O P E R A T I O N S   R E S E A R C H   A N D   D E C I S I O N S

No. 2 2018

DOI: 10.5277/ord180201

Ritu ARORA[1]

Kavita GUPTA[2]

BRANCH AND BOUND ALGORITHM FOR A DISCRETE MULTILEVEL LINEAR FRACTIONAL PROGRAMMING PROBLEM

An algorithm is proposed to find an integer solution for bilevel linear fractional programming problem with discrete variables. The method develops a cut that removes the integer solutions which are not bilevel feasible. The proposed method is extended from bilevel to multilevel linear fractional programming problems with discrete variables. The solution procedure for both the algorithms is elucidated in the paper.

Keywords: *linear fractional programming problem, bilevel programming, multilevel programming, discrete variables, integer solution, branch and bound cut*

1. Introduction

A multilevel programming model deals with decision making problems in a hierarchical system with interactive levels. In a multiple level hierarchical organization, the players at each level optimize their objective functions keeping in mind the actions of the players at other levels. Candler and Norton [7] first used the terms bilevel and multilevel programming. In 1982, Bard and Falk [1] developed an explicit solution to the multilevel programming problem. A multilevel programming model is used for analysing and planning many situations in real life. Candler and Norton [7] presented a version of this problem in the context of economic policy. In 1998, Migdalas et al. [20] published a book on multilevel optimization which is part of a series on non-convex opti-

[1]Department of Mathematics, Keshav Mahavidyalaya, University of Delhi, H-4-5 Zone, Road No. 43, Pitampura Near Sainik Vihar, Delhi 110034, India, e-mail address: ritunarora@yahoo.in

[2]Department of Mathematics, Kirori Mal College, University of Delhi, Delhi 110007, India, e-mail address: gupta_kavita31@yahoo.com

mization and its applications. In 2015, Perez et al. [23] proposed a multi-criteria approach for an urban passenger transport system. In 2016, Lu [16] gave a survey on multilevel decision making. In 2017, Hernandez et al. [14] developed a model by applying multi-criteria techniques to plan the harvest of a forest.

The simplest case of a multilevel programming problem is bilevel programming. The general bilevel programming problem, BLPP, is defined as,

BLPP: $\underset{X_1}{\text{Max}}\, F_1(X_1, X_2)$, where X_2 solves $\underset{X_2}{\text{Max}}\, F_2(X_1, X_2)$, for a given X_1, subject to $(X_1, X_2) \in S$, where

$$S = \{(X_1,\, X_2) : A_1 X_1 + A_2 X_2 = b$$

$$(X_1,\, X_2) \text{ is an integer vector}\}$$

Here, $F_1(X_1,\, X_2)$ and $F_2(X_1,\, X_2)$ can be linear or non-linear. $F_1(X_1,\, X_2)$ and $F_2(X_1, X_2)$ are the objective functions of the leader and the follower, respectively. The feasible region S is assumed to be non-empty and bounded. The vectors of decision variables, $X_1 \in Z^{n_1}$ and $X_2 \in Z^{n_2}$, are under the control of the leader and follower, respectively.

Bilevel/multilevel programming problems with integer variables are of high importance in business and industry. Such problems are of particular importance when fractional units are unrealistic because units are not divisible. Many cutting plane algorithms, like the Dantzing cut, Gomory cut, edge truncating cut, etc., are used to solve such problems. These problems have multiple applications in practical fields. Dempe et al. [11] applied bilevel programming with discrete variables to a problem involving the supply of natural gas. Claassen [9] analysed the use of management tools in paper production.

Several algorithms have been proposed to solve bilevel and multilevel programming problems. The most notable among them are the penalty function approach [26], the ranking method [21], and the branch and bound method [13]. Different methodologies for solving such problems using the branch and bound algorithm have been proposed by various authors. The first known branch and bound algorithm was developed in 1960 by Land and Doig [15] as an application for solving mixed and pure integer programming problems. Bienstock [4] presented computational results for a branch and bound algorithm to solve a family of mixed integer quadratic programming problems. Using various strategies, Bard and Moore [2] proposed a branch and bound algorithm for solving the bilevel programming problem. Luedtke [17] proposed an algorithm for branch and bound cut decomposition to solve chance constrained mathematical programs. The significance of the branch and bound algorithm can be seen in practical situations. Costa et al. [10] used the branch and bound algorithm to determine strategically optimal rules of operation for any type of water distribution system. In his paper, Maroti [18] proposed a branch and bound approach to find robust railway time tables.

In BLPP, linear fractional objective functions are useful in production planning, corporate planning, financial planning and so forth. Fractional programming problems (FP) are a subclass of non-linear programming problems in which the objective function is non-linear, but the constraints are all linear. Charnes and Cooper [8] replaced the FP problem by two equivalent linear programs. Swarup [25] developed a simplex type algorithm for solving (FP) assuming that the solution set is regular and the denominator is strictly positive at all feasible points. Techniques for solving non-linear fractional programming problems have been suggested by Dinkelbach [12]. Schiable [24] listed many applications of fractional programming problems. Mathur and Puri [19] studied the bilevel fractional programming problem. Calvete and Gale [6] solved the bilevel linear fractional programming problem by a penalty method. In 2012, Pal and Gupta [22] solved bilevel fractional programming problems by a fuzzy goal programming approach based on a genetic algorithm. Bhargava[3] solved a multilevel programming problem in which the objective function at each level is linear fractional. The author proposed a *K*-th best algorithm to find the optimal solution.

