OPERATIONS RESEARCH AND DECISIONS

No. 1 2018
DOI: 10.5277/ord180105

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AN APPROXIMATION ALGORITHM FOR MULTI-UNIT AUCTIONS. NUMERICAL AND SUBJECT EXPERIMENTS

In multi-unit auctions for a single item, the Vickrey–Clarke–Groves mechanism (VCG) attains allocative efficiency but suffers from its computational complexity. Takahashi and Shigeno thus proposed a greedy based approximation algorithm (GBA). In a subject experiment there was truly a difference in efficiency rate but no significant difference in seller's revenue between GBA and VCG. It is not clear in theory whether each bidder will submit his or her true unit valuations in GBA. We show, however, that in a subject experiment there was no significant difference in the number of bids that obey "almost" truth-telling between GBA and VCG. As for individual bidding behavior, GBA and VCG show a sharp contrast when a human bidder competes against machine bidders; underbidding was observed in GBA, while overbidding was observed in VCG. Some results in a numerical experiment are also provided prior to reporting those observations.

Keywords: *multi-unit auctions, approximation algorithm, experiment*

1. Introduction

In multi-unit auctions for a single item, the Vickrey–Clarke–Groves mechanism (VCG) attains allocative efficiency but suffers from its computational complexity. In fact, the item allocation problem is known to be *NP*-hard, and thus it is necessary to apply some approximation algorithm to that problem. Kothari et al. [8] considered the item

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allocation problem in reverse auctions as a generalized knapsack problem and proposed a greedy based 2-approximation algorithm with $O(\ell^2)$ time, where ℓ is the total sum of numbers of bidders' anchor values. As far as non-reverse auctions are concerned, however, their algorithm does not necessarily return a solution the approximation ratio of which is not bounded by two. (We can provide such an example upon request.) Takahashi and Shigeno [9] thus proposed another greedy based 2-approximation algorithm (GBA) with $O(\ell^2(\log n + l_{\max}))$ time, where $l_{\max}$ is the maximum number of anchor values among those of bidders.[4]

In GBA, the highest unit bidder is tentatively given the unit and the other unit bids of the tentative winner are updated in the process for determining the final item allocation. The results of this paper are as follows. In a subject experiment there was truly a difference in efficiency rate but no significant difference in seller's revenue between GBA and VCG. It is not clear in theory whether each bidder has an incentive to submit his or her true unit valuations[5] in GBA. In the subject experiment there was no significant difference in the number of bids that obey "almost" truth-telling between GBA and VCG. As for individual bidding behaviour, GBA and VCG show a sharp contrast when a human bidder competes against machine bidders; underbidding was observed in GBA, while overbidding was observed in VCG. Some results in a numerical experiment are also provided prior to reporting our main observations.

In the numerical and subject experiments, it is assumed that for all bidders, each unit valuation is drawn independently of the other unit valuations, i.e., in the random order. We might alternatively assume that unit valuations are given to each bidder in monotone non-increasing order. In a preliminary experiment for VCG conducted prior to our main sessions, however, the standard deviation of seller's revenue observed in monotone non-increasing order of unit valuations is much larger than the one observed in the random order of those. We thus conducted the experiments with unit valuations that were drawn in the random order.

The rest of this paper is organized as follows. Section 2 introduces the model of multi-unit auctions for a single item and describes how GBA and VCG derive allocations of the item. Section 3 displays the results of a numerical experiment on computation time and efficiency rate under the assumption of truth-telling bidding. Section 4 shows our main observations in a subject experiment. Section 5 notes some remarks, referring to a preliminary experiment for VCG and other papers related to our results.

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[4]Takahashi and Shigeno [9] also developed another 2-approximation algorithm, which is based on Dyer's linear time algorithm [4], and they showed in a numerical experiment that the GBA computed faster and approximated better than that alternative algorithm.

[5]The unit valuation is called unit-price by Kothari et al. [8].

