclude ($T_1 \geq 0$, $T_2 \geq 0$, $T \geq 0$), which is not a corner solution, and ($T_1 = 0$, $T_2 = 0$, $T = 0$), where no asset is attacked. We analyse the four solutions: ($T_1 = 0$, $T_2 = 0$, $T \geq 0$), ($T_1 = 0$, $T_2 \geq 0$, $T \geq 0$), ($T_1 \geq 0$, $T_2 = 0$, $T \geq 0$), and ($T_1 \geq 0$, $T_2 \geq 0$, $T = 0$). For the two remaining solutions, in (15) in Section 3.4, we analyse ($T_1 = 0$, $T_2 \geq 0$, $T = 0$) as a special corner solution of ($T_1 \geq 0$, $T_2 \geq 0$, $T = 0$), where $T_1$ decreases to $T_1 = 0$. Then, asset 1 is not attacked due to $C_2 \leq C_1$ against overwhelming general protection $T > 0$ of both assets. Analogously, in (14) in Section 3.4, we analyse ($T_1 \geq 0$, $T_2 = 0$, $T = 0$) as a special corner solution of (($T_1 \geq 0$, $T_2 \geq 0$, $T = 0$), where $T_2$ decreases to $T_2 = 0$. Then, asset 2 is not attacked due to $C_1 \leq C_2$ against overwhelming general protection $t > 0$ of both assets. Combination of the defender's four solutions and the attacker's four solutions gives the $4 \times 4 = 16$ solutions in Table 1.

Table 1 applies $\geq$ instead $>$ since the variables are not necessarily positive, as illustrated in the next sections. In row 1 both players apply only special efforts (Ose), illustrated in the upper right quadrant among the nine quadrants in Fig. 2. The unit costs $c$ and $C$ of general efforts satisfy $c \geq c_1 + c_2$ and $C \geq C_1 + C_2$, causing $t = T = 0$; see Section 3.1. In contrast, in row 6 both players apply only general efforts (Oge), illustrated in the lower left quadrant. The unit costs $c$ and $C$ satisfy $c \leq \text{Min}\{c_1, c_2\}$, $C \leq \text{Min}\{C_1, C_2\}$, causing $t_1 = t_2 = T_1 = T_2 = 0$; see Section 3.2. Such low $c$ and $C$ are realistic when innovation permits both general protection and attack to be superior to special protection and attack. When $\text{Min}\{c_1, c_2\} < c < c_1 + c_2$ and $\text{Min}\{C_1, C_2\} < C < C_1 + C_2$, in rows 11, 12, 15, 16, i.e., when $c$ and $C$ are neither small nor large, the players apply one special effort and general effort (Sge), in the middle quadrant; see Section 3.3.

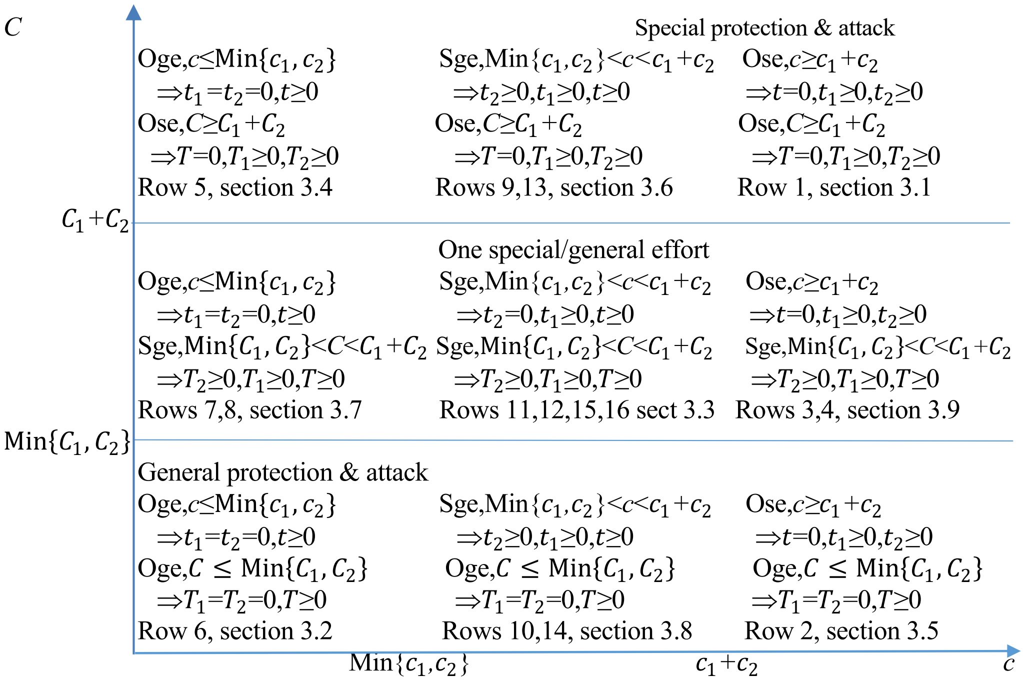



Fig. 2. Portraying 16 solutions in Table 1 in nine regions. The defender's general unit protection cost $c$ varies horizontally. The attacker's general unit attack cost $C$ varies vertically

In Figure 2, $c$ varies along the horizontal axis, marked off at $\text{Min}\{c_1, c_2\}$ and $c_1 + c_2$, and $C$ varies along the vertical axis, marked off at $\text{Min}\{C_1, C_2\}$ and $C_1 + C_2$. This causes nine regions encompassing the 16 rows in Table 1. Each of the four corner regions matches one row. The centre region matches the four rows 11, 12, 15, 16, accounting for $\{c_1 \le c_2, C_1 \le C_2\}$, $\{c_1 \le c_2, C_1 < C_2\}$, $\{c_1 > c_2, C_1 \le C_2\}$, $\{c_1 > c_2, C_1 > C_2\}$, respectively. Each of the four remaining regions matches two rows.

The nine regions in Fig. 2 are analysed in the next nine subsections. Although a region in Fig. 2 may specify certain efforts, they are not applied when they are negative or cause negative expected utility or utilities. It is straightforward to determine the other solutions, but they are omitted since they are too numerous, causing transgression by several times the length requirements for a journal article, and since the structure in Table 1 and Fig. 2 illustrates the main characteristics. An example in each subsection illustrates the solutions.

In subsections with an even number of first-order conditions, i.e., two first-order conditions in Section 3.2 and four first-order conditions in Sections 3.1, 3.3.1, 3.3.2, 3.6, 3.9, the first-order conditions for the defender and attacker are solved against each other. This is possible for general contest intensities $m_1$ and $m_2$ since ratios between the players' efforts are determinable. This applies to five of the nine regions in Fig. 2, i.e., the four regions in the upper right and the lower left region. In contrast, in subsections with an odd number of first-order conditions, i.e., three first-order conditions in Sections 3.4, 3.5, 3.7, 3.8, ratios between the players' efforts are undeterminable. Thus, analytical solutions for general $m_1$ and $m_2$ are unavailable, we set $m_1 = m_2 = 1$, and solve second-order conditions. This applies to the four remaining regions in Fig. 2, i.e., the two outmost regions along the horizontal axis, and the two outmost regions along the vertical axis.

### 3.1. Only special protection and attack, row 1

The upper right quadrant in Table 1 requires $c \ge c_1 + c_2$ and $C \ge C_1 + C_2$. Thus, general protection and attack have higher unit costs than the sum of the two special unit costs for both players. Hence, the players choose special protection and attack; see Appendix A when $t = T = 0$.

$$\text{If } 1-m_i+Q_i^{m_i} \ge 0 \text{ and } 1+(1-m_i)Q_i^{m_i} \ge 0,$$

$$\text{then } t=T=0,\quad T_i=\frac{R_i m_i Q_i^{m_i}}{C_i(Q_i^{m_i}+1)^2},\quad t_i=Q_iT_i,\quad Q_i\equiv\frac{C_i/R_i}{c_i/r_i} \tag{5}$$

$$u=\sum_{i=1}^{2}\frac{r_i(1-m_i+Q_i^{m_i})Q_i^{m_i}}{(Q_i^{m_i}+1)^2},\quad U=\sum_{i=1}^{2}\frac{R_i(1+(1-m_i)Q_i^{m_i})}{(Q_i^{m_i}+1)^2}$$

The inequalities are included to prevent negative expected utilities in (5). The $Q_i$ parameter for asset $i$ expresses the attacker's ratio of unit effort cost $C_i$ and asset valuation $R_i$, divided by the defender's ratio of unit effort cost $c_i$ and asset valuation $r_i$. Hence, small $Q_i < 1$ means that the attacker is advantaged, and large $Q_i > 1$ means that the attacker is disadvantaged. A common benchmark is $Q_i = 1$, where both players are equally advantaged.

Table 2 shows the interpretation of $Q_i^{m_i}$ to ease the understanding of the equations in the remainder of the article.

Table 2. Interpreting $Q_i^{m_i}$ depending on $Q_i$ and $m_i$

| | $Q_i < 1$ | $Q_i > 1$ |
|---|---|---|
| $m_i < 1$ | $Q_i^{m_i} > Q_i$ | $Q_i^{m_i} < Q_i$ |
| $m_i > 1$ | $Q_i^{m_i} < Q_i$ | $Q_i^{m_i} > Q_i$ |

$Q_i < 1$ advantaged attacker, $Q_i > 1$ disadvantaged attacker, $m_i < 1$ contest intensity below 1, $m_i > 1$ contest intensity above one.

Equation (5) applies when both general protection and attack are too expensive. Equation (5), accounting for special protection and attack, is a special case of Hausken's [13] analysis of arbitrarily many assets and, in addition, overarching protection and attack.

**Example 1.** Inserting the common benchmark $Q_i = 1$ into (5) gives

$$\text{If } m_i \le 2, \quad \text{then } t = T = 0, \quad t_i = T_i = \frac{R_i m_i}{4C_i}, \quad u = \sum_{i=1}^{2} \frac{r_i(2-m_i)}{4}, \quad U = \sum_{i=1}^{2} \frac{R_i(2-m_i)}{4} \tag{6}$$

Example 1 shows how the players' efforts are proportional to the attacker's asset valuation $R_i$ and contest intensity $m_i$, and inverse proportional to the attacker's unit effort cost $C_i$ when the players are equally matched in terms of unit effort costs and resources defined such that $Q_i = 1$. When $m_i$ increases above $m_i = 2$, both players withdraw efforts from asset $i$ and (5) no longer applies. Assume that $r_1 + r_2 = r$ and $R_1 + R_2 = R$, where $r$ and $R$ are the system valuations of two assets in parallel or series. Then, the efforts $t_i = T_i$ in (6) equal the efforts in the parallel and series systems analysed by Hausken [14]. When $m_1 = m_2 = m_i$, the expected utilities in (6) are $u = (r_1 + r_2)(2 - m_i)/4$ and $U = (R_1 + R_2) \times(2 - m_i)/4$, i.e., equal utilities $u = U$ when $r_1 + r_2 = R_1 + R_2$. The sum $u = U$ of the expected utilities equals the sum of the expected utilities of both the parallel system, i.e., $u = r(3 - m_i)/4$ and $U = R(1 - m_i)/4$ (where the defender is advantaged) and the series system, i.e., (where the attacker is advantaged) analysed by Hausken [14]. The comparison of Examples 2 and 3 with Hausken [14] is analogous. Comparing these and the subsequent examples are left to the reader as exercises.

### 3.2. Only general protection and attack, row 6

The lower left quadrant in Table 1 requires $c \le \mathrm{Min}\{c_1, c_2\}$ and $C \le \mathrm{Min}\{C_1, C_2\}$. We assume $m_i = m$ for analytical solution. With low unit effort costs the players choose only general protection and attack; see Appendix A when $t_1 = t_2 = T_1 = T_2 = 0$.