In the present paper, we contemplate a new perspective which is different from the approaches developed by previous authors. We have solved the bilevel linear fractional programming problem with discrete variables based on a cut which locates the integer solutions to the problem that are bilevel feasible. This concept is extended to the multilevel linear fractional programming problem with discrete variables.

This paper is divided into the following sections. In Section 2, a methodology is proposed for the bilevel linear fractional programming problem with discrete variables, (DBLFPP). A numerical example is solved illustrating the solution procedure for DBLFPP. In Section 3, the developed procedure is adapted to the multilevel programming problem with discrete variables, DMLLFPP. The method for solving DMLLFPP is illustrated with an example. Finally, Section 4 presents conclusions.

2. Bilevel linear fractional programming problem with discrete variables

The linear fractional bilevel programming problem with discrete variables is mathematically stated as DBLFPP:

$$\underset{X_1}{\text{Max}}\, F_1\left(X_1, X_2\right) = \frac{c_{11}X_1 + c_{12}X_2 + \alpha_1}{d_{11}X_1 + d_{12}X_2 + \beta_1}$$

where X_2 solves

$$\underset{X_2}{\text{Max}}\ F_2\left(X_1, X_2\right) = \frac{c_{21}X_1 + c_{22}X_2 + \alpha_2}{d_{21}X_1 + d_{22}X_2 + \beta_2}, \text{for a given } X_1$$

subject to

$$A_1X_1 + A_2X_2 = b$$

$$X_1 \in Z_+^{n_1}, \quad X_2 \in Z_+^{n_2}$$

$$c_{11}^T, d_{11}^T, \mathrm{c}_{21}^T, d_{21}^T, \in Z^{n_1}, \quad c_{12}^T, d_{12}^T, \mathrm{c}_{22}^T, d_{22}^T \in Z^{n_2}$$

$$A_1 \in Z^{m\times n_1}, \quad A_2 \in Z^{m\times n_2}, \quad b \in Z^m \text{ and } \alpha_1, \alpha_2, \beta_1, \beta_1, \in Z$$

Here

$$(d_{11}X_1 + d_{12}\ X_2 + \beta_1) > 0 \text{ and } (d_{21}X_1 + d_{22}X_2 + \beta_2) > 0 \quad \forall (X_1, X_2) \in S$$

where

$$S = \{(X_1, X_2) : A_1X_1 + A_2X_2 = b \\ (X_1,\ X_2) \text{ is an integer vector}\}$$

Here, S is the feasible region of DBLFPP and it is assumed to be non-empty, compact and contains integer points. Here, $F_1\left(X_1, X_2\right)$ and $F_2\left(X_1, X_2\right)$ are the objective functions of the leader and the follower, respectively.

For each X_1, define the follower's feasible set as

$$S(X_1) = \left\{X_2 \mid X_2 \in Z_+^{n_2}, A_2X_2 = b - A_1X_1\right\}$$

Define the reaction set $R(X_1)$ as the set of all solutions which maximize the follower's objective function

$$R\left(X_1\right) = \arg \underset{X_2}{\text{Max}} \left\{F_2(X_1, X_2) : X_2 \in S(X_1)\right\}$$

The projection of S onto the leader's space is

$$S_1 = \left\{X_1 \in Z_+^{n_1} : \exists\ (X_1, X_2) : A_1X_1 + A_2X_2 = b\right\}$$

The inducible region of the leader's problem is denoted by

$$\text{IR} = \left\{ (X_1, X_2) : X_1 \in Z_+^{n_1}, A_1 X_1 + A_2 X_2 = b, X_2 \in R(X_1) \right\}$$

IR represents the set of all bilevel feasible solutions which are both optimal for the follower and feasible for the leader.

In the bilevel linear fractional programming problem defined above, the objective functions at both levels are linear fractional, hence they are quasi-concave. The inducible region IR of DBLFPP is comprised of the union of connected faces of S and the optimal solution is at an extreme point of the feasible region S. This suggests using a method that searches over the set of extreme points to develop an algorithm for solving DBLFPP. To avoid the ambiguity of multiple optima, assume that for each value of $X_1 \in S_1$, there is a unique solution to the follower's problem. Here, we assume that the IR of DBLFPP is non – empty to ensure the existence of a solution to DBLFPP. Thus, we have the following lemmas as stated and proved in [5].

Lemma 1. The inducible region of the quasi-concave bilevel programming problem is piecewise linear [5].

Lemma 2. There is an extreme point of the feasible region S which is a locally optimal solution to the quasi-concave bilevel programming problem [5].

Note: There is a difference between DBLFPP and the integer linear fractional programming problem. In DBLFPP, a solution (X_1, X_2) in the feasible set S satisfies the condition that (X_1, X_2) should be bilevel feasible, that is $X_2 \in R(X_1)$. An integer solution provides an upper bound on the optimal value for the follower only when it belongs to the inducible region IR.

Now, we define the single level linear fractional programming problem (SLLFPP). It is a relaxed version of DBLFPP in which the follower's objective function is not considered:

$$\underset{X_1, X_2}{\text{Max}}\, F_1\left(X_1, X_2\right) = \frac{c_{11}X_1 + c_{12}X_2 + \alpha_1}{d_{11}X_1 + d_{12}X_2 + \beta_1}$$

subject to $(X_1, X_2) \in S$.

