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APPLYING FUZZY PARAMETERS
IN PRICING FINANCIAL DERIVATIVES INSPIRED
BY THE KYOTO PROTOCOL

The emission trading is proposed in the Kyoto Protocol. An appropriate market and the market of financial derivatives for allowances will be established. Using the neutral martingale method and Monte Carlo simulations, we propose a stochastic model with a pricing formula, which may be useful for an evaluation of derivatives inspired by the Kyoto Protocol.

Keywords: *option pricing, financial derivatives, Kyoto Protocol, martingale method, fuzzy parameters*

1. Introduction

Since its adoption in the text of the Kyoto Protocol, emission trading has been seen as one of the primary tools for international co-operation in reducing emissions of greenhouse gases (GHG). This market-based instrument will be an important mechanism for environmental protection ([9], [21]). Many countries will implement appropriate instruments for the first time, but the first tradable rights were proposed much earlier to aid pollution control ([15], [20], [21]).

The Kyoto Protocol will probably stimulate two levels of markets for the described emissions trading instruments. The first will be on a world-wide scale, i.e. trading

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among the so called Annex A countries, in which the trading mechanism has been implemented. The second is for individual countries or bigger markets, like the European Union. Both of these levels need new financial mechanisms for decreasing risk levels, improving liquidity and the rate of turnover of allowances. We know such instruments from the classical markets of options, futures and other derivatives for money and goods. Hence, there is a need to propose a methodology appropriate to both derivatives and the very specific market of emission allowances.

The paper is organized as follows. We describe the general rules of markets for emission allowances. We propose a stochastic process which may be suitable for modelling these markets and find a mathematical formula for the price of a rather standard option, i.e. a European call option. We also discuss the possibility of adapting our simulations to more complicated financial instruments. Some notes about the necessity of modelling uncertainty via a fuzzy approach are provided. We present some conclusions and possible directions for future research. Other important remarks on the subject discussed may be found in e.g. [4], [6], [8], [12], [14], [17], [19], [22].

2. The Kyoto Protocol and emission trading

International trade in greenhouse gas emissions is specifically provided for in the 1997 Kyoto Protocol ([9]). The main aim of this treaty is to reduce emissions by a fixed percentage below what they were in 1990. These reductions are to be fulfilled during the period 2008–2012. Because the necessary modifications in industrial processes, upgrading factories and changing the course of national economies in many countries may be very costly, the protocol proposed three innovative "flexible mechanisms", among them emission trading.

This emission trading will be based on the same general mechanisms as other financial markets. The main difference is that the tradable "good" will be greenhouse gas emissions permits specified in CO_2-equivalent tons. The buyers of such permits will be companies (or countries in case of the worldwide market) for which the cost of reducing emissions is high. On the contrary, the sellers will be entities, for which this cost is less than the price of the permit, or the overall quantity of their emissions is lower than specified in the Kyoto Protocol. Therefore, a real, functioning market will establish a market price for emissions as long as some standard assumptions are satisfied, like e.g. many enterprises being involved in such trading.

The permits described above may be treated as the primary financial market for some good. It is clear that bonds, options, futures and forward contracts, as well as

other financial derivatives may be established for such a primary market. These instruments and the derivatives market may prove to be very useful, especially for facilitating turnover, increasing flux liquidity and securing against the overestimation of possible emission reductions and the risk connected with such errors. Therefore, it is necessary to use adequate financial mathematics tools to solve problems arising from the issuing or pricing of such instruments. These tools are highly developed in case of the "standard" financial markets, like stock exchanges or futures markets ([3], [5], [10], [11], [18]). But in the case of emissions trading, we should take into account some special attributes, like yearly reports on the fulfilment of reduction requirements. Hence, before any problems can be solved by using financial mathematics instruments, we have to develop an appropriate model for permit prices, which we propose in this article. Obviously, this model should be appropriate to the permit market and should suitably forecast the future behaviour of prices. Hence, we adopt a stochastic process which is a generalization of the widely known Black-Scholes model with additional random jumps at a fixed time of year. Such a jump reflects the possibility of a sudden change in the course of a permit price if a government or organisation announces an annual report on the level of gas emissions and fulfilment of the reduction commitment arising from the Kyoto Protocol for that given country. Usually models for primary financial markets (e.g. commodity markets) involve jumps at random moments of time (e.g. described by Poisson processes), but not jumps at fixed and regular moments of time, as in the approach presented in this paper.