If $1-m+Q^m \ge 0$ and $1+(1-m)Q^m \ge 0$

$$\text{then } T = \frac{(R_1+R_2)mQ^m}{C(Q^m+1)^2},\quad t = QT,\quad Q \equiv \frac{C/(R_1+R_2)}{c/(r_1+r_2)} \tag{7}$$

$$t_1 = t_2 = T_1 = T_2 = 0,\quad u = \frac{(r_1+r_2)Q^m(1-m+Q^m)}{(Q^m+1)^2},\quad U = \frac{(R_1+R_2)(1+(1-m)Q^m)}{(Q^m+1)^2}$$

Equation (7) applies when special protection and attack are too costly. With general protection and attack, the two assets are protected and attacked as one entity. Equation (7) is a special case of Hausken's [14] analysis of arbitrarily many assets (also accounting for overarching protection and attack), and as a special case of Hausken's [11] analysis of a series or parallel system with one asset.

**Example 2.** Inserting $Q = 1$ into (7) gives

$$\begin{aligned} &\text{If } m \le 2,\quad \text{then } t = T = \frac{(R_1+R_2)m}{4C},\quad t_1 = t_2 = T_1 = T_2 = 0 \\ &u = \frac{(r_1+r_2)(2-m)}{4},\quad U = \frac{(R_1+R_2)(2-m)}{4} \end{aligned} \tag{8}$$

### 3.3. One special effort and one general effort, rows 11, 12, 15, 16

The middle quadrant in Table 1 requires $\mathrm{Min}\{c_1, c_2\} < c < c_1 + c_2$ and $\mathrm{Min}\{C_1, C_2\} < C < C_1 + C_2$. In this subsection, general protection and attack have lower unit costs than the sum of the two special unit costs for each player, but larger unit costs than the minimum of the two special unit costs. That is, both players prefer applying general effort, but also prefer applying the special effort with lowest unit cost.

#### 3.3.1. $c_1 \le c_2$ and $C_1 \le C_2$, rows 11, 16

Assume $c_1 \le c_2$ and $C_1 \le C_2$, which also covers $c_1 > c_2$ and $C_1 > C_2$ by interchanging the subscripts 1 and 2. The players apply general effort, special effort for asset 1, and no special effort for asset 2.

**Property 1.** When $\text{Min}\{c_1, c_2\} < c < c_1 + c_2$, $\text{Min}\{C_1, C_2\} < C < C_1 + C_2$, $c_1 \leq c_2$, $C_1 \leq C_2$,

$$\frac{m_1 Q_1^{m_1}}{(Q_1^{m_1}+1)^2 C_1/R_1} \geq \frac{m_2 P_1^{m_2}}{(P_1^{m_2}+1)^2 (C-C_1)/R_2}$$

and $\dfrac{m_1 Q_1^{m_1+1}}{(Q_1^{m_1}+1)^2 C_1/R_1} \geq \dfrac{m_2 P_1^{m_2+1}}{(P_1^{m_2}+1)^2 (C-C_1)/R_2}$

$T_1 \geq 0, \quad t_1 \geq 0, \quad u \geq 0, \quad U \geq 0, \quad$ then

$$T = \frac{m_2 P_1^{m_2}}{(P_1^{m_2}+1)^2 (C-C_1)/R_2}, \quad t = P_1 T, \quad P_1 \equiv \frac{(C-C_1)/R_2}{(c-c_1)/r_2}, \quad t_2 = T_2 = 0$$

$$T_1 = \frac{m_1 Q_1^{m_1}}{(Q_1^{m_1}+1)^2 C_1/R_1} - \frac{m_2 P_1^{m_2}}{(P_1^{m_2}+1)^2 (C-C_1)/R_2} \tag{9}$$

$$t_1 = \frac{m_1 Q_1^{m_1+1}}{(Q_1^{m_1}+1)^2 C_1/R_1} - \frac{m_2 P_1^{m_2+1}}{(P_1^{m_2}+1)^2 (C-C_1)/R_2}$$

$$u = \frac{r_1 Q_1^{m_1}(1-m_1+Q_1^{m_1})}{(Q_1^{m_1}+1)^2} + \frac{r_2 P_1^{m_2}(1-m_2+P_1^{m_2})}{(P_1^{m_2}+1)^2}$$

$$U = \frac{R_1(1+(1-m_1)Q_1^{m_1})}{(Q_1^{m_1}+1)^2} + \frac{R_2(1+(1-m_2)P_1^{m_2})}{(P_1^{m_2}+1)^2}$$

If $\text{Min}\{c_1, c_2\} < c < c_1 + c_2$, $\text{Min}\{C_1, C_2\} < C < C_1 + C_2$, $c_1 \leq c_2$, $C_1 \leq C_2$,

$$\frac{m_1 Q_1^{m_1}}{(Q_1^{m_1}+1)^2 C_1/R_1} < \frac{m_2 P_1^{m_2}}{(P_1^{m_2}+1)^2 (C-C_1)/R_2} \text{ or } \frac{m_1 Q_1^{m_1+1}}{(Q_1^{m_1}+1)^2 C_1/R_1} < \frac{m_2 P_1^{m_2+1}}{(P_1^{m_2}+1)^2 (C-C_1)/R_2}$$

$T_1 \geq 0, \quad t_1 \geq 0, \quad u \geq 0, \quad U \geq 0,$

then three solutions with $t_2 = T_2 = 0$ are possible depending on the parameter values. Either $t_1 = 0$ and $T_1 = 0$ arise as in (23) in Section 3.7 with only general protection but special and general attack, or $t_1 > 0$ and $T_1 = 0$ arise as in (25) in Section 3.8 with special and general protection but only general attack, or $t_1 = T_1 = 0$ arise as in (7) in Section 3.2 with only general protection and attack.

**Proof.** Appendix B.

When the conditions for (9) are satisfied, both players always apply general protection and attack, i.e., $T \geq 0$ and $t \geq 0$. Special protection and attack are not guaranteed. In (9), $t_1$ follows from $T_2$ by multiplying the first term in $T_2$ with $Q_1$ and multiplying the second term in $T_2$ with $P_1$. The attacker applies special effort $T_1$ when three conditions are satisfied. First, the contest intensity $m_1$ for asset 1 should not be too small compared with the contest intensity $m_2$ for asset 2. Second, $Q_1^{m_1}$ should not be too small compared with $P_1^{m_2}$. This occurs when $C$ is only marginally above $C_1$, or $c$ is substantially above $c_1$. Third, opposite to the second, $C_1$ should not be too large compared with $C - C_1$. The sizes of $m_1$ and $m_2$ dictate the relative weights of the second and third conditions. Low $m_1$ and $m_2$ support the more intuitive third condition, where low $C_1$ relative to $C$ more easily justifies special effort $T_1$. The defender applies special effort $t_1$ when the first two of the three conditions are satisfied. Third, for the defender, consistently with the second condition, which is more intuitive for the defender, $c_1$ should not be too large compared with $c$ to justify positive $t_1$. Fourth, for the defender, to ensure positive efforts, both expected utilities have to be positive, which occurs when $m_1$ and $m_2$ are not too large.

**Example 3.** Inserting $Q_1 = P_1 = 1$ into (9) gives

$$\text{If } T_1 \geq 0 \text{ and } m_i \leq 2, \text{ then } t = T = \frac{m_2}{4(C - C_1)/R_2}$$

$$t_1 = T_1 = \frac{m_1}{4C_1/R_1} - \frac{m_2}{4(C - C_1)/R_2} \tag{10}$$

$$t_2 = T_2 = 0, \quad u = \sum_{i=1}^{2} \frac{r_i(2 - m_i)}{4}, \quad U = \sum_{i=1}^{2} \frac{R_i(2 - m_i)}{4}$$

First, when $Q_1 = P_1 = 1$, interpreted so that the players are equally matched, their general efforts are proportional to $m_1$ and $m_2$. Second, special efforts $T_1$ and $t_1$ are applied when $m_1$ is not too small relative to $m_2$, since asset 1 is then more important for both players. Third, the attacker applies more special effort $T_1$ when $C_1$ is small. Fourth, the defender applies special effort $t_1$ when $c_1$ is small.

#### 3.3.2. $c_1 \leq c_2$ and $C_1 > C_2$, rows 12, 15

Assume $c_1 \leq c_2$ and $C_1 > C_2$, which also covers $c_1 > c_2$ and $C_1 \leq C_2$ by interchanging the subscripts 1 and 2. As in Section 3.3.1, the defender applies general protection combined with special protection of asset 1, i.e., $t_2 = 0$. However, the attacker combines general attack with special attack on asset 2, i.e., $T_1 = 0$.

**Property 2.** When $\mathrm{Min}\{c_1, c_2\} < c < c_1 + c_2$, $\mathrm{Min}\{C_1, C_2\} < C < C_1 + C_2$, $c_1 \le c_2$, $C_1 > C_2$, $\frac{R_2 m_2 Q_{21}^{m_2}}{C_2(Q_{21}^{m_2}+1)^2} \ge \frac{R_1 m_1 P_{21}^{m_1}}{(C-C_2)(P_{21}^{m_1}+1)^2}$ and $\frac{R_1 m_1 P_{21}^{m_1+1}}{(C-C_2)(P_{21}^{m_1}+1)^2} \ge \frac{R_2 m_2 Q_{21}^{m_2+1}}{C_2(Q_{21}^{m_2}+1)^2}$, $T_2 \ge 0$, $t_1 \ge 0$, $u \ge 0$, $U \ge 0$, then

$$\begin{aligned}
&T = \frac{R_1 m_1 P_{21}^{m_1}}{(C-C_2)(P_{21}^{m_1}+1)^2}, \quad P_{21} \equiv \frac{(C-C_2)/R_1}{c_1/r_1} \\
&T_2 = \frac{R_2 m_2 Q_{21}^{m_2}}{C_2(Q_{21}^{m_2}+1)^2} - \frac{R_1 m_1 P_{21}^{m_1}}{(C-C_2)(P_{21}^{m_1}+1)^2} \\
&Q_{21} \equiv \frac{C_2/R_2}{(c-c_1)/r_2}, \quad t = Q_{21}\frac{R_2 m_2 Q_{21}^{m_2}}{C_2(Q_{21}^{m_2}+1)^2} \\
&t_1 = \frac{R_1 m_1 P_{21}^{m_1+1}}{(C-C_2)(P_{21}^{m_1}+1)^2} - \frac{R_2 m_2 Q_{21}^{m_2+1}}{C_2(Q_{21}^{m_2}+1)^2}, \quad t_2 = T_1 = 0, \\
&u = \frac{r_1(1-m_1+P_{21}^{m_1})}{(P_{21}^{m_1}+1)^2} + \frac{r_2(1-m_2+Q_{21}^{m_2})}{(Q_{21}^{m_2}+1)^2} \\
&U = \frac{R_1(1+(1-m_1)P_{21}^{m_1})}{(P_{21}^{m_1}+1)^2} + \frac{R_2(1+(1-m_2)Q_{21}^{m_2})}{(Q_{21}^{m_2}+1)^2}
\end{aligned} \tag{11}$$

If $\mathrm{Min}\{c_1, c_2\} < c < c_1 + c_2$, $\mathrm{Min}\{C_1, C_2\} < C < C_1 + C_2$, $c_1 \le c_2$, $C_1 \le C_2$, $\frac{R_2 m_2 Q_{21}^{m_2}}{C_2(Q_{21}^{m_2}+1)^2} < \frac{R_1 m_1 P_{21}^{m_1}}{(C-C_2)(P_{21}^{m_1}+1)^2}$ or $\frac{R_1 m_1 P_{21}^{m_1+1}}{(C-C_2)(P_{21}^{m_1}+1)^2} < \frac{R_2 m_2 Q_{21}^{m_2+1}}{C_2(Q_{21}^{m_2}+1)^2}$, $T_1 \ge 0$, $t_1 \ge 0$, $u \ge 0$, $U \ge 0$, then three solutions with $t_2 = T_1 = 0$ are possible, depending on the parameter values. Either $t_1 = 0$ and $T_2 > 0$ (interchanging the subscripts 1 and 2 for the attacker) arise as in (23) in Section 3.7 with only general protection but special and general attack, or $t_1 > 0$ and $T_2 = 0$ (interchanging the subscripts 1 and 2 for the attacker) arise as in (25) in Section 3.8 with special and general protection but only general attack, or $t_1 = T_2 = 0$ (interchanging the subscripts 1 and 2 for the attacker) arise as in (7) in Section 3.2 with only general protection and attack.