2. Model

2.1. The Vickrey–Clarke–Groves (VCG) mechanism

Consider a multi-unit auction for a single item, where a seller wishes to sell units M of a single item and solicits bids from n buyers. Let $N = \{1, ..., n\}$ be the finite set of buyers (bidders). In this model, each bidder divides a closed interval $[0, M]$ into discontinuous ranges, where the discontinuity points of these ranges are called anchor values. For each bidder $i \in N$, denote the total number of his or her anchor values by ℓ_i and his or her anchor values by $\{d_i^k \mid k = 0, ..., \ell_i\}$, where $d_i^{k-1} < d_i^k$ for all k with $1 \le k \le \ell_i$. Further, for each bidder $i \in N$, denote his or her unit bids by $\{b_i^k \mid k = 1, ..., \ell_i\}$, where b_i^k is a buyer price in half-open range $(d_i^{k-1}, d_i^k]$ for $k = 1, ..., \ell_i$. It is assumed that $d_i^0 = 0$ and $d_i^{\ell_i} \le M$. Let $\ell = \sum_{i \in N} \ell_i$. Define bidder i 's bid function $B_i : \Re_+ \to \Re$ by

$$B_i(y) = \begin{cases} b_i^k y & (d_i^{k-1} < y \le d_i^k,\ k = 1, ..., \ell_i) \\ 0 & (y = d_i^0,\ y > d_i^k) \end{cases} \tag{1}$$

Figure 1 illustrates an example of the bid function.

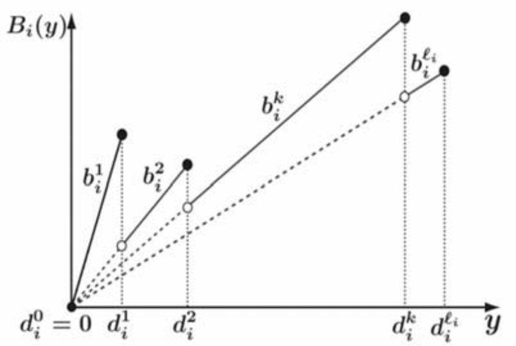

Fig. 1. A bid function. The unit bids represent the gradients of the bid function and the anchor values stand for its discontinuity points

For each bidder $i \in N$, denote the unit valuations of the item by $\{v_i^k \mid k = 1, ..., \ell_i\}$. Define bidder i 's valuation function $V_i : \Re_+ \to \Re$ by

$$V_i(y) = \begin{cases} v_i^k y & (d_i^{k-1} < y \le d_i^k,\ k = 1, \ldots, \ell_i) \\ 0 & (y = d_i^0,\ y > d_i^k) \end{cases} \tag{2}$$

A vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ that satisfies $\sum_{i \in N} x_i \le M$ and $x_i \ge 0$ for any $i \in N$ is called an allocation, where x_i is the unit of the item assigned to bidder $i \in N$ in the allocation. This model may be applied to allocation problems also for a divisible good. In what follows, however, we confine our attention to the case of an indivisible good. Thus, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vector of non-negative integers. An item allocation problem $(AP)_B$ is to find allocations that maximize the total amount of bids, which is formulated by

$$(AP)_B \quad \begin{array}{ll} \max & \sum_{i \in N} B_i(x_i) \\ \text{s.t.} & \sum_{i \in N} x_i \le M \\ & x_i \ge 0 \quad (i \in N) \end{array} \tag{3}$$

Another problem $(AP)_V$ is formulated in the same way by

$$(AP)_V \quad \begin{array}{ll} \max & \sum_{i \in N} V_i(x_i) \\ \text{s.t.} & \sum_{i \in N} x_i \le M \\ & x_i \ge 0 \quad (i \in N) \end{array} \tag{4}$$

in order to find efficient allocations that maximize the total amount of valuations.

The payment scheme is as follows. Denote by $\mathbf{x}^*$ an optimal solution of $(AP)_B$. Let $\mathbf{x}^{-j}$ be an optimal solution of the following restricted item allocation problem $(AP)_B^{-j}$ with the set of bidders $N^{-j} = N \setminus \{ j \}$.

$$(AP)_B^{-j} \quad \begin{array}{ll} \max & \sum_{i \in N^{-j}} B_i(x_i) \\ \text{s.t.} & \sum_{i \in N^{-j}} x_i \le M \\ & x_i \ge 0 \quad (i \in N^{-j}) \end{array} \tag{5}$$

In the VCG, bidder j's payment p_j is determined by

$$p_j = \sum_{i \in N^{-j}} B_i\left(x_i^{-j}\right) - \sum_{i \in N^{-j}} B_i\left(x_i^*\right) \tag{6}$$

Under this payment scheme, it is the dominant strategy for each bidder to truthfully tell his or her unit valuations by bidding. Thus, the optimal solutions of $(AP)_B$ maximize the total amount of valuations in $(AP)_V$. We have to, however, compute as many as O(n) times in $(AP)_B$ to find an optimal solution. It becomes more difficult to compute an allocation and payments in realistic time, as the number of either bidders or units of the item is larger. We thus need to find faster approximation algorithms to solve the item allocation problem.

