O P E R A T I O N S   R E S E A R C H   A N D   D E C I S I O N S

No. 2 2019
DOI: 10.5277/ord190202

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COST ANALYSIS OF SERIES SYSTEMS WITH DIFFERENT STANDBY COMPONENTS AND IMPERFECT COVERAGE

The authors calculate the steady-state availability and the cost-benefit analysis for three different systems with mixed standby (cold standby, warm standby) and imperfect coverage. The coverage factor is the same for an operative-unit failure as that for a warm standby-unit failure. The failure times of the operative unit and the warm standby unit are exponentially distributed while the repair time is arbitrarily distributed. The supplementary variable technique is applied to derive the steady-state availability for three different configurations. For each system, the steady-state availability is calculated according to two different cases for repair time distributions, such as exponential, and k-stage Erlang, where $k = 2, 3$. The configurations are compared as based on availability and cost/benefit at a special numerical value given to the distribution parameters.

Keywords: *availability, cost-usefulness, imperfect coverage, mixed standby*

1. Introduction

Reliability theory is a substantial concept at the planning, design and operation stages of several complicated systems. But, in fact, we deal with a number of complex systems consisting of one or more parts, and a failure of any of the parts results in the decrease of competence of whole systems, and, as a result of it, the reliability of the system decreases. Therefore, the preferable maintenance of such parts produce the best reliability and then only we can realize the market's needs of reliability, effectiveness, price, and performance of that system. On the other hand, it may not be economical to

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always obtain a higher order of reliability through the procedure preventive maintenance, thus the maintenance and repair at the appropriate time may lead to a high degree of availability. Regardless of these, in the literature, many attempts have been made by the researchers to analyze the reliability of the system using various approaches [3–8, 10, 11, 14–17] studied the cost-benefit analysis for three different configurations containing warm standby units with general repair times. Kuo et al. [11, 12] calculated the reliability, the steady-state availability, and the cost/benefit analysis of four different systems with mixed standby components. Wang et al. [17] deduced the optimal system when they examine four different systems with warm standby unit and the switching is imperfect. El-Sherbeny [1, 2] studied the cost function in the presence of mixed standby components. El-Said and El-Sherbeny [9] investigated the impact of preventive maintenance on two different systems, containing two operative units.

The article is devoted to searching for the optimal system from the three studied taking into consideration the existence of mixed standby units and imperfect coverage. This article is based on three main axes The first axis is to present a recursive method, using the supplementary variable technique and treating the supplementary variable as the remaining repair time, to develop the steady-state availability (Av_i) for availability model i, where $i = 1, 2, 3$. The second axis is the explicit expressions for the Av_i for two different repair time distributions such as exponential (M), and k-stage Erlang (E_k), where $k = 2, 3$. The third axis is to compare the three configurations with their cost/benefit ratio as based on assumed numerical values given to the system parameters.

2. Description of the system

The present paper is devoted to considering the requirements of a 10 MW power plant. We assume that generators are available in units of 10 MW and 5 MW. Standby generators are always necessary in case of failure. We also assume that the switchover time from warm standby unit to the operating unit, from cold standby unit to warm standby unit, from failure to repair, or from repair to cold standby unit (or operating unit if the system is short) is instantaneous. Operating units and warm standby units can be considered repairable. Each of the operating units fails independently of the state of the others and has an exponential time-to-failure distribution with parameter λ.

Whenever one operating unit fails, a warm standby moves into operation if any is available, and a cold standby is put on warm standby state if any is available. We now assume that when a warm standby moves into an operating unit state, its failure characteristics will be that of an operating unit, and when a cold standby moves into a warm standby state, its failure characteristics will be that of a warm standby. We also assume that each of the available warm standby units fails independently of the state of all the others and has an exponential time-to-failure distribution with parameter α $\left(0 \le \alpha \le \lambda\right)$.

When an operating unit (or warm standby unit) fails, it may be immediately detected, located, and replaced with a coverage probability c by a standby if one is available. It is assumed that the replacing time is instantaneous. We further assume that the coverage factor is the same for an operating-unit failure as that for a standby-unit failure and is denoted by c. However, we define the unsafe failure state of the system as anyone of the breakdowns is not covered. We continue with the assumption that operating unit failure (or warm standby unit failure) in the unsafe failure state is cleared by a reboot. Reboot delay takes place at the rate β for an operative unit (or warm standby unit) which is exponentially distributed. The system fails when the standby units are emptied which we define as the state of safe failure.

It is assumed that the times to repair of the units are independent and identically distributed (i.i.d.) random variables having a distribution $B(u)\,(u \geq 0)$, a probability density function $b(u)(u \geq 0)$ and mean repair time b_1. If one unit is in repair, then arriving failed units have to wait in the queue until the server is available. Let us assume that failed units arriving at the server form a single waiting line and are served in the order of their arrivals. Suppose that the server can serve only one operating unit (or warm standby unit) at a time and that the service is independent of the arrival of the units. Once a unit is repaired, it is as good as new.

The following configurations are considered. The first configuration is a serial system of one operative 10 MW unit, one warm standby 10 MW unit, and one cold standby 10 MW unit. The second configuration is a serial system of two operative 5 MW units, one warm standby 5 MW unit, and one cold standby 5 MW unit. The last configuration is a serial system of one operative 10 MW unit, two warm standby 10 MW units, and one cold standby 10 MW unit.