**Proof.** Appendix C.

In Properties 2 and 1 both players apply general protection and attack, i.e., $T \ge 0$ and $t \ge 0$, given the specified conditions. But they do not necessarily apply special efforts. In (11), $t_1$ follows from $T_2$ by multiplying the first term in $T_2$ with $-Q_{21}$ and multiplying the second term in $T_2$ with $-P_{21}$. This asymmetry, in contrast to the previous

Section 3.3.1, causes, e.g., $t_1 = -T_2$ when $Q_{21} = P_{21} = 1$, causing one of the players to withdraw special effort.

**Example 4.** Inserting $m_1 = m_2 = R_1 = R_2 = r_1 = r_2 = C_2 = 1$, $C = 2$, $c_1 = 2/3$, $c = 5/3$, which imply $P_{21} = 3/2$ and $Q_{21} = 1$, into (11) gives

$$T = \frac{6}{25},\quad T_2 = \frac{1}{100},\quad t = \frac{1}{4},\quad t_1 = \frac{11}{100},\quad t_2 = T_1 = 0,\quad u = \frac{61}{100},\quad U = \frac{41}{100} \tag{12}$$

### 3.4. Defender Oge, attacker Ose, row 5

The upper left quadrant in Table 1 requires $c \le \mathrm{Min}\{c_1, c_2\}$ and $C \ge C_1 + C_2$. Additionally, assume $m_1 = m_2 = 1$. Hence, the defender chooses only general protection. The attacker chooses only special attack.

**Property 3.** When $c \le \mathrm{Min}\{c_1, c_2\}$, $C \ge C_1 + C_2$, $1 \ge \sqrt{tC_i/R_i}$, then

$$\begin{gathered} t = \left(\frac{r_1\sqrt{C_1/R_1} + r_2\sqrt{C_2/R_2}}{c + r_1C_1/R_1 + r_2C_2/R_2}\right)^2,\ T_i = \sqrt{R_i/C_i}\left(1 - \sqrt{tC_i/R_i}\right)\sqrt{t},\quad t_1 = t_2 = T = 0 \\ u = \frac{\left(r_1\sqrt{C_1/R_1} + r_2\sqrt{C_2/R_2}\right)^2\left(r_1C_1/R_1 + r_2C_2/R_2\right)}{\left(c + r_1C_1/R_1 + r_2C_2/R_2\right)^2},\quad U = \sum_{i=1}^{2} R_i\left(1 - \sqrt{tC_i/R_i}\right)^2 \end{gathered} \tag{13}$$

When $c \le \mathrm{Min}\{c_1, c_2\}$, $C \ge C_1 + C_2$, $C_1 \le C_2$, and $1 < \sqrt{tC_2/R_2}$ in (13), the attacker sets $T_2 = 0$, and

$$\begin{gathered} t = \frac{1}{\left(1 + \frac{c/r_1}{C_1/R_1}\right)^2 C_1/R_1},\quad T_1 = \frac{1}{\left(1 + \frac{C_1/R_1}{c/r_1}\right)^2 c/r_1},\quad t_1 = t_2 = T = T_2 = 0 \\ u = \frac{r_1}{\left(1 + \frac{c/r_1}{C_1/R_1}\right)^2} + r_2,\quad U = \frac{R_1}{\left(1 + \frac{C_1/R_1}{c/r_1}\right)^2} \end{gathered} \tag{14}$$

When $c \le \mathrm{Min}\{c_1, c_2\}$, $C \ge C_1 + C_2$, $C_2 \le C_1$, and $1 < \sqrt{tC_1 / R_1}$ in (13), the attacker sets $T_1 > 0$, and

$$t=\frac{1}{\left(1+\frac{c/r_2}{C_2/R_2}\right)^2 C_2/R_2},\quad T_2=\frac{1}{\left(1+\frac{C_2/R_2}{c/r_2}\right)^2 c/r_2},\quad t_1=t_2=T=T_1=0$$
$$u=\frac{r_2}{\left(1+\frac{c/r_2}{C_2/R_2}\right)^2}+r_1,\quad U=\frac{R_2}{\left(1+\frac{C_2/R_2}{c/r_2}\right)^2} \tag{15}$$

**Proof.** Appendix D.

**Example 5.** Inserting $r_i = R_1 = R_2$ and $C_1 = C_2$ into (13) gives

$$t=\frac{4R_2C_2}{(c+2C_2)^2},\quad T_i=\frac{2R_2c}{(c+2C_2)^2},\quad t_1=t_2=T=0,\quad u=\frac{8R_2C_2^2}{(c+2C_2)^2},\quad U=\frac{2R_2c^2}{(c+2C_2)^2} \tag{16}$$

### 3.5. Defender Ose, attacker Oge, row 2

The lower right quadrant in Table 1 requires $c \geq c_1 + c_2$ and $C \leq \text{Min}\{C_1, C_2\}$. Additionally, assume $m_1 = m_2 = 1$. The defender chooses only special protection. The attacker chooses only general attack.

**Property 4.** When $c \geq c_1 + c_2$, $C \leq \text{Min}\{C_1, C_2\}$, $1 \geq \sqrt{Tc_i/r_i}$, then

$$T=\left(\frac{R_1\sqrt{c_1/r_1}+R_2\sqrt{c_2/r_2}}{C+R_1c_1/r_1+R_2c_2/r_2}\right)^2,\quad t_i=\sqrt{r_i/c_i}\left(1-\sqrt{Tc_i/r_i}\right)\sqrt{T},\quad T_1=T_2=t=0$$
$$u=\sum_{i=1}^{2} r_i(1-\sqrt{Tc_i/r_i})^2,\quad U=\frac{\left(R_1\sqrt{c_1/r_1}+R_2\sqrt{c_2/r_2}\right)^2\left(R_1c_1/r_1+R_2c_2/r_2\right)}{\left(C+R_1c_1/r_1+R_2c_2/r_2\right)^2} \tag{17}$$

When $c \geq c_1 + c_2$, $C \leq \text{Min}\{C_1, C_2\}$, $c_1 \leq c_2$, and $1 < \sqrt{Tc_2/r_2}$, in (17), the defender sets $t_2 = 0$, and

$$T=\frac{1}{\left(1+\frac{C/R_1}{c_1/r_1}\right)^2 c_1/r_1},\quad t_1=\frac{1}{\left(1+\frac{c_1/r_1}{C/R_1}\right)^2 C/R_1},\quad T_1=T_2=t=t_2=0$$
$$u=\frac{r_1}{\left(1+\frac{c_1/r_1}{C/R_1}\right)^2},\quad U=\frac{R_1}{\left(1+\frac{C/R_1}{c_1/r_1}\right)^2}+R_2 \tag{18}$$

When $c \geq c_1 + c_2$, $C \leq \mathrm{Min}\{C_1, C_2\}$, $c_2 \leq c_1$, and $1<\sqrt{Tc_1/r_1}$ in (17), the defender sets $t_1 = 0$, and

$$T=\frac{1}{\left(1+\frac{C/R_2}{c_2/r_2}\right)^2 c_1/r_1},\quad t_1=\frac{1}{\left(1+\frac{c_2/r_2}{C/R_2}\right)^2 C/R_2},\quad T_1=T_2=t=t_1=0$$
$$u=\frac{r_2}{\left(1+\frac{c_2/r_2}{C/R_2}\right)^2},\quad U=\frac{R_2}{\left(1+\frac{C/R_2}{c_2/r_2}\right)^2}+R_1 \tag{19}$$

**Proof.** Appendix E.

Property 4 and (17) follow from Property 3 and (13) by permuting regular and capital letters. That is, the attacker applies general attack, whereas the defender applies special protection if $1\geq\sqrt{Tc_i/r_i}$. Thus, only the asset with high valuation $r_i$ and low unit protection cost $c_i$ may be protected.

**Example 6.** Inserting $R_i = r_1 = r_2$ and $c_1 = c_2$ into (17) gives

$$T=\frac{4r_2c_2}{(C+2c_2)^2},\ t_i=\frac{2r_2C}{(C+2c_2)^2},\ T_1=T_2=t=0,\ u=\frac{2r_2C^2}{(C+2c_2)^2},\ U=\frac{8r_2c_2^2}{(C+2c_2)^2} \tag{20}$$

### 3.6. Defender Sge, attacker Ose, rows 9 and 13

The upper middle quadrant in Table 1 requires $\mathrm{Min}\{c_1, c_2\} < c < c_1 + c_2$ and $C \geq C_1 + C_2$. The defender chooses one special protection and general protection. The attacker chooses only special attack. This section assumes $c_1 \leq c_2$ (row 9), which also covers $c_1 > c_2$ (row 13) by interchanging the subscripts 1 and 2.

**Property 5.** If $\text{Min}\{c_1, c_2\} < c < c_1 + c_2$, $C \geq C_1 + C_2$, $c_1 \leq c_2$, $\frac{r_1 m_1 Q_1^{m_1}}{c_1 (Q_1^{m_1}+1)^2} \geq \frac{R_2 m_2 P_{12}^{m_2}}{C_2 (P_{12}^{m_2}+1)^2}$, $t_1 \geq 0$, $u \geq 0$, $U \geq 0$, then

$$
\begin{aligned}
&T_1 = \frac{R_1 m_1 Q_1^{m_1}}{C_1 (Q_1^{m_1}+1)^2}, \quad T_2 = \frac{R_2 m_2 P_{12}^{m_2}}{C_2 (P_{12}^{m_2}+1)^2}, \quad P_{12} \equiv \frac{(c-c_1)/r_2}{C_2/R_2} \\
&t = \frac{r_2 m_2 P_{12}^{m_2}}{(c-c_1)(P_{12}^{m_2}+1)^2}, \quad t_1 = \frac{r_1 m_1 Q_1^{m_1}}{c_1 (Q_1^{m_1}+1)^2} - \frac{R_2 m_2 P_{12}^{m_2}}{C_2 (P_{12}^{m_2}+1)^2}, \quad t_2 = T = 0 \\
&u = \frac{r_1 (1 - m_1 + Q_1^{m_1}) Q_1^{m_1}}{(Q_1^{m_1}+1)^2} + \frac{r_2 (1 + (1-m_2) P_{12}^{m_2})}{(P_{12}^{m_2}+1)^2} \\
&U = \frac{R_1 (1 + (1-m_1) Q_1^{m_1})}{(Q_1^{m_1}+1)^2} + \frac{R_2 (1 - m_2 + P_{12}^{m_2}) P_{12}^{m_2}}{(P_{12}^{m_2}+1)^2}
\end{aligned}
\tag{21}
$$

If $\text{Min}\{c_1, c_2\} < c < c_1 + c_2$, $C \geq C_1 + C_2$, $c_1 \leq c_2$, $\frac{r_1 m_1 Q_1^{m_1}}{c_1 (Q_1^{m_1}+1)^2} < \frac{R_2 m_2 P_{12}^{m_2}}{C_2 (P_{12}^{m_2}+1)^2}$, $t_1 \geq 0$, $u \geq 0$, $U \geq 0$, then the solution is as in (13) in Section 3.4, with only general protection and only special attack.

**Proof.** Appendix F.

Property 5 states that whereas the attacker applies the two special attacks and the defender applies general protection, the defender additionally applies special protection of asset 1 only when asset 1 is sufficiently valuable and worth fighting for.