The problem SLLFPP is solved on the feasible region S. The variables in this problem are required to be integers. Since SLLFPP is an integer single level linear fractional programming problem, it can be solved using the Branch and bound method. If an integer solution (X_1, X_2) is obtained at some extreme point, then $F_1(X_1, X_2)$ is a lower bound on the optimal value of the leader.

Auxiliary problem for DBLFPP. Define the auxiliary Min–Max problem as LLFPP

$$\underset{X_1,\ X_2}{\text{Min}}\ F_2(X_1,\ X_2) = \frac{c_{21}X_1 + c_{22}X_2 + \alpha_2}{d_{21}X_1 + d_{22}X_2 + \beta_2}$$

subject to $X_1 \in R_+^{n_1}$

$$X_2 \in \arg \underset{X_2}{\text{Max}}\ F_2(X_1, X_2)$$

subject to $A_1X_1 + A_2X_2 = b$

$$X_2 \in R_+^{n_2}$$

If LLFPP is infeasible or unbounded, then DBLFPP is also infeasible or unbounded, as appropriate. Let the value of the objective function of LLFPP be Z_2 corresponding to the optimal solution $(\hat{X}_1,\ \hat{X}_2)$. If $(\hat{X}_1,\ \hat{X}_2)$ is an integer solution, then Z_2 is an upper bound on the value of the lower level objective function for any bilevel feasible solution.

Suppose that the solution $(\hat{X}_1,\ \hat{X}_2)$ obtained on solving LLFPP is not an integer solution. Choose a variable which is not an integer from this solution. This variable is called the branching variable. This branching variable is used to formulate two feasible sets S' and S'' from the original feasible set S. Define the two constrained sets S' and S'' as follows

$$S' = S \cap \left\{X_2 \le \left[\hat{X}_2\right]\right\}, \quad S'' = S \cap \left\{X_2 \ge \left[\hat{X}_2\right] + 1\right\}$$

Thus, the two subproblems generated, namely DBLFPP′ and DBLFPP″, are defined on the feasible sets S' and S'', respectively. The two subproblems are defined as

- DBLFPP′:

$$\underset{X_1}{\text{Max}}\ F_1(X_1, X_2) = \frac{c_{11}X_1 + c_{12}X_2 + \alpha_1}{d_{11}X_1 + d_{12}X_2 + \beta_1}$$

subject to $(X_1, X_2) \in S'$.

- DBLFPP″:

$$\underset{X_1}{\text{Max}}\ F_1(X_1, X_2) = \frac{c_{11}X_1 + c_{12}X_2 + \alpha_1}{d_{11}X_1 + d_{12}X_2 + \beta_1}$$

subject to $(X_1, X_2) \in S''$.

The problem DBLFPP′. Consider DBLFPP′. Let the optimal solution obtained after solving DBLFPP′ be (X_1', X_2'). Two cases arise:

1. If (X_1', X_2') is an integer solution, then solve the follower's problem for $X_1 = X_1'$. Let the solution of the follower's problem be $(X_1', \hat{X}_2')$.

If $X_2' = \hat{X}_2'$, then this solution is bilevel feasible. Also, if $F_1(X_1', X_2')$ is greater than the largest feasible value found so far of $F_1(X_1, X_2)$, let $UB = F_1(X_1', X_2')$. The value of the objective function corresponding to this integer bilevel feasible solution is a lower bound on the optimal value of the leader in DBLFPP.

If $X_2' \neq \hat{X}_2'$, the solution is not bilevel feasible, thus prune the current node.

2. If (X_1', X_2') is not an integer solution, then select a node for branching. Choose a branching variable. Solve the resulting two linear fractional programming problems and proceed as in case 1.

The problem DBLFPP″. Consider the problem DBLFPP″. At any stage, when solving DBLFPP″ there are three possibilities:

1. DBLFPP″ is infeasible, thus the node is closed.
2. S'' is empty. The problem has no solution.
3. S'' contains an optimal solution and DBLFPP″ is solved to find the optimal solution.

If the solution obtained is an integer solution, then it is checked for bilevel feasibility. If the solution obtained is not an integer solution, then that node is pruned.

Solution to the problem DBLFPP. Solve the two problems DBLFPP′ and DBLFPP″ as described above. Accumulate all the integer solutions from these two problems which are bilevel feasible. The integer bilevel feasible solution that gives the maximum value to the objective function $F_1(X_1, X_2)$ is an optimal solution for the problem DBLFPP.

Algorithm for the DBLFPP.

Step 1. Define the LLFPP problem for the bilevel linear fractional programming problem with discrete variables, DBLFPP. Let its optimal solution be $(\hat{X}_1, \hat{X}_2)$. If it is an integer solution, stop.

If $(\hat{X}_1, \hat{X}_2)$ is not an integer solution, choose a branching variable and correspondingly define the feasible sets S' and S'' as

$$S' = S \cap \left\{X_2 \le \left[\hat{X}_2\right]\right\}, \quad S'' = S \cap \left\{X_2 \ge \left[\hat{X}_2\right] + 1\right\}$$

Formulate the respective problems DBLFPP′ and DBLFPP″.

Step 2. Solve the subproblem DBLFPP′. Let its optimal solution be (X_1', X_2').

If it is an integer solution, go to step 4.

If (X_1', X_2') is not an integer solution, go to step 3.

Step 3. Select a node and choose a branching variable to formulate two subproblems.

Go to step 2.

Step 4. Set $X_1 = X_1'$ in the follower's problem. Let the solution so obtained be $(X_1', \hat{X}_2')$.

