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SOLUTIONS FOR NETWORK GAMES AND SYMMETRIC GROUP REPRESENTATIONS

We present the relationship between network games and representation theory of the group of permutations of the set of players (nodes), and also offer a different perspective to study solutions for this kind of problems. We then provide several applications of this approach to the cases with three and four players.

Keywords: *network games, axiomatic solutions, representation theory, symmetric group*

1. Introduction

The organisation of individual agents into networks plays an important role in the determination of the outcome of many social and economic interactions. For instance, friendships and social relationships, communicating information about job openings, business partnerships, international trade agreements and political alliances, etc. What is common to these situations is that the way in which players are connected to each other is important in determining the total productivity or value generated by the group.

Myerson [8] makes a contribution in augmenting a cooperative game by a network structure, specifying which groups of players can communicate and achieve their worth. The feasible groups are the ones whose members can communicate via the given network. There exists an extension of the Shapley value [9] to this kind of cooperative games, providing a simple characterisation of it. This allocation rule is called the Myerson value in the subsequent literature (see, e.g., [1]).

In a more general context, Jackson and Wolinsky [7] introduce a class of games – network games – where the value generated by a group of players depends directly on

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the network structure. They extend the Myerson value to network games and study the stability and efficiency of social and economic networks when self-interested individuals can form or sever links. More recently, Jackson [6] takes an axiomatic point of view for solving network games and presents a family of allocation rules that incorporate information about alternative network structures when allocating value.

In this article, we study the solutions for network games that satisfy the elementary properties of linearity and symmetry, for the cases of three and four players. The paper presents the innovative use of basic representation theory of the group of permutations of the set of players (symmetric group) and provides a different perspective from the more traditional ones.

Roughly speaking, representation theory is a general tool for studying abstract algebraic structures by representing their elements as linear transformations of vector spaces. It makes sense to use it since every permutation may be thought of as a linear map[2] and it presents the information in a more clear and concise way. Some examples of the use of representation techniques in the game theory framework are Hernández--Lamoneda et. al. [4] for games in characteristic function form, and Sánchez-Pérez [5] for games in partition function form. By contrast, for a survey of the ways in which the representation theory of the symmetric group is used in voting theory, see Crisman and Orrison [2].

Our primary goal is to show how certain representation theory tools can be used to make sense of foundational ideas in network games, and how using these tools can, in turn, help us to obtain meaningful information concerning linear symmetric solutions in network games. In short, what we do is to compute direct sum decomposition of the network games space (via the space of value functions) and the space of payoffs into elementary pieces. According to this decomposition, any linear symmetric solution, when restricted to any such elementary piece, is either zero or multiplication by a single scalar. Therefore, all linear symmetric solutions may be written as a sum of trivial maps.

With a global description of all linear and symmetric solutions, it is easy to understand the restrictions imposed by other conditions (e.g., the efficiency axiom). We then use such decomposition to provide, in a very economical way, a characterisation for the class of linear symmetric solutions and the class of all linear, symmetric, and efficient solutions.

The paper is organised as follows. We first recall the main basic features of network games and their solutions in the next section. A decomposition for the space of value functions with three players is introduced in Section 3. In the same section, we show an application of this decomposition by giving characterizations of linear symmetric solutions. In Section 4, we discuss a decomposition for a case with four players, and Section 5 concludes the paper. Long proofs are relegated to Appendix.

[2]The precise statement will be provided in Sec. 3.

To finish this introduction, we give a comment on the methods employed in the article. Although it is true that the characterisation results could be proved without any explicit remark on the basic representation theory of the symmetric group, we believe that this algebraic tool sheds new light on the structure of the space of network games and their solutions. A part of the purpose of the present paper is to share this viewpoint with the reader.

To make the paper as self-reliant as possible, we include an Appendix with some facts we need regarding basic representation theory.

2. Framework and definitions

Let $N = \{1, 2, \ldots, n\}$ be a fixed non-empty finite set, and let the members of N be interpreted as players (or nodes) who are connected in some network relationship. A network is a list of pairs of players that are linked to each other and modeled as a non--directed graph[3].

Definition 1. A network g is a set of unordered pairs of players $\{i, j\}$, where $\{i, j\} \in g$ indicates that i and j are linked under the network g.

When there is no place for confusion and for simplicity, we will write just $I = ij$ to to represent the link $\{i, j\}$. In this way, $ij \in g$ indicates that i and j are linked under the network g. More formally, let g^N be the set of all subsets of N of size 2. In other words, g^N will denote the complete network where all the players are linked with each other.

The set of all possible networks or graphs on N will be denoted by $G(N)$:

$$G(N) = \left\{ g \middle| g \subseteq g^N \right\}$$

The network obtained by adding link ij to an existing network g is denoted g + ij and the network obtained by deleting link ij from an existing network g is denoted $g - ij$.

For $g \in G(N)$, let $N(g)$ be the set of players who have at least one link in g. That is, $N(g) = \{i \mid \exists j \text{ s.t. } ij \in g\}$. Let $n(g) = |N(g)|$ be the number of players involved in g.

Let $L_i(g)$ be the set of links that player i is involved in, so that $L_i(g) = \{ij \mid \exists j \text{ s.t. } ij \in g\}$, and let $\ell_i(g) = |L_i(g)|$.

[3]That is, it is not possible for one individual to link to another without having the second individual also linked to the first.

Given any non-empty subset (coalition) $S \subseteq N$, let g^S be the complete network among the players in S, and let

$$g\big|_S = \{ij \,|\, ij \in g \text{ and } i, j \in S\}$$

Thus $g\big|_S$ is the network found deleting all links except those that are between players in S.

Remark 1. Notice the distinction between the notation g^S which is the complete network among players in S, and $g\big|_S$ which is the network found by starting with some g and then eliminating links involving players outside of S.

Definition 2. A path in a network $g \in G(N)$ between players i and j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$.

From the path relationships in a network, it can be naturally partitioned into different connected subgraphs that are commonly referred to as components.

Definition 3. A component of a network g is a non-empty subnetwork $g' \subseteq g$, such that

a) if $i, j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j,

b) if $i \in N(g')$ and $ij \in g$, then $ij \in g'$.

In this way, the components of a network are the distinct connected subgraphs of a network. The set of components of g will be denoted by $C(g)$.

Notice that $g = \cup_{g' \in C(g)} g'$ and under this definition of a component, a completely isolated player who has no links is not considered a component. Also for a given number of players n, we need to define a set A_n which is used in the sequel. We will say that the pair (k, l) belongs to A_n if k is a conceivable number of links and there exist a network g and a player i such that g has k links and i is involved in l links. Formally:

Definition 4. Let A_n be the set defined by

$$A_n = \left\{ (k, l) \,\middle|\, 1 \leq k \leq \binom{n}{2} \text{ and there exists } g \in G(N) \right.$$

$$\left. \text{and } i \in N \text{ s.t. } |g| = k \text{ and } \ell_i(g) = l \right\}$$

For instance, if $n = 5$, it turns out that $(3, 2) \in A_5$ since there exists a network with 3 links and a player involved in 2 (out of 3) links.

Example 1. For the cases $n = 3$ and $n = 4$, we have A_3

$$A_3 = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 2)\}$$

and

$$A_4 = \{(1, 0), (1, 1), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)$$

$$(3, 3), (4, 1), (4, 2), (4, 3), (5, 2), (5, 3), (6, 3)\}$$

It is interesting to notice that the total productivity[4] of a graph and this notion is captured by a value function.

Definition 5. A value function is a mapping

$$\omega : G(N) \to \mathbb{R}$$

such that $\omega(\varnothing) = 0$. The set of all possible value functions is denoted Ω, i.e.,

$$\Omega = \{\omega : G(N) \to \mathbb{R} \big| \omega(\varnothing) = 0$$

The number $\omega(g)$ specifies the total value generated by a given network structure g. The calculation of value may involve both costs and benefits, and is a richer object than a characteristic function of a cooperative game[5], as it allows the value that accrues to depend on the network structure and not only on the coalition of players involved.