**Example 7.** Inserting $Q_1 = P_{12} = 1$ into (21) gives

$$
\begin{aligned}
&\text{If } t_1 \geq 0 \text{ and } m_i \leq 2, \text{ then } T_i = \frac{R_i m_i}{4C_i}, \quad t = \frac{r_2 m_2}{4(c-c_1)} \\
&t_1 = \frac{r_1 m_1}{4c_1} - \frac{R_2 m_2}{4C_2}, \quad t_2 = T = 0 \\
&u = \sum_{i=1}^{2} \frac{r_i (2-m_i)}{4}, \quad U = \sum_{i=1}^{2} \frac{R_i (2-m_i)}{4}
\end{aligned}
\tag{22}
$$

### 3.7. Defender Oge, attacker Sge, rows 7 and 8

The left middle quadrant in Table 1 requires $c \le \mathrm{Min}\{c_1, c_2\}$ and $\mathrm{Min}\{C_1, C_2\} < C < C_1 + C_2$. Additionally, assume $m_1 + m_2 = 1$. The defender chooses only general protection. The attacker chooses one special attack and general attack. This section assumes $C_1 \le C_2$ (row 7), which also covers $C_1 > C_2$ (row 8) by interchanging the subscripts 1 and 2.

**Property 6.** If $c \le \mathrm{Min}\{c_1, c_2\}$, $\mathrm{Min}\{C_1, C_2\} < C < C_1 + C_2$, $\sqrt{\frac{R_1}{C_1}} \ge \sqrt{\frac{R_2}{C-C_1}}$, $T \ge 0$, $T_1 \ge 0$, $u \ge 0$, $U \ge 0$, then

$$t = \frac{R_1 R_2 \left((C-C_1)R_1 r_2^2 + 2\sqrt{C_1}\sqrt{C-C_1}\, r_1 \sqrt{R_1}\, r_2 \sqrt{R_2} + C_1 r_1^2 R_2\right)}{((C-C_1)R_1 r_2 + (C_1 r_1 + cR_1)R_2)^2}, \quad t_1 = t_2 = T_2 = 0$$

$$T = \left(\sqrt{\frac{R_2}{C-C_1}} - \sqrt{t}\right)\sqrt{t}, \quad T_1 = \left(\sqrt{\frac{R_1}{C_1}} - \sqrt{\frac{R_2}{C-C_1}}\right)\sqrt{t} \tag{23}$$

$$u = r_1\sqrt{\frac{C_1 t}{R_1}} + r_2\sqrt{\frac{(C-C_1)t}{R_2}} - ct, \quad U = R_1 + R_2 + Ct - 2\sqrt{C_1 R_1 t} - 2\sqrt{(C-C_1)R_2 t}$$

If $c \le \mathrm{Min}\{c_1, c_2\}$, $\mathrm{Min}\{C_1, C_2\} < C < C_1 + C_2$, $\sqrt{\frac{R_1}{C_1}} < \sqrt{\frac{R_2}{C-C_1}}$, $T \ge 0$, $T_1 \ge 0$, $u \ge 0$, $U \ge 0$, then the solution is as in (7) in Section 3.2, with only general protection and attack.

**Proof.** Appendix G.

Property 6 shows that when both expected utilities are positive, the defender always applies general protection $t$, whereas the attacker applies special attack $T_1$ only when asset 1 is sufficiently valuable and can be fought for efficiently compared with asset 2, i.e., $\sqrt{R_1/C_1} \ge \sqrt{R_2/(C-C_1)}$.

**Example 8.** Inserting $r_i = R_i$, $C_1 = 1$, $C = 2$ into (23) gives

$$t = \frac{\left(\sqrt{R_1} + \sqrt{R_2}\right)^2}{(c+2)^2}, \quad T = \frac{\left((c+1)\sqrt{R_2} - \sqrt{R_1}\right)\left(\sqrt{R_1} + \sqrt{R_2}\right)}{(c+2)^2}$$

$$T_1 = \frac{(R_1 - R_2)}{c+2}, \quad t_1 = t_2 = T_2 = 0 \tag{24}$$

$$u = \frac{2\left(\sqrt{R_1} + \sqrt{R_2}\right)^2}{(c+2)^2}, \quad U = \frac{(c^2 + 2c + 2)(R_1 + R_2) - 4(c+1)\sqrt{R_1}\sqrt{R_2}}{(c+2)^2}$$

### 3.8. Defender Sge, attacker Oge, rows 10 and 14

The lower middle quadrant in Table 1 requires $\text{Min}\{c_1, c_2\} < c < c_1 + c_2$ and $C \le \text{Min}\{C_1, C_2\}$. Additionally, assume $m_1 = m_2 = 1$. The defender chooses one special protection and general protection. The attacker chooses only general attack. This section assumes $c_1 \le c_2$ (row 10), which also covers $c_1 > c_2$ (row 14) by interchanging the subscripts 1 and 2.

**Property 7.** If $\text{Min}\{c_1, c_2\} < c < c_1 + c_2$, $C \le \text{Min}\{C_1, C_2\}$, $\sqrt{\frac{r_1}{c_1}} \ge \sqrt{\frac{r_2}{c-c_1}}$, $t \ge 0$, $t_1 \ge 0$, $u \ge 0$, $U \ge 0$, then

$$T = \frac{r_1 r_2\left((c-c_1)r_1R_2^2 + 2\sqrt{c_1}\sqrt{c-c_1}R_1\sqrt{r_1}R_2\sqrt{r_2} + c_1R_1^2r_2\right)}{((c-c_1)r_1R_2 + (c_1R_1 + Cr_1)r_2)^2}, \quad T_1 = T_2 = t_2 = 0$$

$$t = \left(\sqrt{\frac{r_2}{c-c_1}} - \sqrt{T}\right)\sqrt{T}, \quad t_1 = \left(\sqrt{\frac{r_1}{c_1}} - \sqrt{\frac{r_2}{c-c_1}}\right)\sqrt{T} \tag{25}$$

$$u = r_1 + r_2 + cT - 2\sqrt{c_1r_1T} - 2\sqrt{(c-c_1)r_2T}, \quad U = R_1\sqrt{\frac{c_1T}{r_1}} + R_2\sqrt{\frac{(c-c_1)T}{r_2}} - CT$$

If $\text{Min}\{c_1, c_2\} < c < c_1 + c_2$, $C \le \text{Min}\{C_1, C_2\}$, $\sqrt{\frac{r_1}{c_1}} < \sqrt{\frac{r_2}{c-c_1}}$, $t \ge 0$, $t_1 \ge 0$, $u \ge 0$, $U \ge 0$, then the solution is as in (7) in Section 3.2, with only general protection and attack.

**Proof.** Appendix H.

Property 7 and (25) follow from Property 6 and (23) by permuting regular and capital letters. That is, the attacker applies general attack $T$, whereas the defender applies special protection $t_1$ only when asset 1 is sufficiently valuable, i.e., $\sqrt{r_1/c_1} \ge \sqrt{r_2/(c-c_1)}$.

**Example 9.** Inserting $R_i = r_i$, $c_1 = 1$, $c = 2$ into (25) gives

$$T = \frac{\left(\sqrt{r_1} + \sqrt{r_2}\right)^2}{(C+2)^2}, \quad t = \frac{\left((C+1)\sqrt{r_2} - \sqrt{r_1}\right)\left(\sqrt{r_1} + \sqrt{r_2}\right)}{(C+2)^2}$$

$$t_1 = \frac{(r_1 - r_2)}{C+2}, \quad T_1 = T_2 = t_2 = 0 \tag{26}$$

$$U = \frac{(C^2 + 2C + 2)(r_1 + r_2) - 4(C+1)\sqrt{r_1}\sqrt{r_2}}{(C+2)^2}, \quad U = \frac{2\left(\sqrt{r_1} + \sqrt{r_2}\right)^2}{(C+2)^2}$$

### 3.9. Defender Ose, attacker Sge, rows 3 and 4

The right middle quadrant in Table 1 requires $c \geq c_1 + c_2$ and $\text{Min}\{C_1, C_2\} < C < C_1 + C_2$. The defender chooses only special protection. The attacker chooses one special attack and general attack. This section assumes $C_1 \leq C_2$ (row 3), which also covers $C_1 > C_2$ (row 4) by interchanging the subscripts 1 and 2.

**Property 8.** If $c \geq c_1 + c_2$ and $\text{Min}\{C_1, C_2\} < C < C_1 + C_2$, $\frac{R_1 m_1 q_1^{m_1}}{C_1(q_1^{m_1}+1)^2} \geq \frac{r_2 m_2 p_{12}^{m_2}}{c_2(p_{12}^{m_2}+1)^2}$, $T_1 \geq 0$, $u \geq 0$, $U \geq 0$, then

$$
\begin{aligned}
&t_1 = \frac{r_1 m_1 q_1^{m_1}}{c_1(q_1^{m_1}+1)^2},\quad q_1 \equiv \frac{c_1/r_1}{C_1/R_1},\quad t_2 = \frac{r_2 m_2 p_{12}^{m_2}}{c_2(p_{12}^{m_2}+1)^2},\quad p_{12} \equiv \frac{(C-C_1)/R_2}{c_2/r_2} \\
&T = \frac{R_2 m_2 p_{12}^{m_2}}{(C-C_1)(p_{12}^{m_2}+1)^2},\quad T_1 = \frac{R_1 m_1 q_1^{m_1}}{C_1(q_1^{m_1}+1)^2} - \frac{r_2 m_2 p_{12}^{m_2}}{c_2(p_{12}^{m_2}+1)^2},\quad T_2 = t = 0 \\
&u = \frac{r_1(1+(1-m_1)q_1^{m_1})}{(q_1^{m_1}+1)^2} + \frac{r_2(1-m_2+p_{12}^{m_2})p_{12}^{m_2}}{(p_{12}^{m_2}+1)^2} \\
&U = \frac{R_1(1-m_1+q_1^{m_1})q_1^{m_1}}{(q_1^{m_1}+1)^2} + \frac{R_2(1+(1-m_2)p_{12}^{m_2})}{(p_{12}^{m_2}+1)^2}
\end{aligned}
\tag{27}
$$

If $c \geq c_1 + c_2$ and $\text{Min}\{C_1, C_2\} < C < C_1 + C_2$, $\frac{R_1 m_1 q_1^{m_1}}{C_1(q_1^{m_1}+1)^2} < \frac{r_2 m_2 p_{12}^{m_2}}{c_2(p_{12}^{m_2}+1)^2}$, $T_1 \geq 0$, $u \geq 0$, $U \geq 0$, then the solution is as in (17) in Section 3.5, with only special protection and only general attack.

**Proof.** Appendix I.

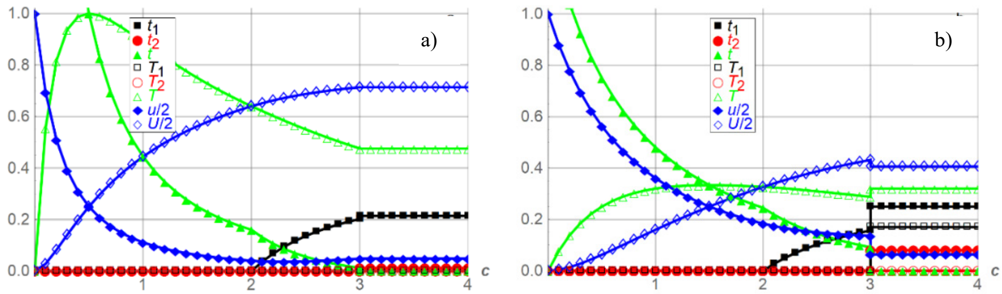
Property 8 and (27) follow from Property 5 and (21) in Section 3.6 by permuting regular and capital letters, except $m_i$. That is, the defender applies the two special protections whereas the attacker applies general attack combined with special attack of asset 1 only when asset 1 is sufficiently valuable.