Table 1. The size-proportional cost for the primary, warm standby and cold standby components

Component	Operative		Warm standby		Cold standby	
	10 MW	5 MW	10 MW	5 MW	10 MW	5 MW
Cost [$]	10×10^6	5×10^6	6×10^6	3×10^6	4×10^6	2×10^6

Table 2. The costs for each configuration i (i = 1, 2, 3)

Configuration	1	2	3
Cost [$]	2×10^6	15×10^6	24×10^6

Cost-benefit factor. We consider the size-proportional costs for the operative, warm standby and cold standby components given in Table 1. Thus we calculate the costs for each configuration $i\,(i = 1, 2, 3)$ shown in Table 2. Let C_i denote the cost of the configuration i, and B_i be the benefit of the configuration i, where B_i is Av_i.

3. Availability analysis of the configurations

We use the following supplementary variable: $U \equiv$ remaining repair time for the component under repair. The state of the system at time t is given by $N(t) \equiv$ a number of working units in the system, and $U(t)$. Let us define

$$P_n(u,t)\,du = P\left\{N(t)=n,\ u<U(t)\le u+du\right\},\ u\ge 0$$

$$P_n(t)=\int_0^\infty P_n(u,t)\,du$$

3.1. Availability for configuration 1

Relating the state of the system at time t and $t+dt$, we obtain

$$\frac{d}{dt}P_3(t)=-(\lambda+\alpha)P_3(t)+P_2(0,t) \tag{1}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_2(u,t)=-(\lambda+\alpha)P_2(u,t)+c(\lambda+\alpha)P_3(u,t)$$
$$+\beta P_{uf_1}(u,t)+b(u)P_1(0,t) \tag{2}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_1(u,t)=-\lambda P_1(u,t)+c(\lambda+\alpha)P_2(u,t)$$
$$+\beta P_{uf_2}(u,t)+b(u)P_0(0,t) \tag{3}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_0(u,t)=\lambda P_1(u,t) \tag{4}$$

$$\frac{d}{dt}P_{uf_1}(t)=-\beta P_{uf_1}(t)+(1-c)(\lambda+\alpha)P_3(t) \tag{5}$$

$$\frac{d}{dt}P_{uf_2}\left(t\right)=-\beta P_{uf_2}\left(t\right)+\left(1-c\right)\left(\lambda+\alpha\right)P_2\left(t\right) \tag{6}$$

where the unsafe failure state *uf* incurs a reboot delay with mean $1/\beta$.

In steady-state, let us define

$$P_n=\lim_{t\to\infty}P_n\left(t\right),\ n=3,\ 2,\ 1,\ 0,\ uf_1,\ uf_2$$

$$P_n\left(u\right)=\lim_{t\to\infty}P_n\left(u,t\right),\ n=3,\ 2,\ 1,\ 0,\ uf_1,\ uf_2$$

and further, define

$$P_3\left(u\right)=b\left(u\right)P_3 \tag{7}$$

$$P_{uf_1}\left(u\right)=b\left(u\right)P_{uf_1} \tag{8}$$

$$P_{uf_2}\left(u\right)=b\left(u\right)P_{uf_2} \tag{9}$$

From (1)–(9), the steady-state equations are given by:

$$0=-\left(\lambda+\alpha\right)P_3+P_2\left(0\right) \tag{10}$$

$$-\frac{\partial}{\partial u}P_2\left(u\right)=-(\lambda+\alpha)P_2\left(u\right)+c\left(\lambda+\alpha\right)b\left(u\right)P_3+\beta b\left(u\right)P_{uf_1}+b\left(u\right)P_1\left(0\right) \tag{11}$$

$$-\frac{\partial}{\partial u}P_1\left(u\right)=-\lambda P_1\left(u\right)+c\left(\lambda+\alpha\right)P_2\left(u\right)+\beta b\left(u\right)P_{uf_2}\left(t\right)+b\left(u\right)P_0\left(t\right) \tag{12}$$

$$-\frac{\partial}{\partial u}P_0\left(u\right)=\lambda P_1\left(u\right) \tag{13}$$

$$0=-\beta P_{uf_1}+\left(1-c\right)\left(\lambda+\alpha\right)P_3 \tag{14}$$

$$0=-\beta P_{uf_2}+\left(1-c\right)\left(\lambda+\alpha\right)P_2 \tag{15}$$

Now, from (10), (14) and (15), we obtain

$$P_2(0) = (\lambda + \alpha) P_3 \tag{16}$$

$$P_{uf_1} = \frac{(1-c)(\lambda + \alpha)}{\beta} P_3 \tag{17}$$

$$P_{uf_2} = \frac{(1-c)(\lambda + \alpha)}{\beta} P_2 = \frac{(1-c)(\lambda + \alpha)}{\beta} P_2^*(0) \tag{18}$$

We further define

$$B^*(s) = \int_0^\infty \mathrm{e}^{-su} dB(u) = \int_0^\infty \mathrm{e}^{-su} b(u)\, du$$

$$P_n^*(s) = \int_0^\infty \mathrm{e}^{-su} P_n(u)\, du$$

$$P_n = P_n^*(0) = \int_0^\infty P_n(u)\, du$$

and

$$\int_0^\infty \mathrm{e}^{-su} \frac{d}{du} P_n(u)\, du = sP_n^*(s) - P_n(0)$$

Taking the LST on both sides of(11)–(13) and using (16), we get

$$(\lambda + \alpha - s) P_2^*(s) = \left(B^*(s) - 1\right) P_2(0) + B^*(s) P_1(0) \tag{19}$$

$$\begin{aligned}(\lambda - s) P_1^*(s) &= c(\lambda + \alpha) P_2^*(s) + B^*(s) \\ &\quad \times (1-c)(\lambda + \alpha) P_2^*(0) + B^*(s) P_0(0) - P_1(0)\end{aligned} \tag{20}$$

$$sP_0^*(s) = P_0(0) - \lambda P_1^*(s) \tag{21}$$

We develop a recursive method to get explicit expressions $P_n^*\left(0\right)\left(n=2,\,1,\,0\right)$. Setting $s=\lambda+\alpha$ and $s=0$ in (16), yields

$$P_1\left(0\right)=\frac{1-B^*\left(\lambda+\alpha\right)}{B^*\left(\lambda+\alpha\right)}P_2\left(0\right)=\frac{\left(\lambda+\alpha\right)\left(1-B^*\left(\lambda+\alpha\right)\right)}{B^*\left(\lambda+\alpha\right)}P_3 \tag{22}$$

and

$$P_2^*\left(0\right)=\frac{1}{\left(\lambda+\alpha\right)}P_1\left(0\right)=\frac{\left(1-B^*\left(\lambda+\alpha\right)\right)}{B^*\left(\lambda+\alpha\right)}P_3 \tag{23}$$