Given $\omega_1, \omega_2 \in \Omega$ and $c \in \mathbb{R}$, we define the sum $\omega_1 + \omega_2$ and the product $\lambda\omega_1$, in Ω in the usual form, i.e.,

$$(\omega_1 + \omega_2)(g) = \omega_1(g) + \omega_2(g) \quad \text{and} \quad (\lambda\omega_1)(g) = \lambda\omega_1(g)$$

respectively. It is easy to verify that Ω is a vector space (over $\mathbb{R}$) with these operations.

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[4]By productivity we mean the utility to the society of players involved. For instance, a buyer's expected utility from trade may depend on how many sellers that buyer is negotiating with, on how many other buyers they are connected to, etc. Similarly, a network where players have very few acquaintances with whom they share information will result in different employment patterns from the one where players have many such acquaintances.

[5]Formally, a cooperative game in characteristic function form is defined as a function $v{:}2^N \to \mathbb{R}$ with the property that $v(\varnothing) = 0$. The elements of 2^N are coalitions and if $S \in 2^N$ then $v(S)$ is the worth of coalition S under the cooperative game v.

For subsequent analysis we will use the notation $G^{(n)}(N)$ and $\Omega^{(n)}$ to emphasise over a particular number n of players considered in the set $G(N)$ and on the space Ω, respectively.

An interesting sub-class of value functions are those where the value of a given component of a network does not depend on the structure of other components. This precludes externalities across (but not within) components of a network.

Definition 6. A value function ω is component additive if for any $g \in G(N)$:

$$\sum_{g' \in C(g)} \omega(g') = w(g)$$

Definition 7. A network game is a pair (N, ω), where N is the set of players and ω is a value function.

In order to know how the total productivity of a network (in a network game) is allocated among the individual nodes, we need to define the notion of a solution[6].

Definition 8. A solution is a function

$$\varphi : G(N) \times \Omega \to \mathbb{R}^n$$

Where $\varphi_i(g, \omega)$ is interpreted as the utility payoff which player i should expect from the network game (N, ω) for a fixed network g.

The previous notion of a solution is the one that we will use for the analysis in this article. In the same sense, it is common to find the concept of an allocation rule in the literature.

Definition 9. An allocation rule is a function $\varphi : G(N) \times \Omega \to \mathbb{R}^n$ such that

$$\sum_{i \in N} \varphi_i(g, \omega) = \omega(g) \quad \forall g \text{ and } \forall \omega \tag{1}$$

Remark 2. Notice the difference between the concepts of solution and allocation rule. While a solution is a more general concept, an allocation rule is a more restrictive one: it is a solution that satisfies the condition imposed by (1), which stands for an efficiency-type property.

[6]This is analogous to that on the set of cooperative games $\Gamma(N) = \{v: 2^N \to \mathbb{R} \mid v(\varnothing) = 0\}$. In this context, $\Gamma(N)$ is a vector space (over $\mathbb{R}$) of dimension $2^n - 1$ and a solution is a function φ: $\Gamma(N) \to \mathbb{R}^n$ where it is interpreted as a rule to divide the common gain among the players of N.

Due to the richness of network games, several solutions (allocation rules) have been given for these problems. For example, as mentioned in the introduction, the first paper that proposes a value concept for network problems is Myerson [8]. It is an allocation rule defined in the context of cooperative games with communication structures, that is a variation of the Shapley value. The following presentation of Myerson's value is due to Jackson and Wolinsky [7], which it is an easy extension of the main theorem of Myerson [8]. Such allocation rule satisfies the following axioms.

Axiom 1 (component balance (CB)). An allocation rule φ satisfies component balance if for any component additive ω and $g \in G(N)$, and $g' \in C(g)$

$$\sum_{i\in N(g')} \varphi_i(g,\omega) = \omega(g')$$

Component balance requires that if a value function is component additive, then the value generated by any component is allocated to the players among that component.

Axiom 2 (equal bargaining power (EBP)). An allocation rule φ satisfies equal bargaining power[7] if for any component additive ω and $g \in G(N)$

$$\varphi_i(g,\omega) - \varphi_i(g-ij,\omega) = \varphi_j(g,\omega) - \varphi_j(g-ij,\omega)$$

This axiom does not require that players split the marginal value of a link; instead, it just requires that they equally benefit or suffer from its addition.

Theorem 1. There exists a unique allocation rule ψ^M that satisfies CB and EBP [7]. Moreover, it is given by

$$\psi_i^M(g,\omega) = \sum_{S\subseteq N-1} \frac{|S|!(|N|-|S|-1)!}{|N|!}\left[\omega(g|_{S+i}) - \omega(g|_S)\right] \tag{2}$$

for all $g \in G(N)$ and any component additive ω.

The previous characterization of the Myerson value can be found in Jackson and Wolinsky ([7], Theorem 4), where it is defined as the Shapley value of an specific cooperative game. This particular game is such that the worth of a coalition is the sum over the worth of the subcoalitions which are those which are intraconnected via the network.

7This was called fairness by Myerson [8].

2.1. The basic axioms

For the study of solutions of network games using representation theory techniches, the reasonable requirements that are necessary to impose are the usual linearity and symmetry axioms. These axioms will be a key ingredient in subsequent developments. Next, we define them.

First, the group of permutations of N, $S_n = \{\theta\colon N \to N \,|\, \theta \text{ is bijective}\}$, acts on $G(N)$ (set of networks) as well as on $\mathbb{R}^n$ (space of payoff vectors) in a natural way, i.e.:

- for $g \in G(N)$ and $\theta \in S_n$:

$$\theta(g) = \{\theta(i)\theta(j) \,|\, ij \in g\}$$

- for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\theta \in S_n$:

$$\theta(x_1, x_2, \ldots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})$$

Moreover, the group S_n acts on the space of value functions Ω in the following way: if $\omega \in \Omega$ and $\theta \in S_n$, then

$$[\theta\omega](g) = \omega[\theta^{-1}(g)]$$

Now, the formal restrictions are the ones below.

Axiom 3 (linearity). The solution φ is linear if for every $g \in G(N)$, every $\omega_1, \omega_2 \in \Omega$ and every $c \in \mathbb{R}$:

$$\varphi(g,\ c\omega_2) = c\varphi(g,\ \omega_1) + \varphi(g,\ \omega_2)$$

Axiom 4 (symmetry). The solution φ is said to be symmetric if and only if

$$\varphi(\theta(g),\ \theta\omega) = \theta\varphi(g, \omega)$$

for every $\theta \in S_n$, every $g \in G(N)$ and every $\omega \in \Omega$.

The axiom of linearity means that in the sharing of benefits (or costs) stemming from two different issues, how much each player obtains does not depend on whether they consider the two issues together or one by one. Hence, the agenda does not affect the final outcome. Also, the sharing does not depend on the unit used to measure the benefits.

In addition, the symmetry axiom means that player's payoffs do not depend on their names and it is only derived from his influence on the value of the networks. The axiom requires that if all that has changed is the labels of the players and the value generated by networks has changed in an exactly corresponding fashion, then the allocation changes only according to the relabeling.

Remark 3. It is not difficult to show that the Myerson value is a solution that satisfies the properties of linearity and symmetry.

3. Representations

Precise definitions and some proofs for this section may be found in the Appendix at the end of the article. Nevertheless, for the sake of easier reading we repeat here a few definitions, sometimes in a less rigorous but more accessible manner.

The group S_n acts naturally on the space of value functions Ω via linear transformations (i.e., Ω is a representation of S_n). That is, each permutation $\theta \in S_n$ corresponds to a linear, invertible transformation, which we still call θ, of the vector space Ω; namely

$$[\theta\omega](g) = \omega[\theta^{-1}(g)]$$

for every $\theta \in S_n$, $\omega \in \Omega$ and $g \in G(N)$.

Moreover, this assignment preserves multiplication (i.e., is a group homomorphism) in the sense that the linear map corresponding to the product of the two permutations $\theta_1\theta_2$ is the product (or composition) of the maps corresponding to θ_1 and θ_2, in that order.

Similarly, the space of payoff vectors $\mathbb{R}^n$ is a representation of S_n:

$$\theta(x_1, x_2, ..., x_n) = (x_{\theta(1)}, x_{\theta(2)}, ..., x_{\theta(n)})$$

Definition 10. Let X_1 and X_2 be two representations for the group S_n. A linear map T: $X_1 \to X_2$ is said to be S_n-equivariant if $T(\theta x) = \theta T(x)$, for every $\theta \in S_n$ and every $x \in X_1$.