**Example 10.** Inserting $q_1 = p_{12} = 1$ into (27) gives

$$
\begin{aligned}
&\text{If } T_1 \geq 0 \text{ and } 2 \geq m_i,\text{ then } t_i = \frac{r_i m_i}{4c_i},\quad T = \frac{R_2 m_2}{4(C-C_1)},\quad T_1 = \frac{R_1 m_1}{4C_1} - \frac{r_2 m_2}{4c_2} \\
&T_2 = t = 0,\quad u = \sum_{i=1}^{2} \frac{r_i(2-m_i)}{4},\quad U = \sum_{i=1}^{2} \frac{R_i(2-m_i)}{4}
\end{aligned}
\tag{28}
$$

# 4. Sensitivity analysis

Figure 3 presents the solution, i.e., the six efforts $t_1$, $t_2$, $t$, $T_1$, $T_2$, $T$ and the expected utilities $u$ and $U$ for the defender and attacker as functions of the defender's general unit cost $c$ of protecting both assets relative to the benchmark parameter values $c_1 = C_1 = m_1 = m_2 = r_1 = r_2 = R_1 = R_2 = 1$, $c_2 = C_2 = 2$. To capture all the nine regions in Fig. 2, panel a assumes $C = 0.5$ for the attacker's general unit cost $C$ of attacking both assets for the bottom three regions. Panel b assumes $C = 1.5$ for the middle three regions, and panel c assumes $C = 3.5$ for the upper three regions. Division of $u$ and $U$ with 2 is for scaling purposes. Unity parameter values were chosen when possible. The values $c_2 = C_2 = 2$ were chosen to make asset 2 twice as costly to protect and attack as assets 1.



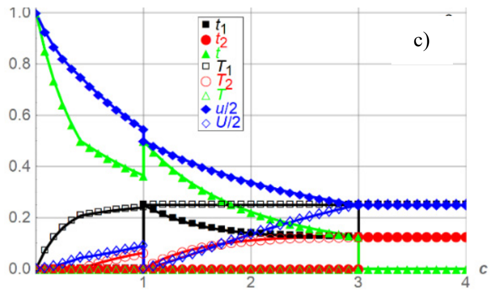




Fig. 3. Efforts $t_1$, $t_2$, $t$, $T_1$, $T_2$, $T$ and expected utilities $u$ and $U$ as functions of the defender's general unit cost $c$ of protecting both assets relative to the benchmark parameter values $c_1 = C_1 = m_1 = m_2 = r_1 = r_2 = R_1 = R_2 = 1$, $c_2 = C_2 = 2$. Division with 2 is for scaling purposes; panel a) $C = 0.5$, panel b) $C = 1.5$, panel c) $C = 3.5$

Figure 3 panel a assumes $C = 0.5$ for the attacker's general unit cost $C$ of attacking both assets, which excludes the attacker from applying costly special efforts since $C_1 = 1 > C = 0.5$ and $C_2 = 2 > C = 0.5$, i.e., $T_1 = T_2 = 0$, and instead to confine attention to general attack $T$. Analogously for the defender, when its general unit cost $c$ of protecting both assets is small, since $c_1 = 1$ and $c_2 = 2$, it chooses zero special efforts, $t_1 = t_2 = 0$, and confines attention to general protection $t$ which decreases convexly from infinity when $c = 0$. The attacker responds with the familiar inverse $U$ shaped general attack $T$ as a function of $c$. That is, the attacker chooses low attack $T$ due to inferiority or weakness when $c$ is low, chooses low attack $T$ due to superiority or strength when $c$ is high, and chooses high attack $T$ when $c$ is intermediate. The attack $T$ reaches a maximum $T = t = 1$ when $c = C = 0.5$, i.e., when the two players have equal unit effort costs and apply equally large efforts. The defender's expected utility $u$ decreases convexly in $c$ (i.e., as $c$ increases), while the attacker's expected utility $U$ increases concavely in $c$. As $c$ increases above $c = 1$, Fig. 2 shows a transition to applying Section 3.8 where $\sqrt{\frac{r_1}{c_1}} \geq \sqrt{\frac{r_2}{c - c_1}}$ in (25) is not satisfied when $c < 2$. Hence Section 3.2, with only general protection and attack, applies for the range $0 \leq c < 2$. Section 3.8 applies when $2 \leq c < 3$. As $c$ increases above $c = 2$, the defender's special protection $t_1$ in (25) of asset 1 increases from $t_1 = 0$, induced by the low unit cost $c_1 = 1$, while its general protection $t$ of both assets continues to decrease. The defender's expected utility $u$ decreases to a minimum and thereafter increases slightly. The attacker's general attack $T$ also decreases due to the attacker's increasing superiority. As $c$ increases above $c = c_1 + c_2 = 3$, Fig. 2 shows a transition to applying Section 3.5 where the defender's general protection $t = 0$ vanishes since it becomes too costly. Instead, the defender relies on its cheap special protection $t_1 = 0.214$ of asset 1, and its twice as expensive special protection $t_2 = 0.012$ of asset 2, determined by (17) which is independent of the irrelevant $c$. The attacker's general attack decreases to $T = 0.478$ when $c = 3$, after which it is also independent of $c$. The defender's expected utility $u$ increases slightly to $u = 0.048$ when $c = 3$, and is thereafter independent of $c$. The attacker's expected utility $U$ decreases to $U = 0.714$ when $c = 3$, and is thereafter independent of $c$. The variables are sometimes continuous and sometimes discontinuous from one region to the next. For example, as $c$ increases from below to above $c = c_1 + c_2 = 3$, general protection $t$ is no longer cost effective, while the special protections $t_1$ and $t_2$ are cost effective. Hence, $t$ decreases discontinuously from $t = 0.012$ to $t = 0$, $t_1$ increases discontinuously from $t_1 = 0.202$ to $t_1 = 0.214$, and $t_2$ increases discontinuously from $t_2 = 0$ to $t_2 = 0.012$.

Figure 3 panel b assumes $C = 1.5$ for the attacker's general unit cost $C$ of attacking both assets. This excludes the attacker from applying costly special effort against asset 2 since $C_2 = 2 > C = 1.5$, i.e., $T_2 = 0$. As we will see, it may or may not exclude the attacker from applying costly special effort against asset 1. Although $C_1 = 1 < C = 1.5$, the general attack $T$ has the added benefit of attacking both assets. The tripling of the attacker's unit cost $C$ in panel b causes qualitatively the same results as in panel a, when $0 \leq c < 3$, as explained below. The reason is that the attacker applies only general attack $T$ in both panels, while the defender applies only general protection $t$ in both panels when $0 \leq c < 2$, and both general protection $t$ and special protection $t_1$ of asset 1 when $2 \leq c < 3$. The transition for the defender when $c = 2$ depends on $\sqrt{\frac{r_1}{c_1}} \geq \sqrt{\frac{r_2}{c - c_1}}$ in (25), which is independent of $C$. The quantitative difference is that the attacker is three times as disadvantaged in panel b as in panel a. Hence the attacker's expected utility $U$ is lower, while the defender's expected utility $u$ is higher, in panel b compared to panel a. Furthermore, the attacker's inverse $U$ shaped general attack $T$ reaches a maximum $T = t = 1/3$ when $c = C = 1.5$, i.e., when the two players have equal unit effort costs and apply equally large efforts. Relating to Fig. 2, when $C = 1.5$, the inequality $c \leq \mathrm{Min}\{c_1, c_2\} = 1$ specifies that Section 3.7 applies. However, since $\sqrt{\frac{R_1}{C_1}} \geq \sqrt{\frac{R_2}{C - C_1}}$ in (23) is not satisfied when $C = 1.5$, Property 6 specifies applying (7) in Section 3.2, with only general protection and attack, as shown in panel b. As $c$ increases above $c = \mathrm{Min}\{c_1, c_2\} = 1$ when $C = 1.5$, the center region in Fig. 2 specifies that Section 3.3 applies. However, the required inequalities for (9) in Property 1 are not satisfied, since $T_1$ and/or $t_1$ are/is negative when $1 \leq c < 2$. Analyzing the three alternative solutions in Property 1, it turns out that, again, (7) in Section 3.2 should be applied, with only general protection and attack, as shown in panel b. When $2 \leq c < 3$, Fig. 2 also specifies that Section 3.3 applies. The required inequalities for (9) in Property 1 are still not satisfied, but now the alternative solution is to apply (25) in Section 3.8 with special and general protection but only general attack, i.e., $t_1 > 0$ and $T_1 = 0$. That turns out to be the same solution as in the bottom middle region in Fig. 2, as shown in panel a when $2 \leq c < 3$. This explains why panels a and b are qualitatively the same when $0 \leq c < 3$. As $c$ increases above $c = c_1 + c_2 = 3$ when $C = 1.5$, the right middle region in Fig. 2 specifies that Section 3.9 applies. General protection is too costly causing $t$ to decrease discontinuously from $t = 0.092$ to $t = 0$ when $c = 3$. All the eight variables in panel b are independent of $c$ when $c \geq 3$. Special protection of asset 1 increases discontinuously from $t_1 = 0.148$ to $t_1 = 0.157$ when $c = 3$. Special protection of asset 2 increases discontinuously from $t_2 = 0$ to $t_2 = 0.08$ when $c = 3$. Applying Section 3.9, the required inequality for $T_1 \geq 0$ is indeed satisfied. More specifically, the attacker's special attack of asset 1 jumps discontinuously from $T_1 = 0$ to $T_1 = 0.17$ when $c = 3$. The attacker's general attack increases discontinuously from $T_1 = 0.288$ to $T_1 = 0.32$ when $c = 3$. (The attacker's special attack of asset 2, naturally, remains at $T_2 = 0$ when $c = 3$ because of the high unit effort cost $C_2 = 2$.) Finally, the defender experiences a discontinuous decrease in its expected utility $u$ as its unit cost $c$ of general protection increases above $c = 3$. The attacker experiences a lower discontinuous decrease in its expected utility $U$ as $c$ increases above $c = 3$.

Figure 3 panel c assumes $C = 3.5$ for the attacker's general unit cost $C$ of attacking both assets. This excludes the attacker from applying costly general effort since $C \geq C_1 + C_2$. Hence, for $c \leq 1$ we cannot get the same result as in panels a and b. Figure 2 specifies using Section 3.4 when $c \leq \mathrm{Min}\{c_1, c_2\} = 1$. Since low defender general unit cost $c$ of protecting both assets causes high general protection $t$, the inequality $1 \geq \sqrt{tC_2/R_2}$ in (13) is not satisfied for asset 2 where $C_2 = 2$. Hence, as specified in (14) in Property 3, the attacker does not protect asset 2, i.e., $T_2 = 0$. The attacker exclusively attacks asset 1 with special effort $T_1$, which increases concavely as $c$ increases, and the defender becomes more disadvantaged. From an intuitive point of view, not attacking asset 2 makes perfect sense. The attacker is overwhelmed and deterred by the substantial protection $t$. However, this result differs from the common result, when applying the ratio form contest success function with one attack effort against and one protection effort of one asset, where both efforts are positive. As $c$ increases and $t$ decreases, a point is reached where $1=\sqrt{tC_2/R_2}$ in (13) is satisfied as an equality. Solving that equality together with the expression for $t$ in (13) (two equations with two unknowns $t$ and $c$) gives

$$c = r_1\sqrt{\frac{C_1}{R_1}}\left(\sqrt{\frac{C_2}{R_2}}-\sqrt{\frac{C_1}{R_1}}\right)=\sqrt{2}-1\approx 0.414 \tag{29}$$

Hence, when $0.414 < c \le 1$, equation (13) in Property 3 applies, and the attacker increases its effort $T_2$ to attack asset 2 from zero, while its attack $T_1$ on asset 1 increases more moderately. As $c$ increases above $c = 1$, Fig. 2 specifies using Section 3.6 where, notably, the defender increases its special protection $t_1$ of asset 1 discontinuously from zero, and also increases its general protection $t$ discontinuously, while avoiding special protection $t_2 = 0$ since the unit effort cost $c_2 = 2$ is too high. The attacker responds by increasing its special attack $T_1$ on asset 1 discontinuously, decreasing its special attack $T_2$ on asset 2 discontinuously to zero, and avoiding general attack $T = 0$ since the unit effort cost $C = 3.5$ is too high. The defender's increased protections $t$ and $t_1$ when $c = 1$ causes the attacker's expected utility $U$ to decrease discontinuously, and thereafter it increases. As $c$ increases from $c = 1$ to $c = 3$, and the defender becomes more disadvantaged, the defender's protections $t$ and $t_1$ decrease, while the attacker's attack $T_1$ is constant and the attack $T_2$ increases, respectively. As $c$ increases above $c = c_1 + c_2 = 3$, Fig. 2 specifies using Section 3.1 where no players exert general effort, $t = T = 0$, and all special efforts thus are independent of $c$. When $c = 3$, the defender's general effort decreases discontinuously to $t = 0$, while its special efforts increase to $t_1 = 0.25$ and $t_2 = 0.125$, respectively. The attacker's general attack remains at zero, $T = 0$, while its special attacks remain at $T_1 = T_2 = 0.125$. Over the entire range of $c$, the defender's expected utility $u$ decreases, while the attacker's expected utility $U$ exhibits overall increase.