Again, setting $s=\lambda$ in (19), it follows that

$$P_2^*\left(\lambda\right)=\frac{\left(\lambda+\alpha\right)\left(B^*\left(\lambda\right)-B^*\left(\lambda+\alpha\right)\right)}{\alpha B^*\left(\lambda+\alpha\right)}P_3 \tag{24}$$

Setting $s=\lambda$ in (20) yields

$$P_0\left(0\right)=\frac{P_1\left(0\right)-c\left(\lambda+\alpha\right)P_2^*\left(\lambda\right)-\left(1-c\right)\left(\lambda+\alpha\right)B^*\left(\lambda\right)P_2^*\left(0\right)}{B^*\left(\lambda\right)} \tag{25}$$

Substituting (22)–(24) in (25), we have

$$P_0\left(0\right)=\left\{\frac{\left(\lambda+\alpha\right)\left[\alpha\left(1-B^*\left(\lambda+\alpha\right)\right)\right]-c\left(\lambda+\alpha\right)\left(B^*\left(\lambda\right)-B^*\left(\lambda+\alpha\right)\right)}{\alpha B^*\left(\lambda\right)B^*\left(\lambda+\alpha\right)}\right.$$
$$\left.-\frac{\left(1-c\right)\left(1-B^*\left(\lambda+\alpha\right)\right)}{B^*\left(\lambda+\alpha\right)}\right\}P_3 \tag{26}$$

Similarly, setting $s=0$ in (20), we obtain

$$\lambda P_1^*\left(0\right)=\left(\lambda+\alpha\right)P_2^*\left(0\right)+P_0\left(0\right)-P_1\left(0\right)$$

From the above equation and (20), we have

$$P_1^*(0) = \frac{1}{\lambda} P_0(0)$$

Thus,

$$P_1^*(0) = \frac{1}{\lambda} \left\{ \frac{(\lambda+\alpha)\left[\alpha\left(1-B^*(\lambda+\alpha)\right)\right] - c(\lambda+\alpha)\left(B^*(\lambda) - B^*(\lambda+\alpha)\right)}{\alpha B^*(\lambda) B^*(\lambda+\alpha)} \right.$$

$$\left. - \frac{(1-c)\left(1-B^*(\lambda+\alpha)\right)}{B^*(\lambda+\alpha)} \right\} P_3 \tag{27}$$

Differentiating (21) with respect to s and setting $s = 0$ in the result, we obtain

$$P_0^*(0) = -\lambda P_1^{*(1)}(0) \tag{28}$$

Differentiating (19) with respect to s and then setting $s = 0$ in the result yields

$$(\lambda+\alpha) P_2^{*(1)}(0) = P_2^*(0) - b_1\left[P_2(0) + P_1(0)\right] \tag{29}$$

Similarly, differentiating (20) with respect to s and setting $s = 0$ in the result, we find that

$$\lambda P_1^{*(1)}(0) = P_1^*(0) + \left[c - b_1(1-c)(\lambda+\alpha)\right] P_2^*(0)$$

$$-b_1\left[c\left(P_2(0) + P_1(0)\right) + P_0(0)\right] \tag{30}$$

Then, using (28), we have

$$P_0^*(0) + P_1^*(0) = b_1\left[c\left(P_2(0) + P_1(0)\right) + P_0(0)\right]$$

$$-\left[c - b_1(1-c)(\lambda+\alpha)\right] P_2^*(0) \tag{31}$$

where $P_2^*(0)$, $P_2(0)$, $P_1(0)$ and $P_0(0)$ are given in (23), (16), (22) and (26), respectively.

Now, using the normalizing condition

$$P_3 + P_2^*(0) + P_1^*(0) + P_0^*(0) + P_{uf_1} + P_{uf_2} = 1$$

From the above equation, we obtain P_3.

We assume that one safe failure state 0 and two unsafe failure states uf_1 and uf_2 are system down states. Then for availability model 1, the explicit expression for the Av_1 is given by

$$Av_1 = 1 - P_0^*(0) - P_{uf_1} - P_{uf_2} = P_3 + P_2^*(0) + P_1^*(0)$$

Using (20) and (24), we obtain the explicit expression for the Av_1

$$Av_1 = P_3 \left[1 + \frac{(\lambda+\alpha)\left(\alpha\left(1 - B^*(\lambda+\alpha)\right) - c(\lambda+\alpha)\left(B^*(\lambda) - B^*(\lambda+\alpha)\right)\right)}{\alpha\lambda B^*(\lambda) B^*(\lambda+\alpha)} \right.$$
$$\left. - \frac{(\lambda+\alpha)(1-c)\left(1 - B^*(\lambda+\alpha)\right)}{\lambda B^*(\lambda+\alpha)} + \frac{\left(1 - B^*(\lambda+\alpha)\right)}{B^*(\lambda+\alpha)} \right] \tag{32}$$