Remark 4. Notice that what we are calling a linear symmetric solution (for network games), in the language of representation theory means a linear map that is S_n-equivariant.

3.1. Decomposition of $\Omega^{(3)}$

Definition 11. Let Y be a subspace of a vector space X.

- Y is invariant (for the action of S_n) if for every $y \in Y$ and every $\theta \in S_n$, we have

$$\theta y \in Y$$

- Y is irreducible if Y itself has no invariant subspaces other than $\{0\}$ and Y itself.

We begin with the decomposition of $\mathbb{R}^n$ into irreducible representations, which is easier, and then proceed to do the same thing for Ω; that is, we wish to write $\mathbb{R}^n$ as a direct sum of subspaces, each invariant for all permutations in S_n in such way that the summands cannot be further decomposed (i.e., they are irreducible).

For this, let

$$U_n = \{(t, t, \ldots, t) \in \mathbb{R}^n \mid t \in \mathbb{R}\} \quad \text{and} \quad \mathrm{V}_n = U_n^{\perp} = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = 0 \right\}$$

The spaces U_n and V_n are usually called the 'trivial' and 'standard' representations, respectively. Notice that U_n is a trivial subspace in the sense that every permutation acts as the identity transformation.

Every permutation fixes every element of U_n, so, in particular, it is an invariant subspace of $\mathbb{R}^n$. Being 1-dimensional, it is automatically irreducible. Its orthogonal complement V_n consists of all vectors such that the sum of their coordinates is zero. Clearly, if we permute the coordinates of any such vector, their sum will still be zero. Hence, V_n is also an invariant subspace.

Proposition 1. The decomposition of $\mathbb{R}^n$ under S_n, into irreducible subspaces is

$$\mathbb{R}^n = U_n \oplus V_n$$

Proof. First, it is clear that $U_n \cap V_n = \{(0, \ldots, 0)\}$. We now prove that $\mathbb{R}^n = U_n + V_n$:

1. If $z \in (U_n + V_n)$, then $z \in \mathbb{R}^n$ since $(U_n + V_n)$ is a subspace of $\mathbb{R}^n$.

2. For $z \in \mathbb{R}^n$, let $\overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$ and z can be written as

$$z = (\overline{z}, \overline{z}, \ldots, \overline{z}) + (z_1 - \overline{z}, z_2 - \overline{z}, \ldots, z_n - \overline{z})$$

and so, $z \in (U_n + V_n)$.

Now, since U_n is 1-dimensional, then it is irreducible. Finally, checking that V_n is also irreducible is an induction argument that can be found in [4]. In this way, this result tells us that $\mathbb{R}^n$ as a vector space with group of symmetry S_n can be written as an orthogonal sum of the subspaces U_n and V_n, which are invariant under permutations and which can no longer be further decomposed.

The decomposition of Ω is carried out in several steps. First, we establish a partition (into distinct classes) of the set of networks as below.

Definition 12. Let $g_1, g_2 \in G(N)\backslash\{\varnothing\}$. We will say that g_1 and g_2 belong to the same class if $\exists\theta \in S_n$ such that $\theta(g_1) = g_2$.

Let $m_{G(N)} \in \mathbb{N}$ be the number of different classes in which the set $G(N)\backslash\{\varnothing\}$ can be partitioned according to Definition 12. Thus, if $G_j(N)$ denotes the set of networks that belong to the class j, then

$$G(N)\backslash\{\varnothing\} = \bigcup_{j=1}^{m_G} G_j(N)$$

where we can notice that $G_j(N) \cap G_k(N) = \varnothing$ if $j \neq k$.

For further analysis, we will asume that $G_1(N)$ is the class of networks with exactly 1 link and $G_{m_{G(N)}}(N)$ is the class of networks that contains the complete network, i.e.

$$G_1(N) = \{g \in G(N) \,\|\, g| = 1\} \text{ and } G_{m_{G(N)}}(N) = \{g^N\}$$

Now, we turn back to the decomposition of Ω. For each $k \in \{1, \ldots, m_{G(N)}\}$, we define the subspace of value functions

$$\Omega_k = \{\omega \in \Omega \,|\, \omega(g) = 0 \quad \text{if} \quad g \notin G_k(N)\} \tag{3}$$

Then, the space Ω has the following decomposition:

$$\Omega = \bigoplus_{k=1}^{m_{G(N)}} \Omega_k \tag{4}$$

Each subspace Ω_k is invariant under S_n and the decomposition is orthogonal with respect to the invariant inner product on Ω given by

$$\langle \omega_1, \omega_2 \rangle = \sum_{g \in G(N)} \omega_1(g)\omega_2(g) \tag{5}$$

Here, invariance of the inner product means that every permutation $\theta \in S_n$ is not only a linear map on Ω, but an orthogonal map with respect to this inner product. Formally, $\langle \theta\omega_1,\ \theta\omega_2 \rangle = \langle \omega_1,\ \omega_2 \rangle$ for every $\omega_1,\ \omega_2 \in \Omega$.

Example 2. For the set of networks of n = 3 nodes, $G^{(3)}(N)$, it turns out that $m_{G^{(3)}(N)} = 3$ and these classes are given by (Fig. 1)

$$G_1^{(3)}(N) = \{\{12\},\ \{13\},\ \{23\}\}$$

$$G_2^{(3)}(N) = \{\{12,\ 13\},\ \{12,\ 23\},\ \{13,\ 23$$

$$G_3^{(3)}(N) = \{\{12\},\ \{13\},\ \{23\}\}$$

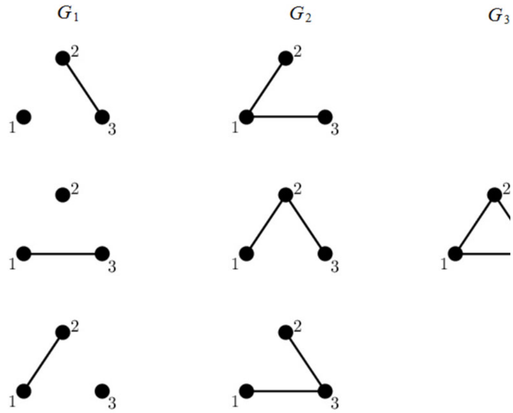

Fig. 1. Partition of $G^{(3)}(N)$

and according to (3), the space of value functions is decomposed as

$$\Omega^{(3)} = \Omega_1^{(3)} \oplus \Omega_2^{(3)} \oplus \Omega_3^{(3)}$$

The next goal is to get a decomposition of each subspace of value functions $\Omega_k^{(3)}$ into irreducible subspaces and so, we will get it for $\Omega^{(3)}$.

The following value functions play an important role in describing the decomposition of the space Ω. For $k \in \{1, \ldots, m_{G(N)}\}$, we define $c_k \in \Omega_k$ as follows:

$$c_k(g) = \begin{cases} 1 & \text{if } g \in G_k(N) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Note that $\Omega_{m_{G(N)}} = \mathbb{R}c_{m_{G(N)}}$.

Also, for each $k \in \{1, \ldots, m_{G(N)}\}$ and for each $z \in \mathbb{R}^n$ we define the value function $z^k \in \Omega_k$ as follows:

$$z_k(g) = \begin{cases} \sum_{ij \in g} (z_i + z_j) & \text{if } g \in G_k(N) \\ 0 & \text{otherwise} \end{cases}$$

Definition 13. Suppose X_1 and X_2 are two representations for the group S_n, i.e., we have two vector spaces X_1 and X_2, where S_n is acting by linear maps. We say that X_1 and X_2 are isomorphic if there is a linear map between them, which is 1–1 and onto and that commutes with the respective S_n-actions. Formally, there is an invertible linear map T: $X_1 \to X_2$ such that $T(\theta T) = \theta T(x)$, for every $\theta \in S_n$ and every $x \in X_1$. We then write $X_1 \approx X_2$.

For our purposes, X_1 will be an irreducible subspace of Ω and X_2 an irreducible subspace of $\mathbb{R}^n$.