# 5. Conclusion

Earlier research has abundantly analysed individual and overarching protection and attack, with as many contests as assets, plus one contest for each level above the individual level. Largely missing in the literature is special and general protection and attack operating additively, with as many contests as assets. That is the focus in [14], analysing two parallel and series assets, and in this article researching two independent assets. The analytical results differ strongly and are most easily seen by placing the two articles next to each other. Generally, whereas a defender is advantaged in a parallel system, and an attacker is advantaged in a series system, no such distinction between the defender and attacker due to the system configuration arises for two independent assets.

Special effort may be a bank vault which prevents the theft of gold but does not prevent bank employees from being taken hostage. General effort may be joint bank protection of both gold and employees. In contrast, breaking through overarching protection may mean breaking the outer barriers of a bank in one separate contest, after which individual assets within the bank may be compromised in separate contests.

For two independent assets, 16 analytical solutions exist. Each player chooses either two special efforts, one general effort, or one special effort and one general effort when the special unit effort cost for one asset is lower than that of the other asset and the general unit effort cost. A player with these three options facing an opponent with three options is positioned in exactly one of the 16 solutions. Realising which of these 16 applies may not be immediately obvious but is clarified in this article.

Some of the solutions are straightforward to interpret, for example, that in which higher efforts are exerted when contest intensities are higher, unit costs lower, and assets valuable. More complex solutions are presented analytically and are illustrated with examples. The solutions constitute tools which the reader may use to determine special solutions for alternative parameter values. Future research should introduce more complexity into the asset network structure, types of special and general efforts, demand requirements for assets, time, incomplete information, and multiple defenders and attackers.

## Appendix A. General solution

Combining the first and second equations in (4) gives

$$\left.\begin{aligned} \frac{m_i(t_i+t)^{m_i-1}(T_i+T)^{m_i-1}}{\left((t_i+t)^{m_i}+(T_i+T)^{m_i}\right)^2} &= \frac{c_i}{r_i(T_i+T)} \\ \frac{m_i(T_i+T)^{m_i-1}(t_i+t)^{m_i-1}}{\left((t_i+t)^{m_i}+(T_i+T)^{m_i}\right)^2} &= \frac{C_i}{R_i(t_i+t)} \end{aligned}\right\} \Rightarrow \frac{(t_i+t)}{(T_i+T)} = \frac{C_i/R_i}{c_i/r_i} = Q_i \tag{30}$$

which are solved to give

$$(t_i+t) = Q_i(T_i+T),\ (T_i+T) = \frac{R_i m_i Q_i^{m_i}}{C_i(Q_i^{m_i}+1)^2} \tag{31}$$

Combining the third and fourth equations in (4), when $m_i = m$, $t_1 = t_2 = t_i$, and $T_1 = T_2 = T_i$, gives

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= \frac{m(r_1+r_2)(t_i+t)^{m-1}(T_i+T)^m}{\left((t_i+t)^m+(T_i+T)^m\right)^2}-c=0,\\ \frac{\partial U}{\partial T} &= \frac{m(R_1+R_2)(T_i+T)^{m-1}(t_i+t)^m}{\left((t_i+t)^m+(T_i+T)^m\right)^2}-C=0\end{aligned}\right\}\Rightarrow \frac{(t_i+t)}{(T_i+T)}=\frac{C/(R_1+R_2)}{c/(r_1+r_2)}=Q \quad (32)$$

which are solved to give

$$(t_i+t)=Q(T_i+T), (T_i+T)=\frac{(R_1+R_2)mQ^m}{C(Q^m+1)^2} \quad (33)$$

Differentiating (4), the players' second-order conditions inserting (31) are

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial t_1^2}+\frac{\partial^2 u}{\partial t_2^2}, \quad \frac{\partial^2 U}{\partial T^2}=\frac{\partial^2 U}{\partial T_1^2}+\frac{\partial^2 U}{\partial T_2^2}\\ \left.\begin{aligned}\frac{\partial^2 u}{\partial t_i^2} &= -\frac{r_i m_i Q_i^{m_i-2}\left(1-m_i+(1+m_i)Q_i^{m_i}\right)}{(1+Q_i^{m_i})^3(T_i+T)^2}\le 0,\\ \frac{\partial^2 U}{\partial T_i^2} &= -\frac{R_i m_i Q_i^{m_i}\left(1+m_i+(1-m_i)Q_i^{m_i}\right)}{(1+Q_i^{m_i})^3(T_i+T)^2}\le 0\end{aligned}\right\}\Rightarrow \begin{aligned}1-m_i+(1+m_i)Q_i^{m_i}\ge 0\\ 1+m_i+(1-m_i)Q_i^{m_i}\ge 0\end{aligned}\end{gathered} \quad (34)$$

Differentiating (4), the players' second-order conditions inserting (33) are

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial t_1^2}+\frac{\partial^2 u}{\partial t_2^2}, \quad \frac{\partial^2 U}{\partial T^2}=\frac{\partial^2 U}{\partial T_1^2}+\frac{\partial^2 U}{\partial T_2^2}\\ \left.\begin{aligned}\frac{\partial^2 u}{\partial t_i^2} &= -\frac{r_i m Q^{m-2}\left(1-m+(1+m)Q^{m}\right)}{(1+Q^{m})^3(T_i+T)^2}\le 0\\ \frac{\partial^2 U}{\partial T_i^2} &= -\frac{R_i m Q^{m}\left(1+m+(1-m)Q^{m}\right)}{(1+Q^{m})^3(T_i+T)^2}\le 0\end{aligned}\right\}\Rightarrow \begin{aligned}1-m+(1+m)Q^{m}\ge 0\\ 1+m+(1-m)Q^{m}\ge 0\end{aligned}\end{gathered} \quad (35)$$

## Appendix B. One special effort and one general effort, $c_1 \le c_2$ and $C_1 \le C_2$, Section 3.3.1

If $\mathrm{Min}\{c_1, c_2\} < c < c_1 + c_2$ and $\mathrm{Min}\{C_1, C_2\} < C < C_1 + C_2$, differentiating (3) when $t_2 = T_2 = 0$ gives

$$\begin{aligned}
\frac{\partial u}{\partial t_1} &= \frac{m_1 r_1 (t_1+t)^{m_1-1}(T_1+T)^{m_1}}{\left((t_1+t)^{m_1}+(T_1+T)^{m_1}\right)^2} - c_1 = 0 \\
\frac{\partial u}{\partial t} &= \frac{m_2 r_2 T^{m_2} t^{m_2-1}}{(t^{m_2}+T^{m_2})^2} + \frac{m_1 r_1 (T_1+T)^{m_1}(t_1+t)^{m_1-1}}{\left((t_1+t)^{m_1}+(T_1+T)^{m_1}\right)^2} - c = 0 \\
\frac{\partial U}{\partial T_1} &= \frac{m_1 R_1 (t_1+t)^{m_1}(T_1+T)^{m_1-1}}{\left((t_1+t)^{m_1}+(T_1+T)^{m_1}\right)^2} - C_1 = 0 \\
\frac{\partial U}{\partial T} &= \frac{m_2 R_2 T^{m_2-1} t^{m_2}}{(t^{m_2}+T^{m_2})^2} + \frac{m_1 R_1 (T_1+T)^{m_1-1}(t_1+t)^{m_1}}{\left((t_1+t)^{m_1}+(T_1+T)^{m_1}\right)^2} - C = 0
\end{aligned} \tag{36}$$

Combining the first and third equations in (36) gives

$$(t_1+t) = Q_1(T_1+T), \quad (T_1+T) = \frac{m_1 Q_1^{m_1}}{(Q_1^{m_1}+1)^2 (C_1/R_1)} \tag{37}$$

which are inserted into the second and fourth equations in (36) to yield

$$\frac{\partial u}{\partial t} = \frac{r_2 T^{m_2} m_2 t^{m_2-1}}{(t^{m_2}+T^{m_2})^2} + c_1 - c = 0, \quad \frac{\partial U}{\partial T} = \frac{R_2 T^{m_2-1} m_2 t^{m_2}}{(t^{m_2}+T^{m_2})^2} + C_1 - C = 0 \tag{38}$$

which cause the expressions for $T$ and $t$ in (9), from which $T_1$ and $t_1$ follow from inserting $T$ and $t$ into (37), assuming $\frac{m_1 Q_1^{m_1}}{(Q_1^{m_1}+1)^2 C_1/R_1} \ge \frac{m_2 P_1^{m_2}}{(P_1^{m_2}+1)^2 (C-C_1)/R_2}$ and $\frac{m_1 Q_1^{m_1+1}}{(Q_1^{m_1}+1)^2 C_1/R_1} \ge \frac{m_2 P_1^{m_2+1}}{(P_1^{m_2}+1)^2 (C-C_1)/R_2}$, and $u$ and $U$ follow from inserting into (3). Differentiating (36), the defender's second-order conditions and Hessian matrix are

$$\frac{\partial^2 u}{\partial t_1^2}=\frac{\partial^2 u}{\partial t_1 \partial t}=\frac{\partial^2 u}{\partial t \partial t_1}=-\frac{c_1^2(Q_1^{m_1}+1)\left(1-m_1+(1+m_1)Q_1^{m_1}\right)}{m_1 Q_1^{m_1} r_1}\leq 0$$

$$P_1=\frac{(C-C_1)/R_2}{(c-c_1)/r_2}$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial t_1^2}-\frac{(c-c_1)^2(P_1^{m_2}+1)\left(1-m_2+(1+m_2)P_1^{m_2}\right)}{m_2 P_1^{m_2} r_2}\leq 0$$

$$\left|H\right|_d=\begin{vmatrix}\dfrac{\partial^2 u}{\partial t_1^2} & \dfrac{\partial^2 u}{\partial t_1 \partial t}\\ \dfrac{\partial^2 u}{\partial t \partial t_1} & \dfrac{\partial^2 u}{\partial t^2}\end{vmatrix}=\frac{\partial^2 u}{\partial t_1^2}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial t_1 \partial t}\frac{\partial^2 u}{\partial t \partial t_1} \tag{39}$$

$$=\frac{c_1^2(Q_1^{m_1}+1)\left(1-m_1+(1+m_1)Q_1^{m_1}\right)}{m_1 Q_1^{m_1} r_1}$$

$$\times\frac{(c-c_1)^2(P_1^{m_2}+1)\left(1-m_2+(1+m_2)P_1^{m_2}\right)}{m_2 P_1^{m_2} r_2}\geq 0$$