3.2. Availability for configuration 2

Following the same procedures as given in the section that analyzes the availability of configuration 1 case, it is easy to set up the following steady-state equations:

$$0 = -(2\lambda + \alpha) P_4 + P_3(0) \tag{33}$$

$$-\frac{\partial}{\partial u} P_3(u) = -(2\lambda+\alpha)P_3(u) + c(2\lambda+\alpha) b(u) P_4$$
$$+ \beta b(u) P_{uf_1} + b(u) P_2(0) \tag{34}$$

$$-\frac{\partial}{\partial u} P_2(u) = -2\lambda P_2(u) + c(2\lambda+\alpha) P_3(u) + \beta b(u) P_{uf_2}(t) + b(u) P_1(t) \tag{35}$$

$$-\frac{\partial}{\partial u} P_1(u) = 2\lambda P_2(u) \tag{36}$$

$$0=-\beta P_{uf_1}+\left(1-c\right)\left(2\lambda+\alpha\right)P_4 \tag{37}$$

$$0=-\beta P_{uf_2}+\left(1-c\right)\left(2\lambda+\alpha\right)P_3 \tag{38}$$

Now, from (35), (39) and (40), we obtain

$$P_3\left(0\right)=\left(2\lambda+\alpha\right)P_4 \tag{39}$$

$$P_{uf_1}=\frac{\left(1-c\right)\left(2\lambda+\alpha\right)}{\beta}P_4 \tag{40}$$

$$P_{uf_2}=\frac{\left(1-c\right)\left(2\lambda+\alpha\right)}{\beta}P_3=\frac{\left(1-c\right)\left(2\lambda+\alpha\right)}{\beta}P_3^*(0) \tag{41}$$

Taking the LST on both sides of (34)–(36) and using (39)–(41), it implies that

$$\left(2\lambda+\alpha-s\right)P_3^*\left(s\right)=\left(B^*\left(s\right)-1\right)P_3\left(0\right)+B^*\left(s\right)P_2\left(0\right) \tag{42}$$

$$\begin{aligned}\left(2\lambda-s\right)P_2^*\left(s\right)=c\left(2\lambda+\alpha\right)P_3^*\left(s\right)+B^*\left(s\right)\left(1-c\right)\left(2\lambda+\alpha\right)\\ \times P_3^*\left(0\right)+B^*\left(s\right)P_1(0)-P_2\left(0\right)\end{aligned} \tag{43}$$

$$sP_1^*\left(s\right)=P_1\left(0\right)-2\lambda P_2^*\left(s\right) \tag{44}$$

Setting $s=2\lambda+\alpha$ and $s=0$ in (42) yields, respectively

$$P_2\left(0\right)=\frac{1-B^*\left(2\lambda+\alpha\right)}{B^*\left(2\lambda+\alpha\right)}P_3\left(0\right)=\frac{\left(2\lambda+\alpha\right)\left(1-B^*\left(2\lambda+\alpha\right)\right)}{B^*\left(2\lambda+\alpha\right)}P_4 \tag{45}$$

and

$$P_3^*\left(0\right)=\frac{1}{\left(2\lambda+\alpha\right)}P_2\left(0\right)=\frac{\left(1-B^*\left(2\lambda+\alpha\right)\right)}{B^*\left(2\lambda+\alpha\right)}P_4 \tag{46}$$

Again, setting $s=2\lambda$ in (42), it follows that

$$P_3^*\left(2\lambda\right)=\frac{\left(2\lambda+\alpha\right)\left(B^*\left(2\lambda\right)-B^*\left(2\lambda+\alpha\right)\right)}{\alpha B^*\left(2\lambda+\alpha\right)}P_4 \tag{47}$$

Setting $s=2\lambda$ in (43) yields

$$P_1\left(0\right)=\frac{P_2\left(0\right)-c\left(2\lambda+\alpha\right)P_3^*\left(2\lambda\right)-\left(1-c\right)\left(2\lambda+\alpha\right)B^*\left(2\lambda\right)P_3^*\left(0\right)}{B^*\left(2\lambda\right)} \tag{48}$$

Substituting (45)–(47) in (48), we have

$$\begin{aligned}P_1\left(0\right)=&\left\{\frac{\left(2\lambda+\alpha\right)\left[\alpha\left(1-B^*\left(2\lambda+\alpha\right)\right)\right]}{\alpha B^*\left(2\lambda\right)B^*\left(2\lambda+\alpha\right)}\right.\\&\left.-\frac{c\left(2\lambda+\alpha\right)\left(B^*\left(2\lambda\right)-B^*\left(2\lambda+\alpha\right)\right)}{\alpha B^*\left(2\lambda\right)B^*\left(2\lambda+\alpha\right)}-\frac{\left(1-c\right)\left(1-B^*\left(2\lambda+\alpha\right)\right)}{B^*\left(2\lambda+\alpha\right)}\right\}P_4\end{aligned} \tag{49}$$

Similarly, setting $s=0$ in (43), we obtain

$$2\lambda P_2^*\left(0\right)=\left(2\lambda+\alpha\right)P_3^*\left(0\right)+P_1\left(0\right)-P_2\left(0\right)$$

From the above equation and (48), we have

$$P_2^*\left(0\right)=\frac{1}{2\lambda}P_1\left(0\right)$$

Now, using (49), we get

$$\begin{aligned}P_2^*\left(0\right)=\frac{1}{2\lambda}&\left\{\frac{\left(2\lambda+\alpha\right)\left[\alpha\left(1-B^*\left(2\lambda+\alpha\right)\right)\right]}{\alpha B^*\left(2\lambda\right)B^*\left(2\lambda+\alpha\right)}\right.\\&\left.-\frac{c\left(2\lambda+\alpha\right)\left(B^*\left(2\lambda\right)-B^*\left(2\lambda+\alpha\right)\right)}{\alpha B^*\left(2\lambda\right)B^*\left(2\lambda+\alpha\right)}-\frac{\left(1-c\right)\left(1-B^*\left(2\lambda+\alpha\right)\right)}{B^*\left(2\lambda+\alpha\right)}\right\}P_4\end{aligned} \tag{50}$$