If $X_2' = \hat{X}_2'$, go to step 5. If $X_2' \neq \hat{X}_2'$, prune the current node.

Step 5. (X_1', X_2') is a bilevel feasible solution. Go to step 9.

Step 6. Check the constrained set S''. If S'' is empty, DBLFPP″ has no solution.

If S'' is non-empty, solve the problem DBLFPP″ and go to step 7.

Step 7. If DBLFPP″ is infeasible, prune the current node. If DBLFPP″ is feasible, solve the problem to find its optimal solution. If the optimal solution so obtained is an integer solution, go to step 8, otherwise prune the current node.

Step 8. Check the integer solution for bilevel feasibility. Go to step 9.

Step 9. Collate the set of bilevel feasible solutions. Out of these solutions, find a solution which maximizes the value of the objective function $F_1(X_1, X_2)$. This is the optimal solution of DBLFPP.

Illustrative example. Consider the following discrete bilevel linear fractional programming problem

$$\operatorname*{Max}_{x_1} F_1\left(x_1, x_2, x_3\right) = \frac{3x_1 + 2x_2 + 2x_3}{4x_1 + 2x_3 + 6}$$

$$\operatorname*{Max}_{x_2, x_3} F_2\left(x_1, x_2, x_3\right) = \frac{x_1 + x_2 + x_3 + 4}{x_3 + 8}$$

subject to $x_2 + 4x_3 \le 9,\ \ x_1 + 2x_2 \le 5,\ \ x_1 + 3x_3 \le 13,\ \ x_1, x_2, x_3 \ge 0$ and integers.

Solution. Define the auxiliary Min–Max problem as LLFPP:

$$\operatorname*{Min}_{x_1, x_2, x_3} F_2(x_1, x_2, x_3) = \frac{x_1 + x_2 + x_3 + 4}{x_3 + 8}$$

subject to $x_1 \geq 0,$

$$x_2, x_3 \in \arg \max_{x_2,\, x_3} F_2\left(x_1, x_2, x_3\right)$$

subject to $x_2 + 4x_3 \leq 9,\ \ x_1 + 2x_2 \leq 5,\ \ x_1 + 3x_3 \leq 13,\ \ x_2, x_3 \geq 0.$

Solving LLFPP, the optimal solution so obtained is $x_1 = 0,\ x_2 = 5/2,\ x_3 = 13/8.$

Here, neither x_2 nor x_3 are integers. We choose x_2 to define the constrained sets S' and S'' as

$$S_1' = S \cap \left\{x_2 \leq 2\right\},\ \ S_2'' = S \cap \left\{x_2 \geq 3\right\}$$

Define DBLFPP$_1'$:

$$\max_{x_1,\, x_2,\, x_3} F_1(x_1, x_2, x_3) = \frac{3x_1 + 2x_2 + 2x_3}{4x_1 + 2x_3 + 6}$$

subject to $x_2 + 4x_3 \leq 9,\ \ x_1 + 2x_2 \leq 5,\ \ x_1 + 3x_3 \leq 13,\ \ x_2 \leq 2,\ \ x_1,\ x_2,\ x_3 \geq 0$ and integers.

DBLFPP$_1''$

$$\max_{x_1,\, x_2,\, x_3} F_1(x_1, x_2, x_3) = \frac{3x_1 + 2x_2 + 2x_3}{4x_1 + 2x_3 + 6}$$

subject to $x_2 + 4x_3 \leq 9,\ x_1 + 2x_2 \leq 5,\ \ x_1 + 3x_3 \leq 13,\ x_2 \geq 3,\ x_1,\ x_2,\ x_3 \geq 0$ and integers.

Solve DBLFPP′. The solution obtained is $x_2 = 2,\ \ x_3 = 7/4.$

Select the branching variable, x_3, and formulate two subproblems as follows

DBLFPP$_2'$:

$$\max_{x_1,\, x_2,\, x_3} F_1(x_1, x_2,\, x_3) = \frac{3x_1 + 2x_2 + 2x_3}{4x_1 + 2x_3 + 6}$$

subject to $x_2 + 4x_3 \leq 9,\ x_1 + 2x_2 \leq 5,\ \ x_1 + 3x_3 \leq 13,\ x_2 \leq 2,\ x_3 \leq 1,\ x_1,\ x_2,\ x_3 \geq 0$ and integers.

DBLFPP$_2''$:

$$\max_{x_1,\, x_2,\, x_3} F_1(x_1, x_2, x_3) = \frac{3x_1 + 2x_2 + 2x_3}{4x_1 + 2x_3 + 6}$$

subject to $x_2 + 4x_3 \le 9,\ x_1 + 2x_2 \le 5,\ x_1 + 3x_3 \le 13,\ x_2 \le 2,\ x_3 \ge 2,\ x_1,\ x_2,\ x_3 \ge 0$ and integers.