Differentiating (36), the attacker's second-order conditions and Hessian matrix are

$$\frac{\partial^2 U}{\partial T_1^2}=\frac{\partial^2 U}{\partial T_1 \partial T}=\frac{\partial^2 U}{\partial T \partial T_1}=-\frac{C_1^2(Q_1^{m_1}+1)\left(1+m_1+(1-m_1)Q_1^{m_1}\right)}{m_1 Q_1^{m_1} R_1}\leq 0$$

$$\frac{\partial^2 U}{\partial T^2}=\frac{\partial^2 U}{\partial T_1^2}-\frac{(C-C_1)^2(P_1^{m_2}+1)\left(1+m_2+(1-m_2)P_1^{m_2}\right)}{m_2 P_1^{m_2} R_2}\leq 0$$

$$\left|H\right|_a=\begin{vmatrix}\dfrac{\partial^2 U}{\partial T_1^2} & \dfrac{\partial^2 U}{\partial T_1 \partial T}\\ \dfrac{\partial^2 U}{\partial T \partial T_1} & \dfrac{\partial^2 U}{\partial T^2}\end{vmatrix}=\frac{C_1^2(Q_1^{m_1}+1)\left(1+m_1+(1-m_1)Q_1^{m_1}\right)}{m_1 Q_1^{m_1} R_1} \tag{40}$$

$$\times\frac{(C-C_1)^2(P_1^{m_2}+1)\left(1+m_2+(1-m_2)P_1^{m_2}\right)}{m_2 P_1^{m_2} R_2}\geq 0$$

## Appendix C. One special effort and one general effort, $c_1 \le c_2$ and $C_1 \le C_2$, Section 3.3.2

If $\mathrm{Min}\{c_1, c_2\} < c < c_1 + c_2$ and $\mathrm{Min}\{C_1, C_2\} < C < C_1 + C_2$, differentiating (3) when $t_2 = T_2 = 0$ gives

$$\begin{aligned}
\frac{\partial u}{\partial t_1} &= \frac{r_1 m_1 T^{m_1}(t_1+t)^{m_1-1}}{((t_1+t)^{m_1}+T^{m_1})^2} - c_1 = 0 \\
\frac{\partial u}{\partial t} &= \frac{r_1 m_1 T^{m_1}(t_1+t)^{m_1-1}}{((t_1+t)^{m_1}+T^{m_1})^2} + \frac{r_2 m_2 t^{m_2-1}(T_2+T)^{m_2}}{(t^{m_2}+(T_2+T)^{m_2})^2} - c = 0 \\
\frac{\partial U}{\partial T_2} &= \frac{R_2 m_2 t^{m_2}(T_2+T)^{m_2-1}}{(t^{m_2}+(T_2+T)^{m_2})^2} - C_2 = 0, \\
\frac{\partial U}{\partial T} &= \frac{R_1 m_1 T^{m_1-1}(t_1+t)^{m_1}}{((t_1+t)^{m_1}+T^{m_1})^2} + \frac{R_2 m_2 t^{m_2}(T_2+T)^{m_2-1}}{(t^{m_2}+(T_2+T)^{m_2})^2} - C = 0
\end{aligned} \tag{41}$$

Combining the second and third equations in (41), inserting the first equation, and solving gives

$$t = Q_{21}(T_2+T), \quad T_2+T = \frac{R_2 m_2 Q_{21}^{m_2}}{C_2(Q_{21}^{m_2}+1)^2}, \quad Q_{21} = \frac{C_2/R_2}{(c-c_1)/r_2} \tag{42}$$

Combining the first and fourth equations in (41), inserting the third equation, and solving gives

$$t_1 = P_{21}T - t, \quad T = \frac{R_1 m_1 P_{21}^{m_1}}{(C-C_2)(P_{21}^{m_1}+1)^2}, \quad P_{21} = \frac{(C-C_2)/R_1}{c_1/r_1} \tag{43}$$

Combining (42) and (43) gives (11). Differentiating (41), the defender's second--order conditions and Hessian matrix, inserting (11), are

$$\begin{aligned}
\frac{\partial^2 u}{\partial t_1^2} &= \frac{\partial^2 u}{\partial t_1 \partial t} = \frac{\partial^2 u}{\partial t \partial t_1} = -\frac{\left(1-m_1+(1+m_1)P_{21}^{m_1}\right)}{c_1^{-2} m_1 r_1 P_{21}^{m_1}(P_{21}^{m_1}+1)^{-1}} \le 0 \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial t_1^2} - \frac{\left(1-m_2+(1+m_2)Q_{21}^{m_1}\right)}{(c-c_1)^{-2} m_2 r_2 Q_{21}^{m_2}(Q_{21}^{m_2}+1)^{-1}} \le 0
\end{aligned} \tag{44}$$

$$\left|H\right|_d = \begin{vmatrix} \frac{\partial^2 u}{\partial t_1^2} & \frac{\partial^2 u}{\partial t_1 \partial t} \\ \frac{\partial^2 u}{\partial t \partial t_1} & \frac{\partial^2 u}{\partial t^2} \end{vmatrix} = \frac{\left(1-m_1+(1+m_1)P_{21}^{m_1}\right)}{c_1^{-2} m_1 r_1 P_{21}^{m_1} (P_{21}^{m_1}+1)^{-1}} \frac{\left(1-m_2+(1+m_2)Q_{21}^{m_1}\right)}{(c-c_1)^{-2} m_2 r_2 Q_{21}^{m_2} (Q_{21}^{m_2}+1)^{-1}} \geq 0$$

Differentiating (41), the attacker's second-order conditions and Hessian matrix, inserting (11), are

$$\frac{\partial^2 U}{\partial T_2^2} = \frac{\partial^2 U}{\partial T_2 \partial T} = \frac{\partial^2 U}{\partial T \partial T_2} = -\frac{\left(1+m_2+(1-m_2)Q_{21}^{m_1}\right)}{C_2^{-2} m_2 R_2 Q_{21}^{m_2} (Q_{21}^{m_2}+1)^{-1}} \leq 0$$

$$\frac{\partial^2 U}{\partial T^2} = \frac{\partial^2 U}{\partial T_2^2} - \frac{\left(1+m_1+(1-m_1)P_{21}^{m_1}\right)}{(C-C_2)^{-2} m_1 R_1 P_{21}^{m_1} (P_{21}^{m_1}+1)^{-1}} \leq 0$$

$$\left|H\right|_a = \begin{vmatrix} \frac{\partial^2 U}{\partial T_2^2} & \frac{\partial^2 U}{\partial T_2 \partial T} \\ \frac{\partial^2 U}{\partial T \partial T_2} & \frac{\partial^2 U}{\partial T^2} \end{vmatrix} = \frac{\left(1+m_2+(1-m_2)Q_{21}^{m_1}\right)}{C_2^{-2} m_2 R_2 Q_{21}^{m_2} (Q_{21}^{m_2}+1)^{-1}} \tag{45}$$

$$\times \frac{\left(1+m_1+(1-m_1)P_{21}^{m_1}\right)}{(C-C_2)^{-2} m_1 R_1 P_{21}^{m_1} (P_{21}^{m_1}+1)^{-1}} \geq 0$$

## Appendix D. Defender Oge, attacker Ose, Section 3.4

If $c \leq \mathrm{Min}\{c_1, c_2\}$ and $C \geq C_1 + C_2$, differentiating (3) when $t_1 = t_2 = T = 0$ and $m_1 = m_2 = 1$ gives

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{2} \frac{r_i T_i}{(t+T_i)^2} - c = 0, \quad \frac{\partial U}{\partial T_i} = \frac{R_i t}{(t+T_i)^2} - C_i = 0 \tag{46}$$

Inserting the second equation for $i = 1, 2$ in (46) into the first equation and solving gives $t$ in (13). Solving the second equation in (46) gives $T_i$ in (13). Differentiating (46), the players' second-order conditions are

$$\frac{\partial^2 u}{\partial t^2} = -\sum_{i=1}^{2} \frac{2 r_i T_i}{(t+T_i)^3} \leq 0, \quad \frac{\partial^2 U}{\partial T_i^2} = -\frac{2 R_i t}{(t+T_i)^3} \leq 0 \tag{47}$$

Assume $C_1 \leq C_2$. If $1 \geq \sqrt{tC_2/R_2}$ in (13) is not satisfied, the attacker sets $T_2 = 0$, which is inserted into (46) and solved to yield (14). Assume $C_2 \leq C_1$. If $1 \geq \sqrt{tC_1/R_1}$ in (13) is not satisfied, the attacker sets $T_1 = 0$, which is inserted into (46) and solved to yield (15).

## Appendix E. Defender Ose, attacker Oge, Section 3.5

If $c \geq c_1 + c_2$ and $C \leq \mathrm{Min}\{C_1, C_2\}$, differentiating (3) when $T_1 = T_2 = t = 0$ and $m_1 + m_2 = 1$ gives

$$\frac{\partial U}{\partial T} = \sum_{i=1}^{2} \frac{R_i t_i}{(t_i + T)^2} - C = 0, \quad \frac{\partial u}{\partial t_i} = \frac{r_i T}{(t_i + T)^2} - c_i = 0 \tag{48}$$

Inserting the second equation for $i = 1, 2$ in (48) into the first equation and solving gives $T$ in (17). Solving the second equation in (46) gives $t_i$ in (17). Differentiating (48), the players' second-order conditions are

$$\frac{\partial^2 U}{\partial T^2} = -\sum_{i=1}^{2} \frac{2R_i t_i}{(t_i + T)^3} \leq 0, \quad \frac{\partial^2 u}{\partial t_i^2} = -\frac{2r_i t}{(t_i + T)^3} \leq 0 \tag{49}$$

Assume $c_1 \leq c_2$. If $1 \geq \sqrt{Tc_2/r_2}$ in (17) is not satisfied, the defender sets $t_2 = 0$, which is inserted into (48) and solved to yield (18). Assume $c_2 \leq c_1$. If $1 \geq \sqrt{Tc_1/r_1}$ in (17) is not satisfied, the defender sets $t_1 = 0$, which is inserted into (48) and solved to yield (19).

## Appendix F. Defender Sge, attacker Ose, Section 3.6

If $\mathrm{Min}\{c_1, c_2\} < c < c_1 + c_2$ and $C \geq C_1 + C_2$, differentiating (3) when $t_2 = T = 0$ gives

$$\begin{gathered}
\frac{\partial u}{\partial t_1} = \frac{r_1 m_1 T_1^{m_1} (t_1 + t)^{m_1 - 1}}{((t_1 + t)^{m_1} + T_1^{m_1})^2} - c_1 = 0, \quad \frac{\partial u}{\partial t} = \frac{r_1 m_1 T_1^{m_1} (t_1 + t)^{m_1 - 1}}{((t_1 + t)^{m_1} + T_1^{m_1})^2} + \frac{r_2 m_2 t^{m_2 - 1} T_2^{m_2}}{(t^{m_2} + T_2^{m_2})^2} - c = 0 \\
\frac{\partial U}{\partial T_1} = \frac{R_1 m_1 T_1^{m_1 - 1} (t_1 + t)^{m_1}}{((t_1 + t)^{m_1} + T_1^{m_1})^2} - C_1 = 0, \quad \frac{\partial U}{\partial T_2} = \frac{R_2 m_2 t^{m_2} T_2^{m_2 - 1}}{(t^{m_2} + T_2^{m_2})^2} - C_2 = 0
\end{gathered} \tag{50}$$

Combining the first and third equations in (50) and solving gives

$$t_1 = Q_1 T_1 - t, \quad Q_1 = \frac{C_1/R_1}{c_1/r_1} \tag{51}$$

and $T_1$ in (21). Combining the second and fourth equations in (50), inserting the first equation, and solving gives

$$T_2 = P_{12} t, \quad P_{12} = \frac{(c-c_1)/r_2}{C_2/R_2} \tag{52}$$

and $t$ in (21). Thereafter $t_1$ follows from (51), assuming $\frac{r_1 m_1 Q_1^{m_1}}{c_1 (Q_1^{m_1}+1)^2} \geq \frac{R_2 m_2 P_{12}^{m_2}}{C_2 (P_{12}^{m_2}+1)^2}$, and $T_2$ follows from (52). Differentiating (50), the defender's second-order conditions and Hessian matrix, inserting (21), are

$$\begin{gathered}
\frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 u}{\partial t_1 \partial t} = \frac{\partial^2 u}{\partial t \partial t_1} = -\frac{c_1^2 \left(1 - m_1 + (1+m_1) Q_1^{m_1}\right)}{m_1 r_1 Q_1^{m_1} (Q_1^{m_1}+1)^{-1}} \leq 0 \\
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial t_1^2} - \frac{(c-c_1)^2 \left(1 + m_2 + (1-m_2) P_{12}^{m_2}\right)}{m_2 r_2 P_{12}^{m_2} (P_{12}^{m_2}+1)^{-1}} \leq 0 \\
\left|H\right|_d = \begin{vmatrix} \frac{\partial^2 u}{\partial t_1^2} & \frac{\partial^2 u}{\partial t_1 \partial t} \\ \frac{\partial^2 u}{\partial t \partial t_1} & \frac{\partial^2 u}{\partial t^2} \end{vmatrix} = \frac{c_1^2 \left(1 - m_1 + (1+m_1) Q_1^{m_1}\right)}{m_1 r_1 Q_1^{m_1} (Q_1^{m_1}+1)^{-1}} \\
\times \frac{(c-c_1)^2 \left(1 + m_2 + (1-m_2) P_{12}^{m_2}\right)}{m_2 r_2 P_{12}^{m_2} (P_{12}^{m_2}+1)^{-1}} \geq 0
\end{gathered} \tag{53}$$