Isomorphic representations are essentially "equal"; not only are they spaces of the same dimension, but the actions are equivalent under some linear invertible map between them.

The next proposition provides a decomposition of the space of value functions for $n = 3$ players (nodes) into irreducible subspaces.

Proposition 2. For $k \in \{1, 2\}$

$$\Omega_k^{(3)} = C(3)_k^{(3)} \oplus R_k^{(3)}$$

where $C_k^{(3)} = \langle c_k \rangle \simeq U_3$ and $R_k^{(3)} = \{z^k \mid z \in V_3\} \simeq V_3$. The decomposition is orthogonal.

Proof. See Appendix.

Remark 5. Recall that $\Omega_3^{(3)}$ is a trivial representation generated by the value function that assigns 1 to the complete network and 0 elsewhere.

Whereas from the above proposition, it is not difficult to verify that for $k \in \{1, 2\}$:

$$C_k^{(3)} = \{\omega \in \Omega_k^{(3)} \mid \omega(g_1) = \omega(g_2) \text{ if } |g_1| = |g_2|\}$$

and

$$R_k^{(3)} = \left\{ \omega \in \Omega_k^{(3)} \,\middle|\, \sum_{\{g \in G(N): |g| = k\}} \omega(g) = 0 \right\}$$

Proposition 2 gives a decomposition of the space of games that is a key ingredient in our subsequent analysis.

Set $C^{(3)} = C_1^{(3)} \oplus C_2^{(3)} \oplus C_3^{(3)}$. This is a subspace of value functions whose value on a given network g depends only on the number of links that form such network.

According to Proposition 2, $C^{(3)}$ is the largest subspace of $\Omega^{(3)}$ where S_3 acts trivially[8]. Let $R^{(3)} = R_1^{(3)} \oplus R_2^{(3)}$. Then

$$\Omega^{(3)} = C^{(3)} \oplus R^{(3)}$$

Thus, given a value function $\omega \in \Omega^{(3)}$, we may decompose it relative to the above as $\omega = c + r$, where, in turn, $u = \sum a_k c_k$ and $r = \sum z_k^k$. This decomposition is very well suited to study the image of ω under any linear symmetric solution. The reason being the following version of the well known Schur's lemma[9].

Theorem 2 (Schur's Lemma). Any linear symmetric solution

$$\varphi : G^{(3)}(N) \times \Omega^{(3)} = G^{(3)}(N) \times [C^{(3)} \oplus R^{(3)}] \to \mathbb{R}^3 = U_3 \oplus V_3$$

satisfies

a) $\varphi[G^{(3)}(N) \times C^{(3)}] \subset U_3$

b) $\varphi[G^{(3)}(N) \times R^{(3)}] \subset V_3$

Moreover,

• for each $k \in \{1, 2, 3\}$, there is a constant $\exists \alpha_k \in \mathbb{R}$ such that for every $(g, \omega) \in G^{(3)}(N) \times C_k^{(3)}$ $\varphi(g, \omega) = \alpha_k (1, 1, 1) \in U_3$,

• for each $k \in \{1, 2\}$, there is a constant $\beta_k \in \mathbb{R}$ such that for every $(g, z^k) \in G^{(3)}(N) \times R_k^{(3)}$ $\varphi(g,\ z^k) = \beta_k z \in V_3$.

For many purposes it suffices to use merely the existence of the decomposition of the value function $\omega \in \Omega^{(3)}$, without having to worry about the precise value of each component. Nevertheless, it will be useful to have it. Thus, we give a formula for computing it.

Proposition 3. Let $\omega \in \Omega^{(3)}$. Then

$$\omega = \sum_{k=1}^{3} a_k c_k + \sum_{k=1}^{2} z_k^k \tag{7}$$

where

[8]i.e., $\theta\omega = \omega$ for every $\theta \in S_3$ and every $\omega \in C^{(3)}$.

[9]See Appendix for a precise statement.

1. a_k is the average of the values $\omega(g)$ with $|g| = k$:

$$a_k \frac{\sum_{|g|=k} \omega(g)}{\left|\{g \in G(N) \,|\, |g| = k\}\right|}$$

2. For every $k \in \{1, 2\}$:

$$(z_k)_i = \sum_{\substack{|g|=k \\ \ell_i(g)=k}} k\omega(g) - \sum_{\substack{|g|=k \\ \ell_i(g)\neq k}} (3-k)\omega(g)$$

Proof. See Appendix.

Notice that the value functions appearing in formula (7) form bases for $C^{(3)}$ and $R^{(3)}$, respectively. In other words, $C^{(3)} = \langle c_1, c_2, c_3 \rangle$ and $R^{(3)} = \langle z_1^1, z_2^2 \rangle$.

3.2. Some applications

In this section, we show how to obtain characterisations of solutions easily by using the decomposition of a value function given by (7) in conjunction with Schur's lemma. We start with providing a characterisation of all linear symmetric solutions φ: $G^{(3)}(N) \times \Omega^{(3)} \to \mathbb{R}^3$ in the way given below.

Proposition 4. The linear symmetric solutions φ: $G^{(3)}(N) \times \Omega^{(3)} \to \mathbb{R}^3$ are precisely those of the form

$$\varphi_i(g, \omega) = \sum_{\substack{(k, l)\in A_3 \\ l\neq 0}} \sum_{\substack{|g|=k \\ \ell_i(g)}} \gamma_{(k, l)}\omega(g) + \sum_{(k, 0)\in A_3} \sum_{|g|=k} \gamma_{(k, 0)}\omega(g) \tag{8}$$

for some real numbers $\{\gamma_{(k, l)} \,|(k, l) \in A_3\}$.

Proof. Let φ: $G^{(3)}(N) \times \Omega^{(3)} \to \mathbb{R}^3$ be a linear symmetric solution. According to Proposition 3, $\omega \in \Omega^{(3)}$ decomposes as

$$\omega = \sum_{k=1}^{3} a_k c_k + \sum_{k=1}^{2} z_k^k$$

where by linearity

$$\varphi_i(g,\ \omega) = \sum_{k=1}^{3} a_k \varphi_i(g,\ c_k) + \sum_{k=1}^{2} \varphi_i(g,\ z_k^k)$$

Now, from Schur's lemma and Proposition 3, we have

$$\varphi_i(g,\ \omega) = \sum_{k=1}^{3} \alpha_k a_k + \sum_{k=1}^{2} \beta_k (z_k)_i = \sum_{k=1}^{3} \alpha_k \frac{\sum_{|g|=k}^{3} \omega(g)}{\left| g \in G(N) \,|\, |g| = k\} \right|}$$

$$+ \sum_{k=1}^{2} \beta_k \left[\sum_{\substack{|g|=k \\ \ell_i(g)=k}} k\omega(g) - \sum_{\substack{|g|=k \\ \ell_i(g)\neq k}} (3-k)\omega(g) \right]$$

Finally, the result follows from grouping terms and by setting

$$\gamma_{(1,\ 0)} = \frac{\alpha_1}{3} - 2\beta_1, \quad \gamma_{(1,\ 1)} = \frac{\alpha_1}{3} + \beta_1,\ \gamma_{(2,\ 1)} = \frac{\alpha_2}{3} - \beta_2,$$

$$\gamma_{(2,\ 2)} = \frac{\alpha_2}{3} + 2\beta_2, \quad \text{and} \ \gamma_{(3,\ 2)} = \alpha_3$$

Corollary 1. The space of all linear and symmetric solutions on $G^{(3)}(N) \times \Omega^{(3)}$ has dimension $|A_3| = 5$.

Once we have such a global description of all linear symmetric solutions, we can understand the restrictions imposed by other conditions or axioms. For example, we can consider that if all players decide to form the complete network (there is a link between any pair of players), then the value $\omega(g^N)$ is allocated among all the players. Formally:

Axiom 5 (efficiency). The solution φ is efficient if and only if for every $\omega \in \Omega$:

$$\sum_{i \in N} \varphi_i(g^N, \quad \omega) = \omega(g^N)$$

Notice that any allocation rule satisfies the efficiency axiom since it is the condition (1) restricted to g^N.

From the point of view of representation theory, the efficiency axiom has the following implications.