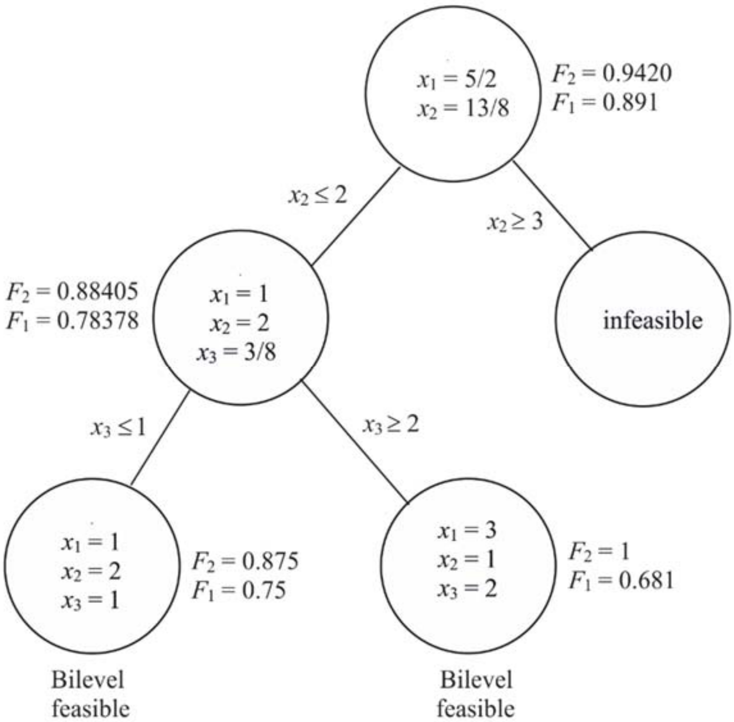
Solve DBLFPP$_2'$. The integer solution so obtained is $x_1 = 1,\ x_2 = 2,\ x_3 = 1$.

To check whether this solution is bilevel feasible, set $x_1 = 1$ in the follower's problem and solve the resulting problem:

$$\underset{x_2,\, x_3}{\text{Max}}\ F_2\left(x_1, x_2, x_3\right) = \frac{x_2 + x_3 + 5}{x_3 + 8}$$

subject to $x_2 + 4x_3 \le 9,\ \ 2x_2 \le 4,\ 3x_3 \le 12,\ x_2 \le 2,\ x_3 \le 1,\ x_2,\ x_3 \ge 0$ and integers.

The solution of the above problem is $x_2 = 2,\ \ x_3 = 1$. (1, 2, 1) is a bilevel feasible solution. Again, solving DBLFPP$_2''$, the bilevel feasible solution is (3, 1, 2). Similarly, if we choose x_3 as the branching variable, we obtain the same set of bilevel feasible solutions. The flow diagram for the above problem is as follows:



Considering the solutions from the above nodes, we have

$$F_2\text{: } 0.875 < 0.884 < 0.9420 < 1,\ F_1\text{: } 0.681 < 0.75 < 0.78 < 0.89$$

We get two sets of bilevel feasible solutions, (1, 2, 1) and (3, 1, 2). The maximum value of $F_1(X_1, X_2)$ corresponds to the integer bilevel feasible solution (1, 2, 1). Hence, the optimal solution for DBLFPP is $x_1 = 1, \ x_2 = 2, \ x_3 = 1,$ for which $F_1 = 0.75$ and $F_2 = 0.875$. We observe that 0.9420 is an upper bound on the value of the lower level objective function.

3. Multilevel linear fractional programming problem with discrete variables

The mathematical representation of the K-level programming problem is as follows:
DMLLFPP:

$$\underset{X_1}{\text{Max}}\ F_1(X) = \frac{c_{11}X_1 + c_{12}X_2 + ... + c_{1K}X_K + \alpha_1}{d_{11}X_1 + d_{12}X_2 + ... + d_{1K}X_K + \beta_1}$$

$$\underset{X_2}{\text{Max}}\ F_2(X) = \frac{c_{21}X_1 + c_{22}X_2 + ... + c_{2K}X_K + \alpha_2}{d_{21}X_1 + d_{22}X_2 + ... + d_{2K}X_K + \beta_2}, \text{ for a given } X_1$$

$$\cdots$$

$$\underset{X_K}{\text{Max}}\ F_K(X) = \frac{c_{K1}X_1 + c_{K2}X_2 + ... + c_{KK}X_K + \alpha_K}{d_{K1}X_1 + x_{K2}X_2 + ... + d_{KK}X_K + \beta_K} \quad \text{for a given } (X_1,\ X_2, ..., X_{K-1})$$

subject to $A_{i1}X_1 + A_{i2}X_2 + ... + A_{iK}X_K = b_i, \ i = 1, 2, ..., m$

$$X_1 \in Z_+^{n_1}, \ \ X_2 \in Z_+^{n_2}, ... \ X_K \in Z_+^{n_K}$$

The above problem has one decision maker at each level, n decision variables and m constraints. $X = (X_1, X_2, ..., X_k), \ \ n = n_1 + n_2 + ... + n_K,$ where the decision vector $X_k \in Z_+^{n_k}$, $k = 1, 2, ..., K$ is under the control of the kth level decision maker who has n_k decision variables.

Here

$$(d_{j1}X_1 + d_{j2}X_2 + ... + d_{jK}X_K + \beta_j) > 0, \ \ j = 1, \ 2, ..., K, \ \ \forall\, (X_1, X_2, ..., \ X_K) \in S_M$$

where

$$S_M = \{(X_1, X_2, ..., X_K); \ \ A_{i1}X_1 + A_{i2}X_2 + ... + A_{iK}X_K = b_i, (X_1, X_2, ..., \ X_K)$$

is a vector of integers.

Here, S_M is the feasible region of the discrete multilevel linear fractional programming problem DMLLFPP. In the problem DMLLFPP, the decision makers at each level solve a linear fractional programming problem. Therefore, it is quasi-concave and its optimal solution will be at an extreme point of S_M.