Because of the lack of past data and relatively short history of the permit market, we may use past experiences from similar markets. Exactly speaking, we use the idea of comparing the GHG market and other emissions allowances markets, e.g. for SO_2. From past data for these markets and general economic theory, it can be seen that geometrical Brownian motion is an appropriate first approximation model for describing the trajectory of allowance prices. However, due to the specification of the market derived from the Kyoto Protocol, there is a need to slightly change such a model, in order to take into account the periodical reports of emission levels.

Because of the imprecise information and uncertainty which characterize such a market (see e.g. [13], [16]), some input parameters of our model cannot always be expected to be known precisely. Therefore, this uncertainty has to be included in the construction of the model. Expert opinions or imprecise estimates of the market parameters may be expressed as fuzzy numbers. Also, statistical estimators may be transformed into fuzzy numbers (see [2]). Therefore, it is natural to consider fuzzy parameters for such a stochastic model. Such a treatment is particularly useful for financial analysts, giving them a tool to pick an option price with an acceptable degree of belief degree for latter use (see [23]).

3. Option pricing

3.1. Mathematical preliminaries

Let $(\Omega, F, (F_t)_{t \in [0,T]}, P)$ be a probability space with filtration. Let $T < \infty$. A stochastic process $H = (H_t)_{t \in [0,T]}$ is cadlag, if its trajectories are right continuous with left limits. A cadlag stochastic process H is a semimartingale, if it has a decomposition $H_t = H_0 + A_t + N_t$, $t \in [0,T]$, where $A = (A_t)_{t \in [0,T]}$ is an F_t-adapted cadlag process with finite variation, $N = (N_t)_{t \in [0,T]}$ is a local martingale and H_0 is the value of the process at moment $t = 0$.

We recall some basic definitions and facts from [18]. Let $\varphi(x) = x I_{\{|x| \leq 1\}}$ be a truncation function. The process $H(\varphi)_t = H_t - \breve{H}(\varphi)_t$, where $\breve{H}(\varphi)_t = \sum_{s \leq t} [\Delta H_s - \varphi(\Delta H_s)]$, is a special semimartingale with canonical decomposition $H(\varphi) = H_0 + N(\varphi) + B(\varphi)$, i.e. $N(\varphi)$ is a local martingale, $B(\varphi)$ is a locally integrable, predictable process with finite variation and H_0 is the value of the process at moment $t = 0$.

DEFINITION 1. The elements of the triplet $T = (B, C, \nu)$ are called the characteristics of H if
i) $B = B(\varphi)$ is a predictable process with finite variation;
ii) $C = \langle H^c \rangle$, where H^c is the continuous martingale part of H;
iii) ν is a predictable random measure on $B(T \times R)$, which is the compensator of the measure describing the jump structure of H, i.e. $\mu^H(dt, dx) = \sum_s I_{\{\Delta H_s \neq 0\}} \varepsilon(dt, dx)$.

DEFINITION 2. A probability measure $\widetilde{P}$ on (Ω, F) is locally absolutely continuous with respect to $P\left(\widetilde{P} \overset{loc}{\ll} P \right)$, if for each $t \in [0,T]$ the probability measure $\widetilde{P}_t = \widetilde{P} | F_t$ is absolutely continuous with respect to $P_t = P | F_t$.

If $\widetilde{P} \overset{loc}{\ll} P$ and $P \overset{loc}{\ll} \widetilde{P}$, then the measures are equivalent ($\widetilde{P} \overset{loc}{\sim} P$).

Let $\widetilde{P} \overset{loc}{\sim} P$. We define the stochastic processes $Z_t = \dfrac{d\widetilde{P}_t}{dP_t}$ and $M_t = \int_0^t \dfrac{dZ_s}{Z_{s-}}$, $t \in [0,T]$.

Let $\beta = \dfrac{d\langle Z^c, H^c\rangle}{d\langle H^c, H^c\rangle} \cdot \dfrac{I(Z_- > 0)}{Z_-}$ and $X = E_{\mu^H}^{P}\left(\dfrac{Z}{Z_-} I(Z_- > 0)\,|\,\widetilde{P}\right)$, where $E_{\mu^H}^{P}$ is

the expectation with respect to the measure $M_{\mu^H}^{P}$ on $F \otimes B(t) \otimes B(R)$ defined by the equality $W * M_{\mu^H}^{P} = E(W * \mu^H)$ for all nonnegative measurable functions $W = W(\omega, t, x)$, and $<.,.>$ is the quadratic characteristic of the processes.

The following theorem (see [10]) is called the Girsanov theorem for semimartingales.