Proposition 5. Let φ: $G^{(3)}(N) \times \Omega^{(3)} \to \mathbb{R}^3$ be a linear symmetric solution. Then φ is efficient if and only if

1. $\varphi_i(g^N,\ c_k) = 0$ for $k \in \{1, 2\}$,
2. $\varphi_i(g^N,\ c_3) = \frac{1}{3}$.

Proof. First of all, $\left(C_3^{(3)}\right)^{\perp}$ is exactly the subspace of value functions ω where $\omega(g^N) = 0$. Of these, those in $R^{(3)}$ trivially satisfy $\sum_{i\in N} \varphi_i(g^N,\ \omega) = 0$, since (by Schur's lemma) $\varphi(G(N) \times R^{(3)} \subset V_3$.

Thus, efficiency needs only be checked in $C^{(3)}$. Since c_k is fixed by every permutation in S_3, we have

$$\sum_{i\in N} \varphi_i(g^N,\ c_k) = 3\varphi_i(g^N,\ c_k)$$

so, φ is efficient if and only if for $k \in \{1, 2\}$,

$$3\varphi_i(g^N,\ c_k) = c_k(g^N) = 0$$

$$3\varphi_i(g^N,\ c_3) = c_3(g^N) = 1$$

Recall that $C^{(3)}$ is a subspace of function whose value on a given network g depends only on the number of links that form such a network. The next corollary characterizes the solutions on network games with these value functions in terms of linearity, symmetry, and efficiency. It turns out that among all linear symmetric solutions, the egalitarian solution is characterised as the unique efficient solution on $C^{(3)}$. Formally:

Corollary 2. Let φ: $G^{(3)}(N) \times \Omega^{(3)} \rightarrow \mathbb{R}^3$ be a linear, symmetric and efficient solution. Then, for all $\omega \in C^{(3)}$:

$$\varphi_i(g^N,\ \omega) = \frac{\omega(g^N)}{3}$$

In other words, all linear, symmetric and efficient solutions (e.g., Myerson's value) coincide with the egalitarian solution when restricted to these type of games $C^{(3)}$.

Now, another immediate application is to provide a characterisation of all linear, symmetric, and efficient solutions.

Theorem 3. The solution φ: $G^{(3)}(N) \times \Omega^{(3)} \rightarrow \mathbb{R}^3$ satisfies linearity, symmetry, and efficiency axioms if and only if it is of the form

$$\varphi_i(g^N,\ \omega) = \frac{\omega(g^N)}{3} + \sum_{k=1}^{2} \beta_k \left[\sum_{\substack{|g|=k \\ \ell_i(g)=k}} k\omega(g) - \sum_{\substack{|g|=k \\ \ell_i(g)=k}} (3-k)\omega(g) \right] \tag{9}$$

for some real numbers $\{\beta_1, \beta_2\}$.

Proof. Let φ: $G^{(3)}(N) \times \Omega^{(3)} \to \mathbb{R}^3$ be a linear, symmetric, and efficient solution; and $\omega \in \Omega^{(3)}$. Then, by proposition 3, Schur's lemma and Proposition 5:

$$\begin{aligned}\varphi_i(g^N,\quad \omega) &= \sum_{k=1}^{3} a_k \varphi_i(g^N,\ c_k) + \sum_{k=1}^{2} \varphi_i(g^N,\ z_k^k) \\ &= a_3 \varphi_i(g^N,\ c_3) + \sum_{k=1}^{2} \beta_k (z_k)_i \\ &= \frac{\omega(g^N)}{3} + \sum_{k=1}^{2} \beta_k \left[\sum_{\substack{|g|=k \\ \ell_i(g)=k}} k\omega(g) - \sum_{\substack{|g|=k \\ \ell_i(g)=k}} (3-k)\omega(g) \right]\end{aligned}$$

Corollary 3. The space of all linear, symmetric, and efficient solutions of $G^{(3)}(N) \times \Omega^{(3)}$ has dimension $\left|\{\beta_1,\ \beta_2\}\right| = 2.$

Example 3. From expression (9), notice that the solution given by (for player 1):

$$\begin{aligned}\varphi_1(g^N,\ \omega) = \frac{\omega(g^N)}{3} &+ \beta_1[\omega(\{12\}) + \omega(\{13\}) - 2\omega(\{23\})] \\ &+ \beta_2[2\omega(\{12,\ 13\}) - \omega(\{12,\ 23\}) - \omega(\{13,\ 23\})]\end{aligned}$$

is linear, symmetric, and efficient for any choice of the parameters β_1 and β_2.

Example 4. The Myerson value ψ^M is a solution that satisfies the axioms of linearity, symmetry, and efficiency. Thus (for n = 3), ψ^M is of the form (9) and its corresponding parameters are $\beta_1 = 1/6$ and $\beta_2 = 0$.

4. The case n = 4

As we have noticed, all previous applications and results follow from the decomposition of the space of value functions into direct sum of irreducible subspaces. In this part, we provide such decomposition for the particular case of four players.

In the case of three players, the set $G^{(3)}(N)$ was partitioned into 3 classes (the jth class contains networks with exactly j links, for $j \in \{1, 2, 3\}$). However, the partition of $G^{(4)}(N)$ does not follow the same line of reasoning. The next example shows that there are networks with the same number of links, however they belong to different classes (recall Definition 12).

Example 5. Let $N = \{1, 2, 3, 4\}$.

• The networks $g_1 = \{12, 13, 24\}$ and $g_2 = \{12, 24, 34\}$ belong to the same class, since there is a permutation $\theta \in S_4$ such that $\theta(g_1) = g_2$. Such a permutation is given by $\theta(1) = 2$, $\theta(2) = 4$, $\theta(3) = 1$ and $\theta(4) = 3$.

• The networks $g_1 = \{24, 34\}$ and $g_2 = \{12, 34\}$ do not belong to the same class, since $\nexists \theta \in S_4$ such that $\theta(g_1) = g_2$.

Notice that $\left|G^{(4)}(N) \backslash \{\varnothing\}\right| = 63$ and, according to Definition 12, $m_{G^{(4)}(N)} = 10$ classes. The following networks are representatives of each class.

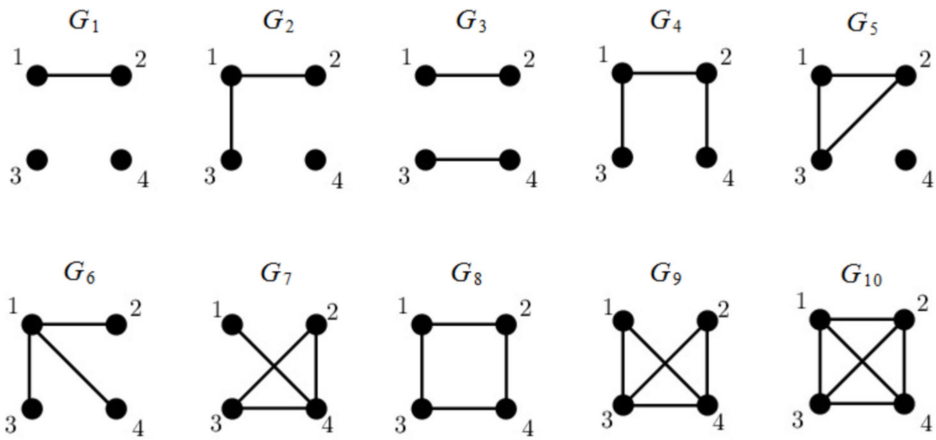

Fig. 2. Representatives of classes in the partition of ( )

The number of networks belonging to each class is shown below.

k	1	2	3	4	5	6	7	8	9	10
$\left\|G_k^{(4)}(N)\right\|$	6	12	3	12	4	4	12	3	6	1

We follow the same line of reasoning as before, i.e., we first obtain a decomposition of each subspace of value functions $\Omega_k^{(4)}$ into irreducible subspaces, and so we will get it for $\Omega^{(4)}$.