Auxiliary problem for DMLLFPP. Define the t-auxiliary problem (t = 2, ..., K) as AMLLFPP:

$$\underset{X_t}{\text{Min}}\, F_t(X) = \frac{c_{t1}X_1 + c_{t2}X_2 + ... + c_{tt}X_K + \alpha_t}{d_{t1}X_1 + d_{t2}X_2 + ... + d_{tt}X_K + \beta_t}$$

subject to $X_1 \in R_+^{n_1}$

$$X_t \in \arg \underset{X_t}{\text{Max}}\, F_t(X), \quad t = 2, ..., K$$

subject to $A_{i1}X_1 + A_{i2}X_2 + ... + A_{iK}X_K = b_i, \;\; i = 1, 2, ..., m,$ where

$$X_t \in R_+^{n_t}, \quad t = 2, ..., K$$

If each of the auxiliary problems (t = 2, ..., K) is feasible, then the problem DMLLFPP is feasible. Let Z_2, Z_3, ..., Z_K be the values of the objective functions corresponding to the optimal solution obtained on solving the auxiliary problems for t = 2, ..., K.

Let $Z^* = M\,\text{ax}\{Z_2, Z_3, ..., Z_K\}$. Then Z^* is an upper bound on the value of the lower level objective functions for any feasible solution. Let the solutions obtained on solving the auxiliary problems AMLLFPP be $(\hat{X}^t),\; t = 2, ..., K,$ where

$$(\hat{X}^t) = (\hat{X}_1^t,\, \hat{X}_2^t,\, \hat{X}_3^t,\, ...,\, \hat{X}_k^t) \quad t = 2, ..., K$$

From these (K – 1) solutions, choose the variables which are not integer variables. These variables are called branching variables. Accordingly, define the constrained sets S_j^1 and S_j^2 as

$$S_j^1 = S_M \cap \left[X_j \le \text{Max}\left\{ \left[\hat{X}_j^2 \right],\; \left[\hat{X}_j^3 \right], ..., \left[\hat{X}_j^k \right] \right\},\;\; j = 2, ..., k \right]$$

$$S_j^2 = S_M \cap \left[X_j \ge \text{Max}\left\{ \left[\hat{X}_j^2 \right] + 1, \left[\hat{X}_j^3 \right] + 1, ..., \left[\hat{X}_j^k \right] + 1 \right\},\;\; j = 2, ..., k \right]$$

Algorithm for multilevel discrete linear fractional programming problem DMLLFPP

Step 1. Consider the problem DMLLFPP.

Step 2. Define the auxiliary problems AMLLFP for $t = 2, ..., K$.

Step 3. Solve the problem AMLLFP for each $t = 2, ..., K$.

If the solution obtained is an integer solution, stop. Go to step 10.

If the solution so obtained is not an integer solution, choose the variables which are not integers as branching variables. Define the constrained sets S_j^1 and S_j^2 as

$$S_j^1 = S_M \cap \left[X_j \le \text{Max} \left\{ \left[\hat{X}_j^2 \right], \left[\hat{X}_j^3 \right], ..., \left[\hat{X}_j^k \right] \right\},\ j = 2, ..., k \right]$$

$$S_j^2 = S_M \cap \left[X_j \ge \text{Max}\ \left\{ \left[\hat{X}_j^2 \right] + 1, \left[\hat{X}_j^3 \right] + 1, ..., \left[\hat{X}_j^k \right] + 1 \right\},\ \ j = 2, ..., k \right]$$

Formulate the problems DMLLFPP_j^1 and DMLLFPP_j^2 corresponding to the sets S_j^1 and S_j^2.

Step 4. Set $j = 2$.

Step 5. Solve the problem $\text{DMLLFPP}^1{}_j$. Let its optimal solution be $(X_1^j,\ X_2^j,\ ..., X_K^j)$. If it is an integer solution, go to step 8.

Otherwise, choose the branching variables to formulate the sub problems and solve till we get an integer solution. Go to step 8.

Step 6. Consider the problem DMLLFPP_j^2 corresponding to the set S_j^2.

If S_j^2 is empty, DMLLFPP_j^2 has no solution. If S_j^2 is non-empty, solve DMLLFPP_j^2. Go to step 7.

Step 7. If DMLLFPP_j^2 is infeasible, prune the current node. If DMLLFPP_j^2 is feasible, find its optimal solution. If the solution so obtained is an integer solution, go to step 8.

Otherwise, prune the current node.

Step 8. Set $X_1 = X_1^j$ in the first follower's problem and solve it. Let the optimal solution be $(X_1^j,\ X_2^j, ..., X_K^j)_2$. If $(X_1^j,\ X_2^j,\ ...,\ X_K^j)_1 = (X_1^j,\ X_2^j,\ ...,\ X_K^j)_2$, go to step 9.

Otherwise, set $j = j + 1$ and go to step 4.

Step 9. Put $X_1 = X_1^j$ and $X_2 = X_2^j$ in the second follower's problem and continue the process. If at any stage $(X_1^j, ..., X_K^j)_s \ne (X_1^j, ..., X_K^j)_l$ $(s = 1, 2, ..., \text{K},\ \ l = 1, 2, ..., K)$, stop.

Set $j = j + 1$, go to step 4.

If $(X_1^j, ..., X_K^j)_s = (X_1^j, ..., X_K^j)_l$, go to step 10.