Theorem 1. Let $\widetilde{P} \overset{loc}{<\!<} P$ and Z, β and X be defined as above. Then $\widetilde{B} = B + \beta C + \varphi(x)(X-1) * \nu$, $\widetilde{C} = C$ and $\widetilde{\nu} = X\nu$ are the characteristics of H with respect to $\widetilde{P}$.

3.2. Fuzzy sets preliminaries

Let X be a universal set and $\widetilde{A}$ be a fuzzy subset of X. We denote its membership function by $\mu_{\widetilde{A}}$, and the α-level set of $\widetilde{A}$ by $\widetilde{A}_\alpha = \{x : \mu_{\widetilde{A}}(x) \geq \alpha\}$. In our paper we assume that X is the set of real numbers.

Let $\widetilde{a}$ be a fuzzy number. Then, under our assumptions, the α-level set $\widetilde{A}$ is a closed interval, which can be denoted by $\widetilde{a}_\alpha = [\widetilde{a}_\alpha^L, \widetilde{a}_\alpha^U]$ (see e.g. [24]).

We can now introduce the arithmetic of any two fuzzy numbers. Let $\widetilde{\bullet}$ be a binary operator on fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$, where the binary operators correspond to $\widetilde{+}$, $\widetilde{/}$, etc. according to the Extension Principle (see [24]). Then the membership function of $\widetilde{a}\,\widetilde{\bullet}\,\widetilde{b}$ is defined by

$$\mu_{\widetilde{a}\,\widetilde{\bullet}\,\widetilde{b}} = \sup_{(x,y):x \bullet y = z} \min\{\mu_{\widetilde{a}}(x), \mu_{\widetilde{b}}(x)\}$$

If $\widetilde{a}$, $\widetilde{b}$ are fuzzy numbers, then the α-level set is given by

$$(\widetilde{a}\,\widetilde{+}\,\widetilde{b})_\alpha = [\widetilde{a}_\alpha^L + \widetilde{b}_\alpha^L, \widetilde{a}_\alpha^U + \widetilde{b}_\alpha^U], \quad (\widetilde{a} \simeq \widetilde{b})_\alpha = [\widetilde{a}_\alpha^L - \widetilde{b}_\alpha^U, \widetilde{a}_\alpha^U - \widetilde{b}_\alpha^L],$$

$$(\widetilde{a}\,\widetilde{*}\,\widetilde{b})_\alpha = [\min\{\widetilde{a}_\alpha^L\widetilde{b}_\alpha^L, \widetilde{a}_\alpha^L\widetilde{b}_\alpha^U, \widetilde{a}_\alpha^U\widetilde{b}_\alpha^L, \widetilde{a}_\alpha^U\widetilde{b}_\alpha^U\}, \quad \max\{\widetilde{a}_\alpha^L\widetilde{b}_\alpha^L, \widetilde{a}_\alpha^L\widetilde{b}_\alpha^U, \widetilde{a}_\alpha^U\widetilde{b}_\alpha^L, \widetilde{a}_\alpha^U\widetilde{b}_\alpha^U\}]$$

and if the α-level set $\widetilde{b}$ does not contain zero, then

$$(\widetilde{a}\ \widetilde{/}\ \widetilde{b})_\alpha = [\min\{\widetilde{a}_\alpha^L / \widetilde{b}_\alpha^L, \widetilde{a}_\alpha^L / \widetilde{b}_\alpha^U, \widetilde{a}_\alpha^U / \widetilde{b}_\alpha^L, \widetilde{a}_\alpha^U / \widetilde{b}_\alpha^U\},$$

$$\max\{\widetilde{a}_\alpha^L / \widetilde{b}_\alpha^L, \widetilde{a}_\alpha^L / \widetilde{b}_\alpha^U, \widetilde{a}_\alpha^U / \widetilde{b}_\alpha^L, \widetilde{a}_\alpha^U / \widetilde{b}_\alpha^U\}]$$

Theorem 2. Let $f : R \to R$ be a function, $Fu(R)$ be the set of all fuzzy subsets of R and $\widetilde{\Lambda} \in Fu(R)$. We assume that the membership function $\mu_{\widetilde{\Lambda}}$ is upper semicontinuous and for all y the set $\{x : f(x) = y\}$ is compact. The function $f(x)$ can induce a fuzzy-valued function $\widetilde{f} : Fu(R) \to Fu(R)$ via the Extension Principle. Then $\widetilde{f}(\widetilde{\Lambda})_\alpha = \{f(x) : x \in \widetilde{\Lambda}_\alpha\}$.