Differentiating (44) with respect to s and setting $s=0$ in the result, we obtain

$$P_1^*\left(0\right) = -2\lambda P_2^{*(1)}\left(0\right) \tag{51}$$

Differentiating (42) with respect to s and then setting $s = 0$ in the result yields

$$\left(2\lambda+\alpha\right)P_3^{*(1)}\left(0\right) = P_3^*\left(0\right) - b_1\left[P_3\left(0\right) + P_2\left(0\right)\right] \tag{52}$$

Likewise, differentiating (43) with respect to s and setting $s = 0$ in the result, we find that

$$\begin{aligned} 2\lambda P_2^{*(1)}\left(0\right) &= P_2^*\left(0\right) + \left[c - b_1\left(1-c\right)\left(2\lambda+\alpha\right)\right]P_3^*\left(0\right) \\ &\quad - b_1\left[c\left(P_3(0)+P_2(0)\right) + P_1(0)\right] \end{aligned} \tag{53}$$

Then, using (51), we have

$$\begin{aligned} P_1^*\left(0\right) + P_2^*\left(0\right) &= b_1\left[c\left(P_3(0)+P_2(0)\right) + P_1(0)\right] \\ &\quad - \left[c - b_1\left(1-c\right)\left(2\lambda+\alpha\right)\right]P_3^*\left(0\right) \end{aligned} \tag{54}$$

where $P_3^*\left(0\right)$, $P_3\left(0\right)$, $P_2\left(0\right)$ and $P_1\left(0\right)$ are given in (46), (39), (45) and (49), respectively.

Now, using the normalizing condition

$$P_4 + P_3^*\left(0\right) + P_2^*\left(0\right) + P_1^*\left(0\right) + P_{uf_1} + P_{uf_2} = 1$$

from the above equation, we obtain P_4.

We assume that one safe failure state 1 and two unsafe failure states uf_1 and uf_2 are system down states. Since states 1, uf_1, and uf_2 are system down states, then for the availability model 2, the explicit expression for the Av_1 is given by

$$Av_2 = 1 - P_1^*\left(0\right) - P_{uf_1} - P_{uf_2} = P_4 + P_3^*\left(0\right) + P_2^*\left(0\right)$$

Using (46) and (50), we obtain the explicit expression for the Av_2

$$Av_2 = P_3\left[1+\frac{(2\lambda+\alpha)\left(\alpha\left(1-B^*(2\lambda+\alpha)\right)-c(2\lambda+\alpha)\left(B^*(2\lambda)-B^*(2\lambda+\alpha)\right)\right)}{\alpha 2\lambda B^*(2\lambda)B^*(2\lambda+\alpha)}\right.$$

$$\left.-\frac{(2\lambda+\alpha)(1-c)\left(1-B^*(2\lambda+\alpha)\right)}{2\lambda B^*(2\lambda+\alpha)}+\frac{\left(1-B^*(2\lambda+\alpha)\right)}{B^*(2\lambda+\alpha)}\right] \tag{55}$$

3.3. Availability for configuration 3

We use the same procedure as above to obtain the steady-state equations as follows

$$0=-(\lambda+2\alpha)P_4+P_3(0) \tag{56}$$

$$-\frac{\partial}{\partial u}P_3(u)=-(\lambda+2\alpha)P_3(u)+c(\lambda+2\alpha)b(u)P_4$$
$$+\beta b(u)P_{uf_1}+b(u)P_2(0) \tag{57}$$

$$-\frac{\partial}{\partial u}P_2(u)=-(\lambda+\alpha)P_2(u)+c(\lambda+2\alpha)P_3(u)+\beta b(u)P_{uf_2}+b(u)P_1(0) \tag{58}$$

$$-\frac{\partial}{\partial u}P_1(u)=-\lambda P_1(u)+c(\lambda+\alpha)P_2(u)+\beta b(u)P_{uf_3}+b(u)P_0(0) \tag{59}$$

$$-\frac{\partial}{\partial u}P_0(u)=\lambda P_1(u) \tag{60}$$

$$0=-\beta P_{uf_1}+(1-c)(\lambda+2\alpha)P_4 \tag{61}$$

$$0=-\beta P_{uf_2}+(1-c)(\lambda+2\alpha)P_3 \tag{62}$$

$$0=-\beta P_{uf_3}+(1-c)(\lambda+\alpha)P_2 \tag{63}$$

Now from (56), and (61)–(63) we obtain

$$P_3(0)=(\lambda+2\alpha)P_4 \tag{64}$$

$$P_{uf_1} = \frac{(1-c)(\lambda+2\alpha)}{\beta} P_4 \tag{65}$$

$$P_{uf_2} = \frac{(1-c)(\lambda+2\alpha)}{\beta} P_3 = \frac{(1-c)(\lambda+2\alpha)}{\beta} P_3^*(0) \tag{66}$$

$$P_{uf_3} = \frac{(1-c)(\lambda+\alpha)}{\beta} P_2 = \frac{(1-c)(\lambda+\alpha)}{\beta} P_2^*(0) \tag{67}$$