Differentiating (50), the attacker's second-order conditions and Hessian matrix, inserting (21), are

$$\begin{gathered}
\frac{\partial^2 U}{\partial T_1^2} = -\frac{C_1^2 \left(1 + m_1 + (1-m_1) Q_1^{m_1}\right)}{m_1 R_1 Q_1^{m_1} (Q_1^{m_1}+1)^{-1}} \leq 0, \quad \frac{\partial^2 U}{\partial T_2^2} = -\frac{C_2^2 \left(1 - m_2 + (1+m_2) P_{12}^{m_2}\right)}{m_2 R_2 P_{12}^{m_2} (P_{12}^{m_2}+1)^{-1}} \leq 0, \\
\frac{\partial^2 U}{\partial T_1 \partial T_2} = \frac{\partial^2 U}{\partial T_2 \partial T_1} = 0, \quad \left|H\right|_a \geq 0
\end{gathered} \tag{54}$$

## Appendix G. Defender Oge, attacker Sge, Section 3.7

If $c \le \mathrm{Min}\{c_1, c_2\}$ and $\mathrm{Min}\{C_1, C_2\} < C < C_1 + C_2$, differentiating (3) when $t_1 = t_2 = T_2 = 0$ and $m_1 = m_2 = 1$ gives

$$\begin{aligned}&\frac{\partial u}{\partial t} = \frac{r_1(T_1+T)}{(t+T_1+T)^2} + \frac{r_2 T}{(t+T)^2} - c = 0, \quad \frac{\partial U}{\partial T_1} = \frac{R_1 t}{(t+T_1+T)^2} - C_1 = 0 \\ &\frac{\partial U}{\partial T} = \frac{R_1 t}{(t+T_1+T)^2} + \frac{R_2 t}{(t+T)^2} - C = 0\end{aligned} \tag{55}$$

Solving the second and third equations in (55) gives

$$T = \sqrt{\frac{R_1 t}{C_1}} - t - T_1 = \left(\sqrt{\frac{R_2}{C-C_1}} - \sqrt{t}\right)\sqrt{t} \Rightarrow T_1 = \left(\sqrt{\frac{R_1}{C_1}} - \sqrt{\frac{R_2}{C-C_1}}\right)\sqrt{t} \tag{56}$$

when $\sqrt{\frac{R_1}{C_1}} \ge \sqrt{\frac{R_2}{C-C_1}}$, which are inserted into the first equation in (55) to yield (23). Differentiating (55), the defender's second-order condition is

$$\frac{\partial^2 u}{\partial t^2} = -\frac{2r_1(T_1+T)}{(t+T_1+T)^3} - \frac{2r_2 T}{(t+T)^3} \le 0 \tag{57}$$

Differentiating (55), the attacker's second-order conditions and Hessian matrix are

$$\begin{aligned}&\frac{\partial^2 U}{\partial T_1^2} = \frac{\partial^2 U}{\partial T_1 \partial T} = \frac{\partial^2 U}{\partial T \partial T_1} = -\frac{2R_1 t}{(t+T_1+T)^3} \le 0 \\ &\frac{\partial^2 U}{\partial T^2} = \frac{\partial^2 U}{\partial T_1^2} - \frac{2R_2 t}{(t+T)^3} \le 0, \quad \left|H\right|_a = \begin{vmatrix} \frac{\partial^2 U}{\partial T_1^2} & \frac{\partial^2 U}{\partial T_1 \partial T} \\ \frac{\partial^2 U}{\partial T \partial T_1} & \frac{\partial^2 U}{\partial T^2} \end{vmatrix} = \frac{4R_1 R_2 t^2}{(t+T_1+T)^3 (t+T)^3} \ge 0\end{aligned} \tag{58}$$

## Appendix H. Defender Sge, attacker Oge, Section 3.8

If $\mathrm{Min}\{c_1, c_2\} < c < c_1 + c_2$ and $C \le \mathrm{Min}\{C_1, C_2\}$, differentiating (3) when $t_2 = T_1 = T_2 = 0$ and $m_1 = m_2 = 1$ gives

$$\frac{\partial u}{\partial t_1} = \frac{r_1 T}{(t_1 + t + T)^2} - c_1 = 0, \quad \frac{\partial u}{\partial t} = \frac{r_1 T}{(t_1 + t + T)^2} + \frac{r_2 T}{(t + T)^2} - c = 0$$
$$\frac{\partial U}{\partial T} = \frac{R_1(t_1 + t)}{(t_1 + t + T)^2} + \frac{R_2 t}{(t + T)^2} - C = 0 \tag{59}$$

Solving the first and second equations in (59) gives

$$t = \sqrt{\frac{r_1 T}{c_1}} - T - t_1 = \left(\sqrt{\frac{r_2}{c - c_1}} - \sqrt{T}\right)\sqrt{T} \Rightarrow t_1 = \left(\sqrt{\frac{r_1}{c_1}} - \sqrt{\frac{r_2}{c - c_1}}\right)\sqrt{T} \tag{60}$$

when $\sqrt{\frac{r_1}{c_1}} \geq \sqrt{\frac{r_2}{c - c_1}}$, which are inserted into the first equation in (59) to yield (25). Differentiating (59), the defender's second-order conditions and Hessian matrix are

$$\frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 U}{\partial t_1 \partial t} = \frac{\partial^2 u}{\partial t \partial t_1} = -\frac{2 r_1 T}{(t_1 + t + T)^3} \leq 0,$$
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial t_1^2} - \frac{2 r_2 T}{(t + T)^3} \leq 0, \quad |H|_d = \begin{vmatrix} \frac{\partial^2 u}{\partial t_1^2} & \frac{\partial^2 u}{\partial t_1 \partial t} \\ \frac{\partial^2 u}{\partial t \partial t_1} & \frac{\partial^2 u}{\partial t^2} \end{vmatrix} = \frac{4 r_1 r_2 T^2}{(t_1 + t + T)^3 (t + T)^3} \geq 0 \tag{61}$$

Differentiating (59), the attacker's second-order condition is

$$\frac{\partial^2 U}{\partial T^2} = -\frac{2 R_1 (t_1 + t)}{(t_1 + t + T)^3} - \frac{2 R_2 t}{(t + T)^3} \leq 0 \tag{62}$$

## Appendix I. Defender Ose, attacker Sge, Section 3.9

If $c \geq c_1 + c_2$ and $\mathrm{Min}\{C_1, C_2\} < C < C_1 + C_2$, differentiating (3) when $t = T_2 = 0$ gives

$$\frac{\partial U}{\partial T_1}=\frac{R_1 m_1 t_1^{m_1}(T_1+T)^{m_1-1}}{((T_1+T)^{m_1}+t_1^{m_1})^2}-C_1=0$$

$$\frac{\partial U}{\partial T}=\frac{R_1 m_1 t_1^{m_1}(T_1+T)^{m_1-1}}{((T_1+T)^{m_1}+t_1^{m_1})^2}+\frac{R_2 m_2 T^{m_2-1} t_2^{m_2}}{(T^{m_2}+t_2^{m_2})^2}-C=0 \tag{63}$$

$$\frac{\partial u}{\partial t_1}=\frac{r_1 m_1 t_1^{m_1-1}(T_1+T)^{m_1}}{((T_1+T)^{m_1}+t_1^{m_1})^2}-c_1=0,\quad \frac{\partial u}{\partial t_2}=\frac{r_2 m_2 T^{m_2} t_2^{m_2-1}}{(T^{m_2}+t_2^{m_2})^2}-c_2=0$$

Combining the first and third equations in (63) and solving gives

$$T_1=q_1 t_1-T,\quad q_1=\frac{c_1/r_1}{C_1/R_1} \tag{64}$$

and $t_1$ in (27), assuming $\frac{R_1 m_1 q_1^{m_1}}{C_1(q_1^{m_1}+1)^2}\geq\frac{r_2 m_2 p_{12}^{m_2}}{c_2(p_{12}^{m_2}+1)^2}$. Combining the second and fourth equations in (63), inserting the first equation, and solving gives

$$t_2=p_{12}T,\quad p_{12}=\frac{(C-C_1)/R_2}{c_2/r_2} \tag{65}$$

Inserting (64) and (65) into the fourth equation in (63) and solving gives

$$T_1=\sqrt{\frac{c_1 c_2 r R^3 T}{(rC_1+Rc_1)^3}}-T \tag{66}$$

and $T$ in (27). Thereafter, $T_1$ follows from (64) and $t_2$ follows (65). Differentiating (63), the defender's second-order conditions and Hessian matrix, inserting (27), are

$$\frac{\partial^2 u}{\partial t_1^2}=-\frac{c_1^2\left(1+m_1+(1-m_1)q_1^{m_1}\right)}{m_1 r_1 q_1^{m_1}(q_1^{m_1}+1)^{-1}}\leq 0$$

$$\frac{\partial^2 u}{\partial t_2^2}=-\frac{c_2^2\left(1-m_2+(1+m_2)p_{12}^{m_2}\right)}{m_2 r_2 p_{12}^{m_2}(p_{12}^{m_2}+1)^{-1}}\leq 0 \tag{67}$$

$$\frac{\partial^2 u}{\partial t_1\partial t_2}=\frac{\partial^2 u}{\partial t_2\partial t_1}=0,\quad \left|H\right|_d\geq 0$$

Differentiating (63), the attacker's second-order conditions and Hessian matrix, inserting (27), are

$$\frac{\partial^2 U}{\partial T_1^2}=\frac{\partial^2 U}{\partial T_1 \partial T}=\frac{\partial^2 U}{\partial T \partial T_1}=-\frac{C_1^2\left(1-m_1+(1+m_1)q_1^{m_1}\right)}{m_1 R_1 q_1^{m_1}(q_1^{m_1}+1)^{-1}}\leq 0$$

$$\frac{\partial^2 U}{\partial T^2}=\frac{\partial^2 U}{\partial T_1^2}-\frac{(C-C_1)^2\left(1+m_2+(1-m_2)p_{12}^{m_2}\right)}{m_2 R_2 p_{12}^{m_2}(p_{12}^{m_2}+1)^{-1}}\leq 0$$

$$\left|H\right|_a=\begin{vmatrix}\dfrac{\partial^2 U}{\partial T_1^2} & \dfrac{\partial^2 U}{\partial T_1 \partial T}\\ \dfrac{\partial^2 U}{\partial T \partial T_1} & \dfrac{\partial^2 U}{\partial T^2}\end{vmatrix}=\frac{C_1^2\left(1-m_1+(1+m_1)q_1^{m_1}\right)}{m_1 R_1 q_1^{m_1}(q_1^{m_1}+1)^{-1}} \tag{68}$$

$$\times\frac{(C-C_1)^2\left(1+m_2+(1-m_2)p_{12}^{m_2}\right)}{m_2 R_2 p_{12}^{m_2}(p_{12}^{m_2}+1)^{-1}}\geq 0$$

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*Received 10 June 2019*
*Accepted 8 January 2020*