For a proof of this proposition see [24].

The triangular fuzzy number $\widetilde{a}$ is defined by its membership function

$$\mu_{\widetilde{a}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{for } a_1 \leq x \leq a_2 \\ \dfrac{x - a_3}{a_2 - a_3} & \text{for } a_2 < x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

where $[a_1, a_3]$ is the supporting interval and the membership function has a peak at a_2.

Left-Right (or L-R) fuzzy numbers are a generalization of triangular fuzzy numbers (see [1], [7]), where the linear functions used in the above formula are replaced by monotonic functions, i.e.

$$\mu_{\widetilde{a}}(x) = \begin{cases} L\left(\dfrac{a_2 - x}{a_2 - a_1}\right) & \text{for } a_1 \leq x \leq a_2 \\ R\left(\dfrac{x - a_2}{a_3 - a_2}\right) & \text{for } a_2 < x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

where L and R are continuous strictly decreasing functions defined on [0, 1] with values in [0, 1] satisfying the conditions $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. An L-R fuzzy number is denoted by $(a_1, a_2, a_3)_{LR}$.

Next we turn to fuzzy estimation based on a statistical approach (see [2]). Let X be a random variable with probability density function $f_\theta(.)$. Assume that the parameter θ is unknown and is to be estimated from a sample $X_1, X_2, ..., X_n$. Let $\hat{\theta}$ be a statistic

based on this sample. Then for a given confidence level β we obtain the confidence interval $[\theta_L(\beta), \theta_R(\beta)]$. If we suppose that $[\theta_L(0), \theta_R(0)] = [\hat{\theta}, \hat{\theta}]$, then we can construct a fuzzy estimator $\widetilde{\theta}$. We place the confidence intervals, one on top of the other, to produce a triangular shaped fuzzy number whose α-cuts are confidence intervals with confidence levels $\beta = 1 - \alpha$. In order to finish the construction, we project the graph of $\widetilde{\theta}$ straight down from e.g. $\alpha = 0.01$ to complete its α-cuts (see [2] for additional details).

3.3. Model of the underlying asset

The price of the underlying asset is given by a process $S = (S_t)_{t \in [0,T]}$ of the form $S_t = S_0 \exp(Y_t)$, where Y_t is an F_t-adapted process defined by the formula

$$Y_t = \mu t + \sigma W_t + \sum_{i=0}^{[t]} \xi_i \,, \tag{3.1}$$

where W_t denotes Brownian motion, $\sigma > 0$, $\mu \in R$ and $\{\xi_i\}_{i=1}^{\infty}$ are sequences of independent random variables with probability density function

$$\rho(x) = \begin{cases} \dfrac{\lambda_1}{2} e^{-\lambda_1 x} & \text{for} & x \geq 0, \\ \dfrac{\lambda_2}{2} e^{\lambda_2 x} & \text{for} & x < 0, \end{cases}$$

where $\lambda_1, \lambda_2 > 0$, [t] denotes the integer part of t.

3.4. The Martingale method of option pricing

Let r be a constant risk-free interest rate and $\Xi_t = e^{-rt} S_t$ be the discounted process describing the price of the underlying asset. Our aim is to find the measure $\widetilde{P}$, locally equivalent to P (see Def. 2), such that Ξ_t is a martingale with respect to $\widetilde{P}$. The next step is to find the form of the process S_t with respect to the new measure $\widetilde{P}$. The price of the derivative with payoff function f is given by the formula:

$$C_t = \exp(-r(T-t)) E^{\widetilde{P}}(f(S) | F_t), \quad t \in [0,T] \tag{3.2}$$

Let $H_t = Y_t - rt$. We will apply the following result from [3] and use the notation from Section 3.1.

Theorem 3. Suppose that Z_t is a positive martingale with $dZ_t = Z_{t-}dM_t$, where M is given by

$$M_t = M_0 + \int_0^t \beta_s dH_s^C + \int_0^t\int_R \overline{W}(.,s,x)d(\mu-\nu), \qquad (3.3)$$

$\overline{W} = X - 1 + \dfrac{\hat{X} - a}{1-a} I(a<1)$, where $a = a_t(\omega) = \nu(\omega, \{t\}\times R)$, $\hat{X}_t =$ $\int_R X(\omega, s, x)\nu(\omega, \{t\}\times dx)$ and β and $\overline{W}$ satisfy the corresponding integrability conditions ([10]). Moreover, assume that $E|Z_t| = 1$. Then in the cases where $\nu(\omega, \{t\}\times R) \in \{0,1\}$, the condition

$$K_t + \int_0^t \beta_s d\langle H^C\rangle_s + \int_0^t\int_R (X-1)(e^x-1)d\nu = 0, \quad t\leq T, \qquad (3.4)$$

where $K_t = B_t + \dfrac{1}{2}\langle H^C\rangle_t + \int_0^t\int_R (e^x - 1 - \varphi(x))d\nu$, implies the existence of a measure

$\widetilde{P}_T$, constructed as above, equivalent to the measure P_T, for which Ξ is a local martingale. In the above formula, B_t is the first local characteristic of the process H (see Def. 1).