Taking the LST on both sides of (57)–(60) and using (64)–(67), it implies that

$$(\lambda+2\alpha-s) P_3^*(s) = (B^*(s)-1) P_3(0) + B^*(s) P_2(0) \tag{68}$$

$$(\lambda+\alpha-s) P_2^*(s) = c(\lambda+2\alpha) P_3^*(s) + B^*(s)(1-c)(\lambda+2\alpha) P_3^*(0)$$
$$+ B^*(s) P_1(0) - P_2(0) \tag{69}$$

$$(\lambda-s) P_1^*(s) = c(\lambda+\alpha) P_2^*(s) + B^*(s)(1-c)(\lambda+\alpha) P_2^*(0)$$
$$+ B^*(s) P_0(0) - P_1(0) \tag{70}$$

$$sP_0^*(s) = P_0(0) - \lambda P_1^*(s) \tag{71}$$

Setting $s = \lambda + 2\alpha$ and $s = 0$ in (68), respectively, yields

$$P_2(0) = \frac{1-B^*(\lambda+2\alpha)}{B^*(\lambda+2\alpha)} P_3(0) = \frac{(\lambda+2\alpha)(1-B^*(\lambda+2\alpha))}{B^*(\lambda+2\alpha)} P_4 \tag{72}$$

$$P_3^*(0) = \frac{1}{(\lambda+2\alpha)} P_2(0) = \frac{(1-B^*(\lambda+2\alpha))}{B^*(\lambda+2\alpha)} P_4 \tag{73}$$

Again, setting $s = \lambda + \alpha$ and $s = \lambda$ in (68), it follows that

$$P_3^*(\lambda+\alpha) = \frac{(\lambda+2\alpha)(B^*(\lambda+\alpha)-B^*(\lambda+2\alpha))}{\alpha B^*(\lambda+2\alpha)} P_4 \tag{74}$$

$$P_3^*(\lambda) = \frac{\left(B^*(\lambda)-1\right)P_3(0) + B^*(\lambda)P_2(0)}{2\alpha} \tag{75}$$

Setting $s = \lambda + \alpha$ in (69) yields

$$P_1(0) = \frac{P_2(0) - c(\lambda+2\alpha)P_3^*(\lambda+\alpha) - (1-c)(\lambda+\alpha)B^*(\lambda+\alpha)P_3^*(0)}{B^*(\lambda+\alpha)} \tag{76}$$

Similarly, setting $s = 0$ in (69), we obtain

$$P_2^*(0) = \frac{(\lambda+2\alpha)P_3^*(0) + P_1(0) - P_2(0)}{(\lambda+\alpha)} \tag{77}$$

Again, setting $s = \lambda$ in (69), it follows that

$$P_3^*(\lambda) = \frac{c(\lambda+2\alpha)P_3^*(\lambda) + (1-c)(\lambda+2\alpha)B^*(\lambda)P_3^*(0)}{\alpha} + \frac{B^*(\lambda)P_1(0) - P_2(0)}{\alpha} \tag{78}$$

Setting $s = \lambda$ in (70) yields

$$P_0(0) = \frac{P_1(0) - c(\lambda+\alpha)P_2^*(\lambda) - (1-c)(\lambda+\alpha)B^*(\lambda)P_3^*(0)}{B^*(\lambda)} \tag{79}$$

Likewise, setting $s = 0$ in (70), we obtain

$$P_1^*(0) = \frac{(\lambda+\alpha)P_2^*(0) + P_0(0) - P_1(0)}{\lambda} \tag{80}$$

Differentiating (71) with respect to s and setting $s = 0$ in the result, we obtain

$$P_0^*(0) = -\lambda P_1^{*(1)}(0) \tag{81}$$

Differentiating (68) with respect to s and then setting $s = 0$ in the result yields

$$\left(\lambda+2\alpha\right)P_3^{*(1)}\left(0\right)=P_3^{*}\left(0\right)-b_1\left[P_3\left(0\right)+P_2\left(0\right)\right] \tag{82}$$

Differentiating (68) with respect to s and then setting $s = 0$ in the result, it follows that

$$\begin{aligned}\left(\lambda+\alpha\right)P_2^{*(1)}\left(0\right)&=P_2^{*}\left(0\right)+c\left(\lambda+2\alpha\right)P_3^{*(1)}\left(0\right)\\&\quad-b_1\left(P_1(0)+\left(1-c\right)\left(\lambda+2\alpha\right)P_3^{*}\left(0\right)\right)\end{aligned} \tag{83}$$

Likewise, differentiating (70) with respect to s and setting $s = 0$ in the result, we find that

$$\lambda P_1^{*(1)}\left(0\right)=P_1^{*}\left(0\right)+c\left(\lambda+\alpha\right)P_2^{*(1)}\left(0\right)-b_1\left[P_0(0)+\left(1-c\right)\left(\lambda+\alpha\right)P_2^{*}\left(0\right)\right] \tag{84}$$

Then, using (81), we have

$$P_1^{*}\left(0\right)+P_2^{*}\left(0\right)=b_1\left[P_0(0)+\left(1-c\right)\left(\lambda+\alpha\right)P_2^{*}\left(0\right)\right]-c\left(\lambda+\alpha\right)P_2^{*(1)}\left(0\right) \tag{85}$$

Now, using the normalizing condition

$$P_4+P_3^{*}\left(0\right)+P_2^{*}\left(0\right)+P_1^{*}\left(0\right)+P_0^{*}\left(0\right)+P_{uf_1}+P_{uf_2}+P_{uf_3}=1$$

From the above equation we obtain P_4.