For that purpose, let $z \in V_4$ and for $k \in \{1, 4, 5, 6, 9\}$ define the value functions $z_k \in \Omega_k^{(4)}$ as

$$z^k(g) = \begin{cases} \sum_{ij\in g} (z_i + z_j) & \text{if } g \in G_k^{(4)}(N) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

also define $z^2, z^{2'} \in \Omega_2^{(4)}$ and $z^7, z^{7'} \in \Omega_7^{(4)}$ as

$$z^2(g) = \begin{cases} \sum_{ij\in g} (z_i + z_j) & \text{if } g \in G_2^{(4)}(N) \text{ and } \ell_1(g) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$z^{2'}(g) = \begin{cases} \sum_{ij\in g} (z_i + z_j) & \text{if } g \in G_2^{(4)}(N) \text{ and } \ell_1(g) \neq 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

$$z^7(g) = \begin{cases} \sum_{ij\in g} (z_i + z_j) & \text{if } g \in G_7^{(4)}(N) \text{ and } \ell_1(g) = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$z^{7'}(g) = \begin{cases} \sum_{ij\in g} (z_i + z_j) & \text{if } g \in G_7^{(4)}(N) \text{ and } \ell_1(g) \neq 2 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

The nature of the previous value functions will be justified in the decomposition of $\Omega^{(4)}$, presented in the following:

Proposition 6. For $k \in \{1, \ldots, 9\}$, the decomposition of each $\Omega_k^{(4)}$ (under S_4) into irreducible subspaces is

$$\Omega_k^{(4)} = C_k^{(4)} \oplus R_k^{(4)} \oplus T_k^{(4)}$$

where

- $C_k^{(4)} = \langle c_k \rangle \simeq U_4$ if $k \in \{1, \ldots, 9\}$,

- $R_k^{(4)} = \begin{cases} \{z^k \mid z \in V_4\} & \text{if} \quad k \in \{1,\ 4,\ 5,\ 6,\ 9\} \\ \{z^k \mid z \in V_4\} \cup \{z^{k'} \mid z \in V_4\} & \text{if} \quad k \in \{2,\ 7\} \end{cases}$

- $T_k^{(4)} = \left(C_k^{(4)} \oplus R_k^{(4)}\right)^{\perp}$ does not contain any summands isomorphic to neither U_4 nor V_4.

The decomposition is orthogonal.

Proof. See Appendix.

It is not difficult to verify that $\Omega_{10}^{(4)} = \langle c_{10} \rangle \simeq U_4$ is a trivial representation generated by the value function that assigns 1 to the complete network and 0 elsewhere.

On the other hand, from the above proposition, it turns out that for $k \in \{1, \ldots, 9\}$:

$$C_k^{(4)} = \{\omega \in \Omega_k^{(4)} \mid \omega(g_1) = \omega(g_2) \text{ if } \quad g_1,\ g_2 \in G_k^{(4)}(N)\}$$

Remark 6. Proposition 6 does not quite give a decomposition of $\Omega_k^{(4)}$ into irreducible summands. The subspace $C_k^{(4)}$ is irreducible and $R_k^{(4)}$ is a direct sum of irreducible subspaces. Whereas $T_k^{(4)}$ may or may not be irreducible (depending on k), but, as we shall see, the exact nature of this subspace plays no role in the study of linear symmetric solutions since it lies in the kernel of any solution of this kind.

Set $C^{(4)} = \bigoplus_{k=1}^{10} C_k^{(4)}$. This is a subspace of value functions whose value on a given network g depends only on the 'shape' of such network[10]. Let $R^{(4)} = \bigoplus_{k \in \{1,\ 2,\ 4,\ 5,\ 6,\ 7,\ 9\}} R_k^{(4)}$ and $T^{(4)} = \bigoplus_{k=1}^{10} T_k^{(4)}$. Then,

$$\Omega^{(4)} = C^{(4)} \oplus R^{(4)} \oplus T^{(4)}$$

Corollary 4. If $\varphi : G^{(4)}(N) \times \Omega^{(4)} \to \mathbb{R}^4$ is a linear symmetric solution, then for every $(g,\ \omega) \in G^{(4)}(N) \times T^{(4)}$:

$$\varphi(g,\ \omega) = 0$$

Proof. Let $\varphi : G^{(4)}(N) \times \Omega^{(4)} = G^{(4)}(N) \times [C^{(4)} \oplus R^{(4)} \oplus T^{(4)}] \to \mathbb{R}^4 = U_4 \oplus V_4$ be a linear symmetric solution. Assume $X \subset T^{(4)}$ is an irreducible summand in the decomposition

[10]Following Jackson and Wolinsky [7], the value functions in $C^{(4)}$ are known as anonymous.

of $T^{(4)}$ (even while we do not know the decomposition of $T^{(4)}$ as a sum of irreducible subspaces, it is known that such a decomposition exists). Let p_1 and p_2 denote orthogonal projection of $\mathbb{R}^4$ onto U_4 and V_4, respectively. Now, $\varphi : G^{(4)}(N) \times \Omega^{(4)} \to \mathbb{R}^4 = U_4 \oplus V_4$ may be written as $\varphi = (p_1 \circ \varphi, p_2 \circ \varphi)$. Denote by $\iota: X \to G^{(4)}(N) \times \Omega^{(4)}$ the inclusion, then, the restriction of φ to X may be expressed as $\varphi_{|X} = \varphi \circ \iota = (p_1 \circ \varphi \circ \iota,\ p_2 \circ \varphi \circ \iota)$.

On the other hand, $p_1 \circ \varphi \circ \iota :\ X \to U_4$ and $p_2 \circ \varphi \circ \iota : X \to V_4$ are linear symmetric maps; since X is not isomorphic to either of these two spaces, thus Schur's lemma (see Appendix for the statement) implies that $p_1 \circ \varphi \circ \iota$ and $p_2 \circ \varphi \circ \iota$ must be zero. Since this is true for every irreducible summand X of $T^{(4)}$, φ is zero on all of $T^{(4)}$.

Remark 7. According to Proposition 6 and Corollary 4, in order to study linear symmetric solutions, one needs to look only at those value functions inside $C^{(4)} \oplus R^{(4)}$ (i.e., one has to take care of those copies of U_4 and V_4 contained in $\Omega^{(4)}$).

Example 6. From Proposition 6, we know the number of copies of U_4 (trivial representation) and V_4 (standard representation), inside of each $\Omega_k^{(4)}$:

$\Omega_k^{(4)}$	$\Omega_1^{(4)}$	$\Omega_2^{(4)}$	$\Omega_3^{(4)}$	$\Omega_4^{(4)}$	$\Omega_5^{(4)}$	$\Omega_6^{(4)}$	$\Omega_7^{(4)}$	$\Omega_8^{(4)}$	$\Omega_9^{(4)}$	$\Omega_{10}^{(4)}$
# of copies of U_4	1	1	1	1	1	1	1	1	1	1
# of copies of V_4	1	2	0	1	1	1	2	0	1	0

As we have already pointed out, in the case of four players we can also obtain characterisations for the class of linear and symmetric solutions, as well as for the class of linear, symmetric, and efficient solutions. Once again, the key is the decomposition of $\Omega^{(4)}$ into irreducible subspaces (Proposition 6), together with Schur's lemma.

5. Concluding remarks

We have noticed that the point of view we adopt in this article depends heavily on a decomposition of the space of value functions as a direct sum of "special" subspaces. In the cases when $n = 3, 4$, it was decomposed as a direct sum of three orthogonal subspaces: a subspace of anonymous value functions, another subspace which we call $R^{(n)}$, and a subspace $T^{(n)}$ (wich is zero for the case of $n = 3$ nodes) that plays only the role of the common kernel of every linear symmetric solution. Although $R^{(n)}$ does not have a natural definition in terms of well-known network theoretical considerations, it has a simple characterisation in terms of vectors all of whose entries add up to zero.

Characterisations of solutions follow from such decomposition in an very economical way. So, an open challenge is to obtain the general decomposition for $\Omega^{(n)}$ into direct sum of irreducible subspaces since, mathematically, the general case seems to have a much more complicated structure.

Although it is true that the characterisation results could be proved without any explicit mention to the representation theory of the symmetric group, we feel that by doing that we would be withholding valuable information from the reader. This algebraic tool, we believe, sheds new light on the structure of the space of network games and their solutions. A part of the purpose of the present paper is to share this viewpoint with the reader. There is, however, much more work that could be done (e.g., extend the theory for any number of players), and we ecourage interested readers to consider how they might use these and other ideas to contribute to the understanding of network games and their solutions.