For $\gamma > 2$ we define

$$\widetilde{\rho}(x) = \begin{cases} \dfrac{\gamma}{2}e^{-\gamma x} & \text{for} \quad x \geq 0, \\ \dfrac{\gamma-2}{2}e^{(\gamma-2)x} & \text{for} \quad x < 0. \end{cases} \qquad (3.5)$$

Theorem 4. For a European call option with payoff function $f(x) = (x - K^*)_+$, $K^* > 0$,

$$C_0 = \exp(-rT)\int_{\ln K - \left(r-\frac{\sigma^2}{2}\right)T}^{\infty} \left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+x} - K^*\right)dG^{[T]}(x), \qquad (3.6)$$

where $G^{[T]}(x) = G * \overbrace{\Psi * ... * \Psi}^{[T]}(x)$, G is the cumulative distribution function of the normal distribution $N(0, \sigma\sqrt{T})$ and Ψ is the cumulative distribution function corresponding to density $\tilde{\rho}$. $G^{[T]}$ has density function $g^{[T]}$ and the above formula can be written as follows:

$$C_0 = e^{-\frac{1}{2}\sigma^2 T} \int_{\ln\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}^{\infty} e^x g^{[T]}(x)dx - e^{-rT}K^*\left(1 - G^{[T]}\left(\ln\frac{K^*}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T\right)\right).$$

Proof: The characteristics B and C of the process H_t are of the form $B_t = (\mu - r)t + [t]\int_R \varphi(x)\rho(x)dx$ and $C_t = \sigma^2 t$. Since H is a process with independent increments, formula (3.3) shows the decomposition of M. All the assumptions of Theorem 3 are satisfied. Since

$$K_t = (\mu - r)t + [t]\int_R \varphi(x)\rho(x)dx + \frac{1}{2}\sigma^2 t + [t]\int_R (e^x - 1 - \varphi(x))\rho(x)dx,$$

equation (3.4) is equivalent to

$$\left(\mu - r + \frac{1}{2}\sigma^2\right)t + \sigma^2\int_0^t \beta_s ds + [t]\left(\int_R X(\omega, 1, x)(e^x - 1)\rho(x)dx\right) = 0, \quad t \le T. \quad (3.7)$$

The sum $I_1 = \left(\mu - r + \frac{1}{2}\sigma^2\right)t + \sigma^2\int_0^t \beta_s ds$ is continuous in t and $I_2 = [t]\int_R X(\omega, 1, x)(e^x - 1)\rho(x)dx$ has jumps at $t = 1, 2, ...$ We obtain the following solution of (3.7):

$$\beta = -\frac{(\mu - r)}{\sigma^2} - \frac{1}{2},$$

$$X(\omega, t, x) = \begin{cases} \dfrac{e^{(-\gamma + \lambda_1)x}}{R_1} & \text{for} \quad t \in N, x \ge 0 \\ \dfrac{e^{(\gamma - 2 - \lambda_2)x}}{R_2} & \text{for} \quad t \in N, x < 0 \\ 1 & \text{for} \quad t \notin N \end{cases}$$

for $R_1 = 2Ee^{(-\gamma + \lambda_1)\xi_1}I_{\{\xi_1 \ge 0\}}$ and $R_2 = 2Ee^{(\gamma - 2 - \lambda_2)\xi_1}I_{\{\xi_1 < 0\}}$.