We assume that one safe failure state 0 and three unsafe failure states unf_1, unf_2 and unf_3 are system down states. Since states 0, unf_1, unf_2, and unf_3 are system down states. Then for availability model 3, the explicit expression for the Av_3 is given by

$$Av_3=1-P_0^{*}\left(0\right)-P_{uf_1}-P_{uf_2}-P_{uf_3}=P_4+P_3^{*}\left(0\right)+P_2^{*}\left(0\right)+P_1^{*}\left(0\right)$$

Using above equations, we obtain the explicit expression for the Av_3

$$Av_3 = P_4\left[1+\frac{\left(1-B^*(\lambda+2\alpha)\right)}{B^*(\lambda+2\alpha)}+\frac{(\lambda+2\alpha)}{2\alpha^2 B^*(\lambda+2\alpha)B^*(\lambda+\alpha)}\left\{\frac{-2\alpha}{\lambda+\alpha}\right.\right.$$

$$\times\Big(\Big(B^*(\lambda+\alpha)(1+c)-1+\alpha B^*(\lambda+2\alpha)\Big(1-B^*(\lambda+\alpha)+c\big(B^*(\lambda+\alpha)-2\big)\Big)\Big)$$

$$+c\lambda\Big(B^*(\lambda+\alpha)-B^*(\lambda+2\alpha)\Big)\Big)-\frac{1}{\lambda B^*(\lambda)}\Big(2\alpha^2\Big[1-B^*(\lambda+\alpha)+B^*(\lambda+2\alpha)$$

$$\times\Big(B^*(\lambda+\alpha)(c-1)^2-1+2c\Big)\Big]\Big(B^*(\lambda)-1\Big)-c\alpha\lambda\Big[B^*(\lambda)\Big(B^*(\lambda+\alpha)(2+c)-2\Big)$$

$$+B^*(\lambda+2\alpha)\Big(2-2B^*(\lambda+\alpha)+3cB^*(\lambda+\alpha)-4cB^*(\lambda)\Big)\Big]-c^2\Big(B^*(\lambda+2\alpha)$$

$$\left.\left.\times\lambda^2\Big(B^*(\lambda+\alpha)-2B^*(\lambda)+B^*(\lambda+\alpha)B^*(\lambda)\Big)\Big)\Big)\right\}\right] \tag{86}$$

4. Comparison of the three configurations

The purpose of this section is to present specific comparisons for the $Av_i\ (i=1,2,3)$ for two different repair time distributions: exponential and k-stage Erlang, using an efficient Mathematica computer program. Basically, we consider the following values:

$$\frac{1}{\lambda}=2500 \text{ days},\quad \frac{1}{\alpha}=4000 \text{ days},\quad \frac{1}{\mu}=10 \text{ days},\quad \frac{1}{\beta}=\frac{10}{24} \text{ days}=10 \text{ h}$$

4.1. Comparison of all availability models

We first consider the following four cases to perform a comparison for the Av of the configurations 1, 2, 3 when the repair time distribution is exponential, or 2-stage Erlang, or 3-stage Erlang.

Case 1. We fix $\alpha=0.00025$, $\mu=0.1$, $\beta=2.4$, $c=0.9$ and vary the values of λ from 0.0004 to 0.01.

Case 2. We fix $\lambda=0.0004$, $\alpha=0.00025$, $\beta=2.4$, $c=0.9$ and vary the values of μ from 0.01 to 0.18.

Case 3. We fix $\lambda=0.0004$, $\alpha=0.00025$, $\mu=0.1$, $c=0.9$ and vary the values of β from 1 to 10.

Case 4. We fix $\lambda=0.0004$, $\alpha=0.00025$, $\mu=0.1$, $\beta=2.4$ and vary the values of c from 0.5 to 1.

Numerical results of the $Av_i(M)$ and $Av_i(E_k)$ for each availability model $i(i=1,2,3)$ are shown in Tables 3–6 for cases 1–4, respectively.

4.2. Comparison of all availability models based on their cost/benefit ratios

We consider that the various configurations may have different costs when comparing all configurations. We assume that the size-proportional costs for the operative units, cold standby units and warm standby units are given in Table1. With this, we calculate the costs for each configuration $i\ (i=1, 2, 3)$ shown in Table 2. Let C_i be the cost of the configuration *i*, and B_i be the benefit of the configuration *i*, where B_i is the Av_i. Under the cost/benefit (C_i/Av_i) ratio, comparisons are made based on assumed numerical values given to the system parameters, and to the costs of configurations. Numerical results of (C_i/Av_i) for configurations $\forall i=1, 2, 3$ are shown in Tables 7–10 for cases 1–4, respectively.

Table 3. Comparison of the availability models 1, 2, 3 for *Av* (case 1)

Range of λ	Result
1. Exponential repair time	
$0.0004 < \lambda < 0.002$	$Av_1(M) > Av_3(M) > Av_2(M)$
$0.002 < \lambda < 0.001$	$Av_3(M) > Av_1(M) > Av_2(M)$
2. 2-stage Erlange repair time	
$0.0004 < \lambda < 0.00257$	$Av_1(E_2) > Av_3(E_2) > Av_2(E_2)$
$0.00257 < \lambda < 0.01$	$Av_3(E_2) > Av_1(E_2) > Av_2(E_2)$
3. 3-stage Erlange repair time	
$0.0004 < \lambda < 0.00283$	$Av_1(E_3) > Av_3(E_3) > Av_2(E_3)$
$0.00283 < \lambda < 0.01$	$Av_3(E_3) > Av_1(E_3) > Av_2(E_3)$

Table 4. Comparison of the availability models 1, 2, 3 for *Av* (case 2)

Range of μ	Result
1. Exponential repair time	
$0.01 < \mu < 0.0246$	$Av_3(M) > Av_1(M) > Av_2(M)$
$0.0246 < \mu < 0.2$	$Av_1(M) > Av_3(M) > Av_2(M)$
2. 2-stage Erlange repair time	
$0.01 < \mu < 0.02$	$Av_3(E_2) > Av_1(E_2) > Av_2(E_2)$
$0.02 < \mu < 0.2$	$Av_1(E_2) > Av_3(E_2) > Av_2(E_2)$
3. 3-stage Erlange repair time	
$0.01 < \mu < 0.0184$	$Av_3(E_3) > Av_1(E_3) > Av_2(E_3)$
$0.0184 < \mu < 0.2$	$Av_1(E_3) > Av_3(E_3) > Av_2(E_3)$