Appendix

A reference for basic representation theory is Fulton and Harris [3]. Nevertheless, we recall all basic facts that we need.

The symmetric group S_n acts on Ω via linear transformations (i.e., Ω is a representation of S_n). That is, there is a group homomorphism ρ: $S_n \to GL(\Omega)$, where $GL(\Omega)$ is the group of invertible linear maps in Ω. This action is given by

$$(\theta\omega)(g) := [\rho(\theta)(\omega)](g) = \omega\theta^{-1}(g)]$$

for every $\theta \in S_n$, $\omega \in \Omega$ and $g \in G(N)$.

Definition 14. Let H be an arbitrary group. A representation for H is a homomorphism ρ: $H \to GL(X)$, where X is a vector space and $GL(X) = \{T\colon X \to X \mid T \text{ linear and invertible}\}$. In other words, a representation of H is a map assigning to each element $h \in H$ a linear map $\rho(h)$: $X \to X$ that respects multiplication

$$\rho(h_1h_2) = \rho(h_1)\rho(h_2)$$

for all $h_1, h_2 \in H$.

One usually abuses notation and talks about the representation X without explicitly mentioning the homomorphism ρ. Thus, when applying the linear transformation corresponding to $h \in H$ on the element $x \in X$, we write hx rather than $(\rho(h))(x)$.

The space of payoff vectors $\mathbb{R}^n$ is also a S_n-representation

$$\theta(x_1, x_2, ..., x_n) := [\rho(\theta)](x_1, x_2, ..., x_n) = (x_{\theta(1)}, x_{\theta(2)}, ..., x_{\theta(n)})$$

Definition 15. Let X_1 and X_2 be two representations for the group H.

• A linear map $T: X_1 \to X_2$ is said to be H-equivariant if $T(hx) = hT(x)$ for every $h \in H$ and every $x \in X_1$.

• X_1 and X_2 are said to be isomorphic H-representations, $X_1 \simeq X_2$, if there exists an H-equivariant isomorphism between them.

Thus, two representations that are isomorphic are, as far as all problems dealing with linear algebra with a group of symmetries, the same. They are vector spaces of the same dimension where the actions are seen to correspond under a linear isomorphism.

Definition 16. A representation X is irreducible if it does not contain a nontrivial invariant subspace. That is, if $Y \subset X$ is also a representation for H (meaning that $hy \in Y$ $\forall h \in H$), then Y is either $\{0\}$ or all of X.

Proposition 7. For any representation X of a finite group H, there is a decomposition

$$X = X_1^{\oplus a_1} \oplus X_2^{\oplus a_2} \oplus \cdots \oplus X_j^{\oplus a_j}$$

where the X_i are distinct irreducible representations. The decomposition is unique, as are the X_i that occur and their multiplicities a_i.

This property is called ‘complete reducibility’ and the extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique is one of the consequences of the following:

Theorem 4 (Schur’s lemma). Let X_1, X_2 be irreducible representations of a group H. If $T: X_1 \to X_2$ is H-equivariant, then $T = 0$ or T is an isomorphism.

Moreover, if X_1 and X_2 are complex vector spaces, then T is unique up to multiplication by a scalar $\lambda \in \mathbb{C}$.

The previous theorem is one of the reasons why it is worth carrying around the group action when there is one. Its simplicity hides the fact that it is a very powerful tool.

Following Fulton and Harris [3], the only three irreducible representations of S_3 are the trivial U_3, the standard V_3, and alternating representation[11] U'. Then, for an arbitrary representation X of S_3, we can write

$$X = U_3^{\oplus a} \oplus U'^{\oplus b} \oplus V_3^{\oplus c} \tag{13}$$

[11]Here, the action is given by $\theta x = \mathrm{sgn}(\theta)x$, for $\theta \in S_3$ and $x \in \mathbb{R}$.

and there is a way to determine the multiplicities a, b and c; in terms of $\tau = (123)$ and $\sigma = (12)$, which generates S_3. c, for example, is the number of independent eigenvectors for τ with eigenvalue ε[12] whereas $a + c$ is the multiplicity of 1 as an eigenvalue of σ, and $b + c$ is the multiplicity of -1 as an eigenvalue of σ.

Proof of Proposition 2. We start with showing that $\Omega_k^{(3)}$ has exactly 1 copy of U_3 and 1 copy of V_3, if $k \in \{1, 2\}$.

It is clear that $\mathfrak{B} = \{\omega_{\tilde{g}} \mid \tilde{g} \in G(N) \setminus \{\varnothing\}\}$ form a basis for $\Omega^{(3)}$, where

$$\omega_{\tilde{g}}(g) = \begin{cases} 1 \text{ if } g = \tilde{g} \\ 0 \text{ otherwise} \end{cases} \tag{14}$$

For $\Omega^{(3)}$, it is easy to verify that $[\tau]_{\mathfrak{B}}$ has the characteristic polynomial

$$p(x) = [(x-1)(x-\varepsilon)(x-\varepsilon^2)]^2(x-1)$$

and $[\sigma]_{\mathfrak{B}}$ has the characteristic polynomial

$$p(x) = (x+1)^2(x-1)^5$$

From these and (13), we have that $c = 2$, $a + c = 5$ and $b + c = 2$. Then,

$$\Omega^{(3)} = U_3^{\oplus 3} \oplus V_3^{\oplus 2}$$

This implies directly that if $k \in \{1, 2\}$, then every Ω_k has exactly 1 copy of U_3 and 1 copy of V_3, since $\Omega_{m_{G(N)}} = \mathbb{R}c_{m_{G(N)}} \simeq U_3$ and $\dim \Omega_k = 3$.

Now, we define the map $T^k \colon \mathbb{R}^n \to \Omega_k$ by $T^k(z) = z^k$. This map is an isomorphism between $C_k^{(3)}$ and U_3 (similarly, between $R_k^{(3)}$ and V_3) since it is linear, S_3-equivariant and 1–1. From Proposition 1 we have the splitting $\mathbb{R}^3 = U_3 \oplus V_3$. Thus, inside Ω_k, we have the images of these two subspaces: $C_k^{(3)} = T^k(U_3)$ and $R_k^{(3)} = T^k(V_3)$.

Finally, the invariant inner product $\langle , \rangle$ gives an equivariant isomorphism, in particular, it must preserve the decomposition. This implies orthogonality of the decomposition.

---

[12]Denoting by 1, ε, ε^2 the cube roots of unity.

Proof of Proposition 3. We start with computing the orthogonal projection of ω onto $C^{(3)}$. Notice that $\{c_k\}$ is an orthogonal basis for $C^{(3)}$, and that

$$\|c_k\|^2 = \left|\{g \in G^{(3)}(N) \,|\, |g| = k\}\right|$$

Thus, the projection of ω onto $C^{(3)}$ is $\sum_{k=1}^{3} \frac{\langle \omega,\, c_k \rangle}{\langle c_k,\, c_k \rangle} c_k$,

and so

$$a_k = \frac{\langle \omega,\, c_k \rangle}{\langle c_k,\, c_k \rangle} = \frac{\sum_{|g|=k} \omega(g)}{\left|\{g \in G^{(3)}(N) \,|\, |g| = k\}\right|}$$

Now, for each $k \in \{1, 2, 3\}$, we define h^k: $\Omega^{(3)} \to \mathbb{R}^{(3)}$ as

$$h_i^k(\omega) = \sum_{\substack{|g|=k \\ \ell_i(g)=k}} \omega\,(g)$$

where each h^k is S_3-equivariant, and observe that $h^3(\omega) = \omega(g^N)(1, 1, 1)$. Let $z \in V_3$, then $h^k(z^l) = 0$ if $k \neq l$, whereas (by Schur's lemma) for $k \in \{1, 2, 3\}$ $\exists \lambda_k \in \mathbb{R}$ such that $h^k(z^k) = \lambda_k z$.

Let p: $\mathbb{R}^3 \to V_3$ be the projection of $\mathbb{R}^3$ onto V_3 given by

$$p_i(x) = x_i - \overline{x}$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. This projection is equivariant, sends U_3 to zero and it is the identity on V_3.