2.2. A greedy based approximation algorithm (GBA)

The GBA proposed by Takahashi and Shigeno [9] uses the slope function [6] $p_i^k : \Re_+ \to \Re$, for any $i \in N$ and all k with $0 < k < \ell_i$. Denote by $p_i^k(y)$ the gradient of bid function B_i between a unit of y and each anchor value d_i^k, i.e.,

$$p_i^k(y) = \frac{(B_i(d_i^k) - B_i(y))}{(d_i^k - y)} \tag{7}$$

The GBA takes the following process of four steps.

Step 1. Set $x_i = 0$ for any $i \in N$.

Step 2. Find a pair (i^*, k^*) such as $p_{i^*}^{k^*}(x_{i^*}) = \max\left\{ p_i^k(x_i) \mid i \in N,\, x_i < d_i^k \right\}$.

If $p_{i^*}^{k^*}(x_{i^*}) \le 0$, then return $\mathbf{x}$, otherwise, update $x_{i^*} = d_{i^*}^{k^*}$.

Step 3. If $\sum_{i \in N} x_i < M$, go to Step 2.

Step 4. Make two solutions $\hat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ by

$$\hat{\mathbf{x}}_i = \begin{cases} x_i & (i \ne i^*) \\ M - \sum_{j \ne i^*} x_j & (i = i^*) \end{cases} \text{ and } \tilde{\mathbf{x}}_i = \begin{cases} 0 & (i \ne i^*) \\ x_{i^*} & (i = i^*) \end{cases}$$

If $\sum_{i \in N} B_i(\hat{x}_i) > \sum_{i \in N} B_i(\tilde{x}_i)$, then return $\hat{\mathbf{x}}$, otherwise, return $\tilde{\mathbf{x}}$.

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[6]Since the slope function values only particular anchor values which satisfies $y < d_i^k$, this function never has a value of 0/0.

The process is initialized in Step 1. In Step 2, GBA finds a pair (i^*, k^*) which maximizes the slope function. This algorithm employs a priority queue to find such a pair in case of ties; the pair with smaller indices is given priority to others. If GBA stops in Step 2, i.e., $p_{i^*}^{k^*}(x_{i^*}) \le 0$, no solution can improve the objective value from the current solution. The returned solution is thus optimal. GBA iterates Step 2 until $\sum_{i \in N} x_i \le M$. In Step 4, GBA makes two solutions $\hat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$. The residual units $M - \sum_{j \ne i^*} x_j$ is allocated to bidder i^* in $\hat{\mathbf{x}}$, while no unit is allocated to any bidder $j \in N^{-i^*}$ in $\tilde{\mathbf{x}}$. GBA compares the objective values of these two solutions and returns the larger one.

The Ausubel auction [1] also has a similar process of updating the other unit bids of the tentative winners, although the updates are made in dynamic ascending-bid auctions. The GBA makes the updates in static auctions, as shown above. This feature of GBA reduces computation time when the number of bidders or the total sum of numbers of bidders' anchor values. The GBA finds an approximate solution of $(AP)_B$, and the objective value obtained by the approximate solution is at least a half of the optimal objective value in $(AP)_B$. These are formally stated as the following theorem.

Theorem 1. GBA finds a 2-approximation solution of $(AP)_B$ in $O(\ell^2(\log n + l_{\max}))$ time, where $l_{\max} = \max_{i \in N} \ell_i$ [9].