Table 5. Comparison of the availability models 1, 2, 3 for *Av* (case 3)

Range of β	Result
1. Exponential repair time $1 < \beta < 10$	$Av_1(M) > Av_3(M) > Av_2(M)$
2. 2-stage Erlange repair time $1 < \beta < 10$	$Av_1(E_2) > Av_3(E_2) > Av_2(E_2)$
3. 3-stage Erlange repair time $1 < \beta < 10$	$Av_1(E_3) > Av_3(E_3) > Av_2(E_3)$

Table 6. Comparison of the availability models 1, 2, 3 for *Av* (case 4)

Range of C	Result
1. Exponential repair time	
$0.5 < C < 0.9935$	$Av_1(M) > Av_3(M) > Av_2(M)$
$0.9935 \le C \le 1$	$Av_1(M) = Av_3(M) > Av_2(M)$
2. 2-stage Erlange repair time	
$0.5 < C < 0.9953$	$Av_1(E_2) > Av_3(E_2) > Av_2(E_2)$
$0.9953 \le C < 0.99601$	$Av_1(E_2) > Av_3(E_2) = Av_2(E_2)$
$0.99601 \le C < 0.99758$	$Av_1(E_2) = Av_3(E_2) > Av_2(E_2)$
$0.99758 \le C \le 1$	$Av_1(E_2) = Av_3(E_2) = Av_2(E_2)$
3. 3-stage Erlange repair time	
$0.5 < C < 0.9951$	$Av_1(E_3) > Av_3(E_3) > Av_2(E_3)$
$0.9951 \le C < 0.99601$	$Av_1(E_3) > Av_3(E_3) = Av_2(E_3)$
$0.99601 \le C < 0.99732$	$Av_1(E_3) = Av_3(E_3) > Av_2(E_3)$
$0.99732 \le C \le 1$	$Av_1(E_3) = Av_3(E_3) = Av_2(E_3)$

From the Tables (7–10), we can predict that the optimal system using $\cos t/Av_i$ value is system 2. It should be noted that the optimal configuration using the $\cos t/Av_i$ value does not depend on distributions of repair time and the ranges of λ, μ, β, and c.

Table 7. Rank of (C_i/Av_i) for $\alpha = 0.00025$, $\mu = 0.1$, $\beta = 2.4$, $v = 0.9$

Repair time distribution	Range of λ	Rank (C_i/Av_i)
1. Exponential repair time	$0.0004 < \lambda < 0.01$	$C3/Av_3(M) > C1/Av_1(M) > C2/Av_2(M)$
2. 2-stage Erlange repair time		$C3/Av_3(E_2) > C1/Av_1(E_2) > C2/Av_2(E_2)$
3. 3-stage Erlange repair time		$C3/Av_3(E_3) > C1/Av_1(E_3) > C2/Av_2(E_3)$

Table 8. Rank of (C_i/Av_i) for $\lambda = 0.0004$, $\alpha = 0.00025$, $\mu = 0.1$, $\beta = 2.4$, $v = 0.9$

Repair time distribution	Range of μ	Rank (C_i/Av_i)
1. Exponential repair time	$0.01 < \mu < 0.2$	$C_3/Av_3(M) > C_1/Av_1(M) > C_2/Av_2(M)$
2. 2-stage Erlange repair time		$C_3/Av_3(E_2) > C_1/Av_1(E_2) > C_2/Av_2(E_2)$
3. 3-stage Erlange repair time		$C_3/Av_3(E_3) > C_1/Av_1(E_3) > C_2/Av_2(E_3)$

Table 9. Rank of (C_i/Av_i) for $\lambda = 0.0004$, $\alpha = 0.00025$, $\mu = 0.1$, $c = 0.9$

Repair time distribution	Range of β	Rank (C_i/Av_i)
1. Exponential repair time	$1 < \beta < 10$	$C_3/Av_3(M) > C_1/Av_1(M) > C_2/Av_2(M)$
2. 2-stage Erlange repair time		$C_3/Av_3(E_2) > C_1/Av_1(E_2) > C_2/Av_2(E_2)$
3. 3-stage Erlange repair time		$C_3/Av_3(E_3) > C_1/Av_1(E_3) > C_2/Av_2(E_3)$

Table 10. Table 9. Rank of (C_i/Av_i) for $\lambda = 0.0004$, $\alpha = 0.00025$, $\mu = 0.1$, $\beta = 2.4$

Repair time distribution	Range of c	Rank (C_i/Av_i)
1. Exponential repair time	$0.5 \le c \le 1$	$C_3/Av_3(M) > C_1/Av_1(M) > C_2/Av_2(M)$
2. 2-stage Erlange repair time		$C_3/Av_3(E_2) > C_1/Av_1(E_2) > C_2/Av_2(E_2)$
3. 3-stage Erlange repair time		$C_3/Av_3(E_3) > C_1/Av_1(E_3) > C_2/Av_2(E_3)$

5. Conclusions

The authors develop analytic steady-state results for availability systems with mixed standby components and imperfect coverage. However, the first objective of this article was to provide a recursive method, using the supplementary variable technique, to derive the steady-state availability for three systems. The second objective was to verify the explicit expressions for two different repair time distributions such as exponential distribution (*M*), and *k*-stage Erlang (E_k). Finally, we provided the cost/benefit analysis of three availability models, and rank three availability models for two different repair time distributions. We conclude that the optimal system using value is system 2.

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Received 24 March 2018
Accepted 2 May 2019