Since $(p \circ h^k)(\omega) = \lambda_k z_k$, then $z_k = \frac{1}{\lambda_k} p(h^k(\omega))$. Thus, we evaluate

$$p(h^k(\omega)) = \lambda_k z_k = \sum_{\substack{|g|=k \\ \ell_i(g)=k}} k\omega(g) - \sum_{\substack{|g|=k \\ \ell_i(g)\neq k}} (3-k)\omega(g)$$

There is a remarkably effective technique for decomposing any given finite dimensional representation into its irreducible components. The tool is character theory. In the analysis of the representations of S_3, the key was to study the eigenvalues of the actions of individual elements of S_3. This is the starting point of character theory.

Finding individual eigenvalues, however, is difficult. Luckily, it is sufficient to consider their sum, the trace, which is much easier to compute.

Definition 17. Let ρ: $H \to GL(X)$ be a representation. The character of X is the complex-valued function χ_X: $H \to \mathbb{C}$, defined as

$$\chi_X(h) = Tr(\rho(h))$$

The character of a representation is easy to compute. If H acts on an n-dimensional space X, we write each element h as an $n \times n$ matrix according to its action expressed in some convenient basis, then sum up the diagonal elements of the matrix for h to get $\chi_X(h)$. For example, the trace of the identity map of an n-dimensional vector space is the trace of the $n \times n$ identity matrix, or n. In fact, $\chi_X(e) = \dim X$ for any finite dimensional representation X of any group.

Notice that, in particular, we have $\chi_X(h) = \chi_X(ghg^{-1})$ for g, $h \in H$, so that χ_X is constant on the conjugacy classes of H; such a function is called a class function.

Definition 18. Let $C_{\text{class}}(H) = \{f\colon H \to C \mid f \text{ is a class function on } H\}$. If $\chi_1, \chi_2 \in \mathbb{C}_{\text{class}}(H)$, we define an Hermitian inner product on $\mathbb{C}_{\text{class}}(H)$ by

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|H|} \sum_{h \in H} \overline{\chi_1(h)} \chi_2(h) \tag{15}$$

The character of a representation of a group H is really a function on the set of conjugacy classes in H. This suggests expressing the basic information about the irreducible representations of a group H in the form of a character table. This is a table with the conjugacy classes $[h]$ of H listed across the top, usually given by a representative h, with the number of elements in each conjugacy class over it; the irreducible representations of H listed on the left and, in the appropriate box, the value of the character on the conjugacy class $[h]$. For example, if $H = S_4$ and we only focus on the irreducible representations U_4 and V_4, then[13]:

	1	6	8	6	3
S_4	$[e]$	$[(12)]$	$[(123)]$	$[(1234)]$	$[(12)(34)]$
U_4	1	1	1	1	1
V_4	3	1	0	–1	–1

[13]In fact, there are five irreducible representations for S_4.

Finally, the multiplicities of irreducible subspaces in a representation can be calculated via:

Proposition 8. If $Z = Z_1^{\oplus a_1} \oplus Z_2^{\oplus a_2} \oplus \cdots \oplus Z_j^{\oplus a_j}$, then the multiplicity Z_i (irreducible representation) in Z, is:

$$a_i = \left\langle \chi_z,\ \chi_{z_i} \right\rangle$$

Where $\langle , \rangle$ is the inner product given by (15).

Proof of Proposition 6. First, $\left\langle \chi_{\Omega_k^{(4)}},\ \chi_{U_4} \right\rangle$ and $\left\langle \chi_{\Omega_k^{(4)}},\ \chi_{V_4} \right\rangle$ are the number of subspaces isomorphic to the trivial (U_4) and standard representation (V_4) within $\Omega_k^{(4)}$, respectively. The characters for each $\Omega_k^{(4)}$ are given by[14]:

	1	6	8	6	3
S_4	[(1)]	[(12)]	[(123)]	[(1234)]	[(12)(34)]
$\Omega_1^{(4)},\ \Omega_9^{(4)}$	6	2	0	0	2
$\Omega_2^{(4)},\ \Omega_7^{(4)}$	12	2	0	0	0
$\Omega_3^{(4)},\ \Omega_8^{(4)}$	3	1	0	1	3
$\Omega_4^{(4)}$	12	0	0	0	4
$\Omega_5^{(4)},\ \Omega_6^{(4)}$	4	2	1	0	0
$\Omega_{10}^{(4)}$	1	1	1	1	1

Thus from (15) $\left\langle \chi_{\Omega_k^{(4)}},\ \chi_{U_4} \right\rangle = 1$ for each $k \in \{1, \ldots, 10\}$ and

$$\left\langle \chi_{\Omega_1^{(4)}},\ \chi_{V_4} \right\rangle = \begin{cases} 1 \text{ if } k \in \{1,\ 4, 5, 6, 9\} \\ 2 \text{ if } k \in \{2, 7\} \\ 0 \text{ if } k \in \{3,\ 8, 10\} \end{cases}$$

The last part is to identify such copies of U_4 and V_4 inside $\Omega_k^{(4)}$. To this end, for $k \in \{1, \ldots, 10\}$ let $f_k : U_4 \to C_k^{(4)}$ be given by $f_k(u) = \omega_k$ in which there exists $t \in \mathbb{R}$ such that $u = t(1, 1, 1, 1)$ and $\omega_k(g) = t$ if $g \in G_k(N)$ and $\omega_k(g) = 0$ otherwise. The function f_k

[14]In which a convenient basis is the one given in (14).

is an isomorphism between U_4 and $C_k^{(4)}$ since it is linear, S_4-equivariant, and one to one. Thus, $\Omega_k^{(4)}$ contains the image of this subspace: $C_k^{(4)} = f_k(U_4)$.

Now, for $k \in \{1, 4, 5, 6, 9\}$ define the functions $L_k : V^4 \to \Omega_k^{(4)}$ by $L_k(z) = z^k$ (given by (10)). These maps are isomorphisms between $R_k^{(4)}$ and V_4, and $R_k^{(4)} = L_k(V_4)$.

In the same way, for $k \in \{2, 7\}$ define the functions L_k, $L_{k'}$: $V^4 \to \Omega_k^{(4)}$ by $L_k(z) = z^k$ and $L_{k'}(z) = z^{k'}$ (given by (11) and (12)), respectively. Thus, $R_k^{(4)} = L_k(V_4) \cup L_{k'}(V_4)$.

Orthogonality of the decomposition follows again from the fact that the invariant inner product $\langle , \rangle$ gives an equivariant isomorphism, which preserves the decomposition.

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References

[1] AUMANN R., MYERSON R., *Endogenous formation of links between players and coalitions: an application of the Shapley value*, [In:] A. Roth (Ed.), *The Shapley Value*, Cambridge University Press, 1988, 175–191.

[2] CRISMAN K., ORRISON M., *Representation theory of the symmetric group in voting theory and game theory*, Cont. Math., 2017, 685, 97–116.

[3] FULTON W., HARRIS J., *Representation Theory. A first course*, Springer-Verlag Graduate Texts in Mathematics, New York 1991, 129.

[4] HERNÁNDEZ-LAMONEDA L., JUÁREZ R., SÁNCHEZ-SÁNCHEZ F., *Dissection of solutions in cooperative game theory using representation techniques*, Int. J. Game Theory, 2007, 35 (3), 395–426.

[5] SÁNCHEZ-PÉREZ J., *A decomposition for the space of games with externalities*, Int. J. Game Theory, 2017, 46 (1), 205–233.

[6] JACKSON M., *Allocation rules for network games*, Games Econ. Behav., 2005, 51, 128–154.

[7] JACKSON M., WOLINSKY A., *A strategic model of social and economic networks*, J. Econ. Theory, 1996, 71, 44–74.

[8] MYERSON R., *Graphs and cooperation in games*, Math. Oper. Res., 1977, 2 (3), 225–229.

[9] SHAPLEY L., *A value for n-person games*, [In:] A.W. Tucker, R.D. Luce (Eds.), *Contribution to the Theory of Games*, Ann. Math. Studies, Princeton University Press, Princeton 1953, 307–317.

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