The intuition of Theorem 1 on computation time is explained as follows. The number of iteration in GBA is clearly at most ℓ. If we store $\max \left\{ p_i^k(x_i) \mid x_i < d_i^k \right\}$ for all $i \in N$ in a heap, Step 2 can be performed in $O(\log n)$. After Step 2, we need to compute $\max \left\{ p_{i^*}^k(x_{i^*}) \mid x_{i^*} < d_{i^*}^k \right\}$ for updated x_{i^*}, which runs in $O(\ell^2)$. The total running time of Step 2 is thus bounded by $O(\log n + l_{\max})$. Therefore, the total running time is bounded by $O(\ell^2(\log n + l_{\max}))$.

The payment scheme is as follows. Denote by $\breve{\mathbf{x}}$ an allocation determined by GBA. Let $\breve{\mathbf{x}}^{-j}$ be an allocation determined by GBA when the set of bidders is restricted to $N^{-j} = N \setminus \{ j \}$. In GBA, bidder j's payment $\breve{p}_j$ is determined by

$$\breve{p}_j = \sum_{i \in N^{-j}} B_i(\breve{x}_i^{-j}) - \sum_{i \in N^{-j}} B_i(\breve{x}_i) \tag{8}$$

Under this payment scheme, it is not clear in theory whether each bidder will truthfully submit his or her unit valuations by bidding in GBA. It is, however, shown later that in a subject experiment there was no significant difference in the number of bids that obey “almost” truth-telling between GBA and VCG. The definition of the almost truth-telling bidding is noted later in Section 4.

In this section, we did not explain the intuition of Theorem 1 on approximation ratio. Instead, the approximation ratio of GBA against VCG can be confirmed as well as the computation time in a numerical experiment, the results of which are shown in the next section.

3. Numerical experiment

This section displays the results of a numerical experiment which functions as a control group against part of the observations in the corresponding subject experiment, where bidders are all truth-telling machine bidders. All computations were conducted on a personal computer with Core i7 CPU (3.4 GHz) and 16GB memory, and the code was written with Python 2.6.5. (The code is available upon request.) We fix *n* or *M* for instances, varying the values of the other variables.

For each bidder $i \in N$, the number of his or her anchor values, ℓ_i, was independently drawn from the set of 15 integers $\{1, ..., 15\}$ with equal probability. Then, for any bidder $i \in N$, anchor values, the number of which is ℓ_i, were drawn independently from the set of integers $\{1, ..., M\}$ with equal probability, arranged in ascending order, and indexed from 0 to ℓ_i, to construct $\{d_i^k \mid k = 0, ..., \ell_i\}$[7]. As is mentioned in Section 1, for any bidder $i \in N$, each unit valuation v_i^k is independently drawn from the set of 100 integers $\{1, ..., 100\}$ with equal probability. Both in GBA and in VCG, truth-telling bidders were assumed.

We conducted this numerical experiment by using dynamic programming. Consider an arbitrary ordering on *n* bidders. For the first k bidders and m units with $0 \le m \le M$, define

$$T[k, m] := \max \left\{ \sum_{j=1}^{k} V_j(x_j) \mid \sum_{j=1}^{k} x_j \le m,\ x_j \ge 0,\ (1 \le j \le k) \right\},\ (k \in N,\ 0 \le m \le M) \quad (9)$$

The following recurrence relation describes how to solve problem (9) with dynamic programming.

[7]There was no case of a tie observed in this numerical experiment. The description of a tie-break rule has been thus omitted.

Table 1. Averages of computation time
and approximation ratio in GBA against VCG

Instance	Computation time [s]		Approximation ratio
	GBA	EXACT	GBA/EXACT
(50, 50)	0.00227	0.10300	0.727
(50, 100)	0.00201	0.38864	0.736
(50, 150)	0.00190	0.83885	0.623
(50, 200)	0.00217	1.52568	0.723
(50, 250)	0.00202	2.36149	0.633
(50, 300)	0.00222	3.40518	0.775
(50, 350)	0.00215	4.70473	0.815
(50, 400)	0.00203	5.91917	0.874

Each instance is represented by the numbers of bidders and units, i.e., (n, M), where $n = 50$. VCG is noted as EXACT. As the number of units of the item increases, the computation time in VCG increases, whereas GBA suppresses the increase in computation time. The expected value of the upper bound of computation time in GBA, $O(\ell^2(\log n + l_{\max}))$, is kept intact, because n, the expected value of $\ell = \sum_{i \in N} \ell_i$ and the expected value of $l_{\max}$ are not changed.