Step 10. From all the integer solutions so obtained, formulate a set of feasible solutions. From this set, find a solution which maximizes the value of the objective function $F_1(X)$. This is the optimal solution of the problem DMLLFPP.

Illustrative example. Consider the following trilevel discrete linear fractional programming problem, DMLLFPP:

$$\underset{x_1}{\text{Max}}\ F_1(x_1, x_2, x_3, x_4) = \frac{2x_1 + 4x_2 + x_4}{2x_1 - 2x_2 + x_3 + 3x_4 + 1}$$

$$\underset{x_2}{\text{Max}}\ F_2(x_1,\ x_2, x_3, x_4) = \frac{3x_2 - x_4 + 12}{-x_1 + 2x_2 + 2} \text{ for a given } x_1$$

$$\underset{x_3, x_4}{\text{Max}}\ F_3(x_1, x_2, x_3, x_4) = \frac{x_1 + 2x_4 + 11}{x_3 + 1} \text{ for a given } x_2$$

subject to $x_2 + x_3 + 2x_4 \le 20$, $7x_2 + 4x_4 \le 36$, $-x_2 + 4x_4 \le 14$, $x_1 + 3x_2 + 2x_4 \le 11$, $x_1 + 2x_2 \le 5$, $x_1,\ x_2,\ x_3,\ x_4 \ge 0$ and integers.

Solution. Here, we have 2-auxiliary problems, corresponding to F_2 and F_3. Solving these problems, we obtain (0, 5/2, 7/4) with $Z_2 = 0.5238$ and (0, 8/7, 79/7, 53/14) with $Z_3 = 1.2380$, respectively. Therefore, $Z^* = \text{Max}\ (Z_2, Z_3) = \text{Max}\ (0.5238, 1.2380) = 1.2380$. Therefore, $Z^* = 1.2380$ is an upper bound on the value of the lower level objective functions for any feasible solution. Here, x_2 and x_4 are the branching variables.

Define the constrained sets as

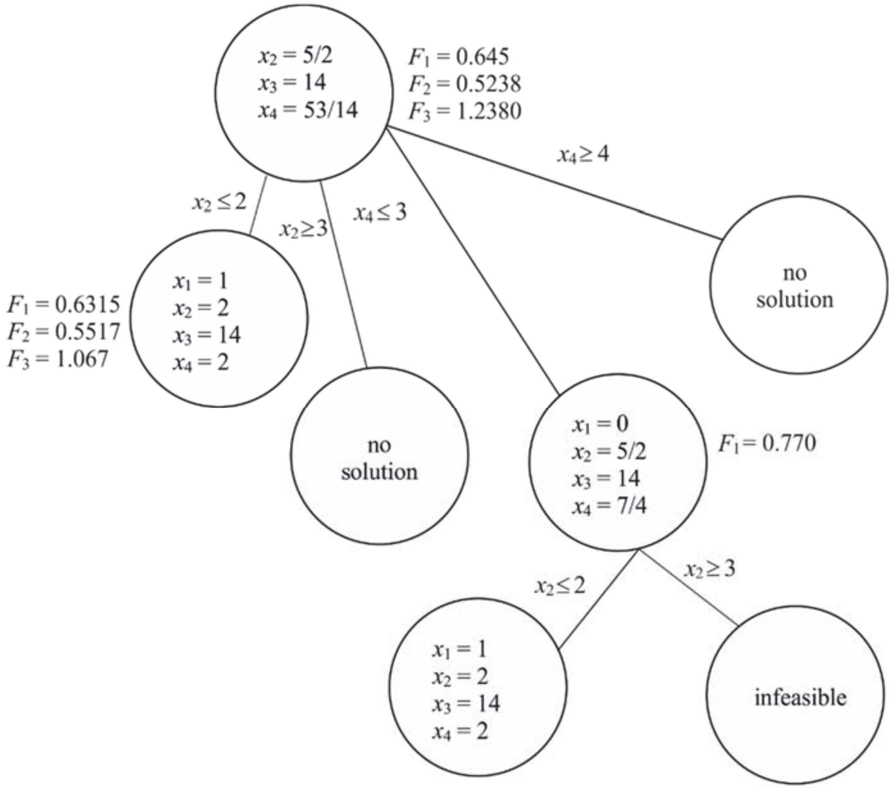
$$S_2^1 = S_M \cap \left[x_2 \le \text{Max}\left\{ \left[\frac{5}{2}\right], \left[\frac{8}{7}\right] \right\} \right]$$

$$S_2^2 = S_M \cap \left[x_2 \ge \text{Max}\left\{ \left[\frac{5}{2}\right] + 1, \left[\frac{8}{7}\right] + 1 \right\} \right]$$

$$S_4^1 = S_M \cap \left[x_4 \le \text{Max}\left\{ \left[\frac{7}{4}\right], \left[\frac{53}{14}\right] \right\} \right]$$

$$S_4^2 = S_M \cap \left[x_4 \ge \text{Max}\left\{ \left[\frac{7}{4}\right] + 1, \left[\frac{53}{14}\right] + 1 \right\} \right]$$

Applying the algorithm to the feasible sets defined above, we obtain



The optimal solution to the problem DMLLFPP is (1, 2, 14,2) with $\text{Max}F_1 = 0.6315$, $\text{Max}F_2 = 0.5517 < Z^*$ and Max $F_3 = 1.067 < Z^*$.