Applying Theorem 1 for the characteristics of H with respect to $\widetilde{P}_T$, the characteristics of Y with respect to $\widetilde{P}_T$ are of the form

$$\widetilde{B}_t^Y = \left(r - \frac{1}{2}\sigma^2 \right)t + [t]\int_R \varphi(x)X(\omega,1,x)\rho(x)dx,$$

$$\widetilde{C}_t^Y = \sigma^2 t,$$

$$\widetilde{v}^Y(\omega,(0,t]\times dx) = [t]X(\omega,1,x)\rho(x)dx.$$

Therefore the form of the process Y with respect to $\widetilde{P}_T$ is described by the formula $Y_t = \left(r - \frac{1}{2}\sigma^2 \right)t + \sigma\widetilde{W}_t + \sum_{i=0}^{[t]}\widetilde{\xi}_i$, where $\widetilde{W}_t$ is a Brownian motion and $\{\widetilde{\xi}_i\}_{i=1}^\infty$ are independent random variables with density $\widetilde{\rho}(x) = X(\omega,1,x)\rho(x)$. If $\Psi(x) = \int_{-\infty}^x \widetilde{\rho}(x)dx$ is the cumulative distribution function of $\widetilde{\xi}_i$ and $\Psi^{[T]}(x) = \overbrace{\Psi*...*\Psi}^{[T]}(x)$, then $G^{[T]}(x) = G*\overbrace{\Psi*...*\Psi}^{[T]}(x)$ is the cumulative distribution function of $Y_T - \left(r - \frac{1}{2}\sigma^2 \right)T$. Therefore, $Y_T - \left(r - \frac{1}{2}\sigma^2 \right)T$ has a probability density function.

From (3.2) it follows that

$$C_0 = \exp(-rT)E^{\widetilde{P}}f(S_T) = \exp(-rT)[E^{\widetilde{P}}S_T I_{S_T > K^*} - K^*\widetilde{P}(S_T > K^*)]$$

$$= e^{-\frac{1}{2}\sigma^2 T}\int_{\ln\frac{K^*}{S_0} - \left(r - \frac{\sigma^2}{2} \right)T}^{\infty} e^x g^{[T]}(x)dx - e^{-rT}K^*\left(1 - G^{[T]}\left(\ln\frac{K^*}{S_0} - \left(r - \frac{\sigma^2}{2} \right)T \right) \right). \qquad (3.8)$$

3.5. Fuzzy approach

Let us denote the fuzzy counterparts of the parameters describing process (3.1) by $r', \sigma', \lambda_1', \lambda_2'$. We will treat them as L-R fuzzy numbers which are obtained from experts or estimated using a statistical approach (see Section 3.2). Then, after the change of measure, the stochastic process has the following fuzzy form

$$\widetilde{Y}_t = \left(r' \simeq \frac{1}{2} \widetilde{*} \, \sigma' \, \widetilde{*} \, \sigma' \right) \widetilde{*} \, t \, \widetilde{\mp} \, \sigma' \, \widetilde{*} \, \widetilde{W}_t \, \widetilde{\mp} \, \sum_{i=0}^{[t]} \widetilde{\xi}_i. \tag{3.9}$$

Formula (3.9) can be applied to pricing derivatives via simulation methods (see Section 3.6). In particular, Monte Carlo simulations can be used for complex derivatives for which it is impossible to describe their prices using a closed formula.

The pricing formula for a European call option in Theorem 4 has a closed analytical form. Therefore, one can write its fuzzy version, replacing crisp valued parameters by their fuzzy counterparts and arithmetic operators by $\widetilde{+}$, $\simeq$, $\widetilde{*}$ and $\widetilde{/}$. For the integral $I(z) = \int_z^\infty e^x g^{[T]}(x)dx$ and the cumulative distribution function $G^{[T]}$ one can use Theorem 2, since the integrals can be expressed as antiderivatives with crisp values and it is possible to apply the Extension Principle directly to them. A European call option can be priced via α-cuts. Since the pricing formula is complex, one has to use numerical methods to find an appropriate price.

Let us denote the fuzzy price of a European call option by $\widetilde{C}_0$. From the equality

$$\mu_{\widetilde{C}_0}(c) = \sup_{0 \leq \alpha \leq 1} \alpha I_{(\widetilde{C}_0)_\alpha}(c),$$

we obtain the membership function of $\widetilde{C}_0$. If c is the option price and $\mu_{\widetilde{C}_0}(c) = \alpha$, then the value of the membership function may be treated by a financial analyst as the degree of belief in c. Therefore, the fuzzy approach, together with the application of α-cuts (for α close to 1), to pricing derivatives enables us to obtain intervals for prices with a high degree of belief.

3.6. Monte Carlo simulations

As mentioned in Section 3.5, the stochastic process which describes the price of an allowance is rather complicated for direct calculations, especially due to the complicated formula for the density $G^{[T]}(x)$. Even in the simplest case of a European call function with strictly crisp parameters, the price formula (3.6) may not be analytically tractable.