Table 2. Averages of computation time
and approximation ratio in GBA against VCG

Instance	Computation time [s]		Approximation ratio
	GBA	EXACT	GBA/EXACT
(10, 200)	0.00119	0.70154	0.915
(50, 200)	0.00409	3.44612	0.804
(100, 200)	0.00768	6.81375	0.837
(200, 200)	0.01440	13.54417	0.700
(400, 200)	0.02869	27.49757	0.715
(800, 200)	0.05754	55.84550	0.760
(1000, 200)	0.06933	69.44206	0.760
(5000, 200)	0.34113	348.4999	0.753
(10000, 200)	0.72498	698.56540	0.622

Each instance is represented by the numbers of bidders and units, i.e., (n, M), where $M = 200$. VCG is noted as EXACT. As the number of bidders, n, increases, the upper bound of computation time, $O(\ell^2(\log n + l_{\max}))$, is expected to go up in GBA, because the expected value of $\ell = \sum_{i \in N} \ell_i$ increases, although $l_{\max}$ is always equal to or lower than 15. Even in that case, however, GBA completes the computation much faster than VCG.

$$T[0, m] = 0, \ (0 \le m \le M) \tag{10}$$

$$T[k, m] = \max \begin{cases} T[k-1, m] \\ \max\limits_{1 \le z \le m} \left\{ T[k-1, m-z] + V_k(z) \right\} \end{cases}, (k \in N, 0 \le m \le M) \tag{11}$$

The optimal objective value of $(AP)_V$ can be obtained by

$$\max_{0 \le m \le M} \{T[n, m]\} \tag{12}$$

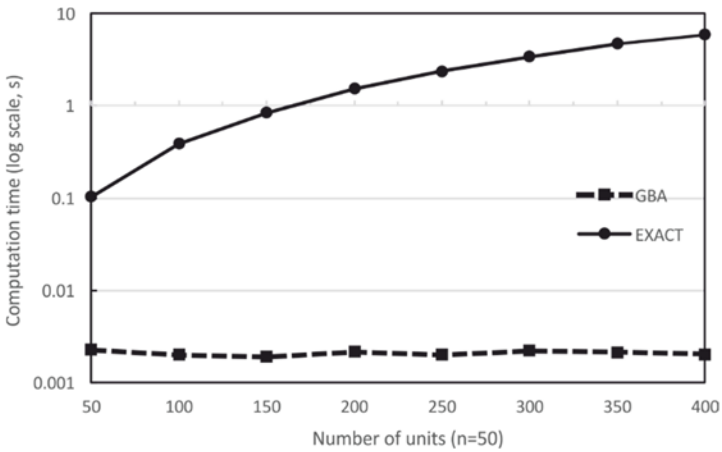

Fig. 2. Number of units of the item and computation time;
$n = 50$, VCG is noted as EXACT

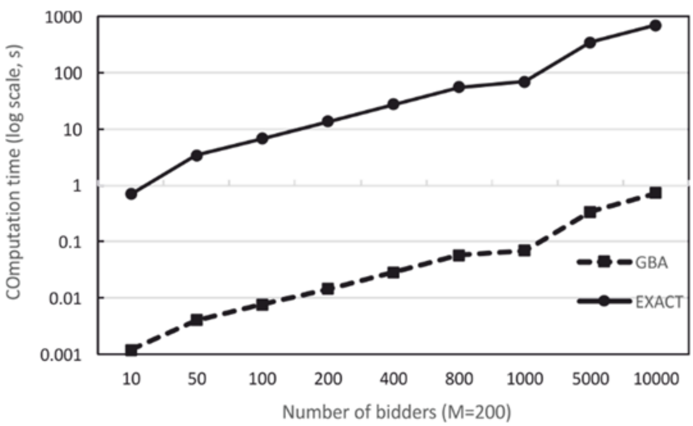

Fig. 3. Number of bidders and computation time;
$M = 200$, VCG is noted as EXACT

Let $n = 50$ or $M = 200$. Tables 1 and 2 show the averages of computation time and approximation ratio in GBA against VCG, where the approximation ratio is defined by

$$\frac{\text{approximate value of } (AP)_V}{\text{optimal value of } (AP)_V} \tag{13}$$

which actually measures the efficiency rate of GBA against VCG. In Tables 1 and 2, VCG is denoted as EXACT in order to indicate that the optimal values are used there. In subject experiments, human bidders do not necessarily behave in such a way that the optimal values of $(AP)_V$ are derived even in VCG. Figures 2 and 3 depict the average computation time which correspond to the instances listed in Tables 1 and 2, respectively.