4. Conclusions

In this paper, we discussed discrete bilevel linear fractional programming problem DBLFPP. A branch and bound algorithm to solve DBLFPP has been presented. In this method, an inequality is developed which is used to cut the integer points which are not bilevel feasible. This inequality is used to define two subproblems, DBLFPP′ and DBLFPP″. In this algorithm, iterations of branch and bound cut are applied to both the problems DBLFPP′ and DBLFPP″. It has been observed that DBLFPP′ is more likely to contain the optimal solution. This paper has also presented an extension of this branch and bound algorithm to Multilevel Linear Fractional Programming Problem with Discrete Variables DMLLFPP. The formulation presented above corresponds to an optimistic approach from the point of view of the leader. According to this optimistic approach, the leader assumes that the followers are willing to support him. This means that out of the set of optimal solutions for a lower level problem, the follower will select a solution which is best from the leader's point of view.

Acknowledgements

The authors are extremely thankful to the Editor and Reviewers for their valuable suggestions to improve the quality of this paper.

References

[1] BARD J.F., FALK J.E., *An explicit solution to the multilevel programming problem*, Comp. Oper. Res., 1982, 9, 77–100.
[2] BARD J.F., MOORE J.J., *A branch and bound algorithm for the bilevel programming problem*, SIAM J. Sci. Stat. Comp., 1990, 11, 281–292.
[3] BHARGAVA S., *Solving linear fractional multilevel programs*, Oper. Res. Dec., 2014, 24,1, 1–17.
[4] BIENSTOCK D., *Computational study of a family of mixed integer quadratic programming problems*, Math. Progr., 1996, 74, 121–140.
[5] CALVETE H.I., GALE C., *Local optimality in quasiconcave bilevel programming*, Monog. Semin. Matem. Gracia de Galdeano, 2003, 27, 153–160.
[6] CALVETE H.I., GALE C., *A penalty method for solving bilevel linear fractional/linear programming problem*, Asia Pacific J. Oper. Res., 2004, 21, 2, 207–224.
[7] CANDLER W., NORTON R., *Multilevel programming*, Technical Report 20, World Bank Development Research Centre, Washington, D.C., 1977.
[8] CHARNES A., COOPER W.W., *Programming with linear fractional functional*, Naval Res. Log. Quart., 1962, 9, 181–186.
[9] CLAASSEN G.D.H., *Mixed integer (0–1) fractional programming for decision support in paper production industry*, Omega, 2014, 43, 21–29.
[10] COSTA L.H.M., PRATA B.A., RAMOS H.M., CASTRO M.A.H., *A branch and bound algorithm for optimal pump scheduling in water distribution networks*, Water Res. Manage., 2016, 30 (3), 1037–1052.
[11] DEMPE S., KALASHNIKOV V., RIOS-MERCADO R.Z., *Discrete bilevel programming. Application to a natural gas cash-out problem*, Eur. J. Oper. Res., 2005, 166 (2), 469–488.
[12] DINKELBACH W., *On non-linear fractional programming*, Manage. Sci., 1967, 13, 492–498.
[13] FALK J.E., SOLAND R.M., *An algorithm for solving separable non-convex programming problems*, Manage. Sci., 1969, 15, 550–569.
[14] HERNANDEZ M., GOMEZ T., MOLINA J., LEON M.A., *Applications of multi-criteria techniques to plan the harvest of a forest taken into account different kinds of objectives*, Intelligence Systems in Environmental Management. Theory and Applications, Part of the Intelligent Systems Reference Library Book Series, ISRL, 2017, 113, 367–383.
[15] LAND A.H., DOIG A.G., *An automatic method for solving discrete programming problems*, Econometrica, 1960, 28, 497–520.

[16] LU J., HAN J., HU Y., ZHANG G., *Multilevel decision making. A survey*, Inf. Sci., 2016, (346–347), 463–487.
[17] LUEDTKE J., *A branch and cut decomposition algorithm for solving chance-constrained mathematical programs with finite support*, Math. Prog., 2014, 146 (1–2), 219–244.
[18] MAROTI G., *A branch and bound approach for robust railway time tabling*, Public Transp., 2017, 9 (1–2), 73–94.
[19] MATHUR K., PURI M.C., *On bilevel fractional programming*, Optimization, 1995, 35, 215–226.
[20] MIGDALAS A., PARDALOS P.M., VARBRAND P., *Multilevel optimization. Algorithms and Applications*, Vol. 20, Springer, 1998.
[21] MURTY K.G., *Solving the fixed charge problem by ranking the extreme points*, Oper. Res., 1969, 16, 268–279.
[22] PAL B.B., GUPTA S., *A genetic algorithm based fuzzy goal programming approach for solving fractional bilevel programming problems*, Int. J. Oper. Res., 2012, 14, 4, 453–471.
[23] PEREZ J.C., CARRILLO M.H., MONTOYA-TORRES J.R., *Multi-criteria approaches for urban passenger transport systems. A literature review*, Ann. Oper. Res., 2015, 226 (1), 69–87.
[24] SCHIABLE S., *Fractional programming applications and algorithms*, Eur. J. Oper. Res., 1981, 7, 111–120.
[25] SWARUP K., *Linear fractional functional programming*, Oper. Res., 1965, 13 (6), 1029–1036.
[26] WHITE D.J., ANANDALINGAM G., *A penalty function approach for solving bilevel linear programs*, J. Global Opt., 1993, 3, 397–419.

Received 13 November 2017
Accepted 14 May 2018