Hence, there is a need to use numerical or simulation methods, in order to find a computable formula. Such methods could also be helpful for pricing other, more complex, derivatives based on the proposed process (3.1) and its martingale modification obtained in Sec. 3.4. They may also be applied using a fuzzy approach. We use Monte Carlo simulations, which are very flexible and efficient even for large portfo-

lios of financial instruments. This approach may be especially suitable for entities which calculate the possible costs and revenues from using various strategies for emissions reduction based on various instruments, e.g. not only financial, but also on changing industrial processes, or the scope of production, etc. In such a case, simulations of possible scenarios are a natural way to incorporate these different mechanisms and to provide the basis for a decision support system.

Using Monte Carlo simulation, we find the following price for the derivative (compare with (3.2))

$$\hat{C}_0 = \exp(-rT)E(FV(f(S_t))),$$ (3.10)

which is the discounted expected value of future cash flows of the payment function $f(.)$ based on the underlying asset trajectory S_t modelled by the given stochastic process. Therefore, we have to simulate n trajectories $S_t^{(1)}, S_t^{(2)}, ..., S_t^{(n)}$, where n is called the number of trajectories and then calculate the classical estimator based on the average value. For a European call function, we have

$$\hat{C}_0 = \exp(-rT)\frac{1}{n}\sum_{i=1}^{n}(S_T^{(i)} - K^*)_+ \,,$$

where $S_T^{(i)}$ is the price for the i-th underlying asset trajectory at moment T.

If we divide the time interval $[0, T]$ into m equal steps $t_0 = 0, t_1, ..., t_m = T$, then we can discretize the trajectory $S_t^{(i)}$ into segments $S_{t_0}^{(i)}, S_{t_1}^{(i)}, ..., S_{t_m}^{(i)}$, where m is the number of steps. From the martingale modification of process (3.1) and the form of S_t, we have

$$S_{t_{j+1}}^{(i)} = S_{t_j}^{(i)} \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t_j + \sigma N_{i,j}\Delta t_j + [\Delta t_j]P_{i,j}\right)$$ (3.11)

where $\Delta t_j = t_{j+1} - t_j$, $N_{i,j}$ are *i.i.d.* (independent, identically distributed) random variables from N(0;1) (the standard normal distribution), $P_{i,j}$ are *i.i.d.* random variables with distribution given by (3.5) and $[\Delta t_j]$ is the number of "jump events" in the interval Δt_j, i.e. $[\Delta t_j] := [t]_{[\Delta t_j]}$. Therefore, from (3.11), which is similar to Euler's known scheme for the Black–Scholes model, we can simulate the necessary steps $S_{t_0}^{(i)}, S_{t_1}^{(i)}, ..., S_{t_m}^{(i)}$ for any trajectory $S_t^{(i)}$. Substituting the data from these trajectories into (3.10), we can find an estimator of the price for any given kind of derivative.

In order to conduct the simulations using formula (3.11), we need crisp values for our parameters. Therefore, using a fuzzy approach we first find the α-level sets for L-R fuzzy numbers $r', \sigma', \lambda_1', \lambda_2'$ for a given value of α, e.g. $\alpha = 0.95$. Then we randomly pick a value for each parameter of the model treating the appropriate α-level sets as in-

tervals with uniform distribution. We repeat this procedure for each new trajectory $S_t^{(1)}, S_t^{(2)}, ..., S_t^{(n)}$.

We have carried out appropriate simulations to illustrate our considerations. An example of a price trajectory generated by Monte Carlo simulations is shown by Figure 1. As we can see, such a trajectory is similar to one obtained from the classical Black–Scholes model (i.e. modelled by a geometrical Wiener process), but with occasional "shocks" added by the $P_{i,j}$ part in (3.11).
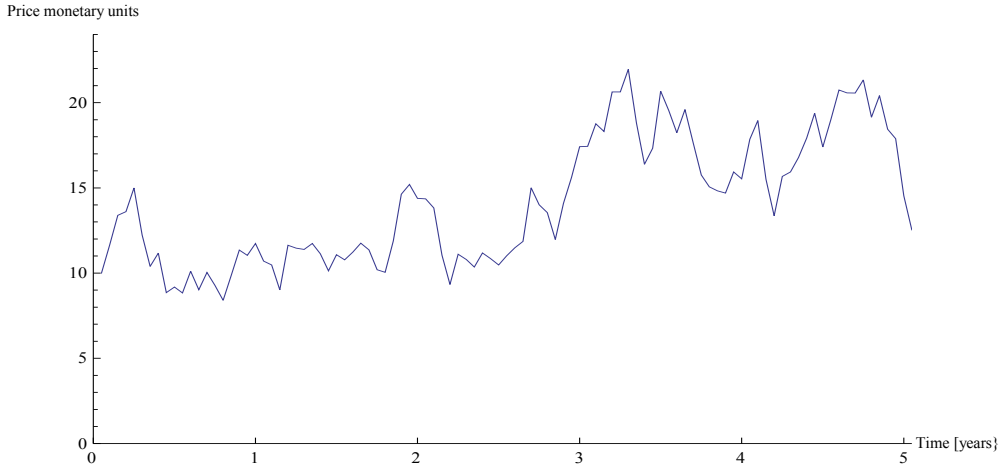
Price monetary units

Fig. 1. Example of a path of the process generated by Monte Carlo simulations

We have conducted a set of simulations for a European call option. In our experiments the $r', \sigma', \lambda_1', \lambda_2'$ were given as triangular fuzzy numbers, where $a_2 = (a_1 + a_3)/2$. We set $S_0 = 10$, $T = 5$, $m = 10$, $n = 10000$, $\lambda_1' = [2, 2.5]$, $\lambda_2' = [2, 2.5]$, where each interval denotes an α-level set for $\alpha = 0.95$. Other parameters and the prices obtained are given in Table 1.

Table 1. Prices of European call options calculated via Monte Carlo simulations

r	σ	K^*	Price	Confidence interval for price
[0.05, 0.075]	[0.25, 0.5]	12	3.97825	[3.76493, 4.19158]
[0.05, 0.075]	[0.1, 0.2]	12	2.47049	[2.38683, 2.55415]
[0.05, 0.075]	[0.1, 0.2]	8	4.40962	[4.31257, 4.50666]
[0.05, 0.1]	[0.2, 0.3]	12	3.41014	[3.27577, 3.54452]

Apart from the crisp values of prices, we also calculated 95% confidence intervals based on the central limit theorem, which may be applied in this case.

4. Conclusions

In this paper we have discussed some general issues concerning the market of emissions allowances arising from the Kyoto Protocol. We point out that the introduction of such financial instruments, like options, futures and forward contracts, known as derivatives, will help in decreasing the level of risk and improving the liquidity of the evolving emissions market. Such derivatives, based on emissions allowances, are similar to the analogous instruments for classical financial markets.

However, there are some differences, and one of them is the model for the trajectory of the underlying asset. We propose such a model based on a generalization of the classical Black–Scholes model. We also discuss the possibility of applying simulations to finding an estimator for the price of an option. We present some calculations based on simulations for a European call option. We have also discussed the necessity for a fuzzy approach and present some calculations for this case.

Possible directions for future research include comparison of the predictions based on our model with the real market, estimation of the parameters required in the stochastic process described, as well as the generalization of and problems with the fitting of this model.

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Zastosowanie parametrów rozmytych w wycenie pochodnych instrumentów finansowych związanych z Protokołem z Kioto

Jednym z celów Protokołu z Kioto jest redukcja emisji gazów cieplarnianych. Aby zrealizować to zamierzenie, zaproponowano mechanizm handlu pozwoleniami na emisje. Pozwolenia te mogą być traktowane tak jak inne towary, co zaowocuje powstaniem odpowiedniego rynku, który pozwoli na określenie ich ceny. Co więcej, może się również rozwinąć rynek instrumentów pochodnych, takich jak opcje, kontrakty futures i forward, znanych w odniesieniu do innych dóbr. Dlatego niezbędne jest wykorzystanie odpowiedniej metodologii, w której weźmie się pod uwagę zarówno specyfikę rynku pozwoleń na emisję, jak i charakterystyczne właściwości instrumentów pochodnych. Istotne jest również uwzględnienie niepewności i nieprecyzyjności związanej z opisywanym rynkiem. W artykule proponujemy stochastyczny model o parametrach rozmytych, który może zostać wykorzystany do wyceny instrumentów pochodnych związanych z Protokołem z Kioto. Skupiamy się przede wszystkim na problemie znajdowania ceny derywatyw, a nie na modelowaniu samego rynku. Stosując metodę martyngałową oraz symulacje Monte Carlo, znajdujemy formułę wyceny europejskiej opcji kupna dla pozwolenia na emisję.

Słowa kluczowe: *wycena opcji, pochodne instrumenty finansowe, Protokół z Kioto, metoda martyngałowa, parametry rozmyte*