Table 1 shows that as the number of units of the item increases, the computation time in VCG remarkably increases, whereas the increase in computation time is suppressed in GBA. When the number of bidders, n, does not change, the expected values of $\ell = \sum_{i \in N} \ell_i$ and $l_{\max}$ also do not change, respectively. The expected value of the upper bound of computation time in GBA, O($\ell^2(\log n + l_{\max})$), is then kept intact, as far as n is fixed. As the number of bidders, n, increases, the upper bound of computation time, O($\ell^2(\log n + l_{\max})$), is expected to go up in GBA, because the expected value of $\ell = \sum_{i \in N} \ell_i$ increases, although $l_{\max} \le 15$ always. Even in this case, Table 2 shows that GBA finds the solution of $(AP)_B$ much faster than VCG.

Note that the approximation ratios shown in Tables 1 and 2 are bounded by 0.62 and 0.92. As is mentioned above, those ratios actually measure efficiency rates of GBA against VCG. In the next section, we show that the average rates of efficiency in GBA observed in the subject experiment were more than 0.93, although there was truly a difference between GBA and VCG. We also show, however, that there was no significant difference in seller's revenue between GBA and VCG. The rate of efficiency is defined with the observed value and the optimal value (EXACT) both in GBA and in VCG.

4. Subject experiment

Unlike truth-telling machine bidders in the numerical experiment (Section 3), human bidders might not necessarily report their true unit valuations of the item even in VCG. It may be too complicated for the subjects to understand how VCG works. Even in that case, the VCG mechanism may achieve the higher efficiency due to its algorithmic feature, when bidders take on the behaviour of almost truth-telling. Then, what would be the efficiency and seller's revenue in GBA?

In this section, the rates of efficiency and seller's revenue are defined as the observed value of $(AP)_V$ against the optimal value of $(AP)_V$ and the observed total amount of payments against the optimal total amount of payments, where optimal values are computed with truth-telling machine bidders in VCG.

4.1. Experimental design

In the experiment, 5 units of an item are auctioned off to 3 bidders, where the item is a virtual object, i.e., $n = 3$ and $M = 5$. Each session consists of 20 rounds in total, and 2 sessions are paired; In a session GBA is applied in the first 10 rounds and VCG is applied in the second 10 rounds, and the order of GBA and VCG is reversed in another session. Every subject thus bids in both treatments, although he or she is assigned to only one session. (In analysis, the data should be merged in order to cancel the effect of the order of treatments on the results.) We will explain how subjects were recruited and assigned to a session later at the beginning of Subsection 4.2. Below are the contents described in the instruction of this experiment.

For every bidder *i*, the number of anchor values is set as $\ell_i = 5,$ and thus his or her anchor values are $d_i^0 = 0,$ $d_i^1 = 1$, ..., $d_i^5 = 5$. As is mentioned in Section 1, for each bidder i and for each unit of the item, his or her unit valuation for k units of the item, v_i^k $(k = 1, ..., 5),$ is independently and uniformly distributed over integers between 1 and 200. At the beginning of each round, each bidder i is given his or her unit valuations $\left\{ v_i^k \mid k = 0, ..., \ell_i \right\}$ by a computer, which are his or her private information. (The reasons why each unit valuation was randomly drawn independently of the others will be explained in Section 5.) Each bidder i submits his or her unit bids $\left\{ B_i^k \mid k = 1, ..., \ell_i \right\}$. When k units of the item are allocated to bidder *i*, he or she receives the points that amounts $v_i^k k$ minus his or her payment.

In each round, there is a 120-second time limit for submitting unit bids. If none of three bidders bids within the time limit, every bidder of those three then obtains zero point for that round. The units assigned to a bidder and his or her payment are shown to the bidder in 5 seconds at the end of each round. The cumulative points of bidders are not shown to them. It is thus prohibited for subjects to take notes throughout the session.

Subjects are informed that they will be paid according to the total points they obtain in 6 rounds (3 from the first 10 rounds and 3 from the subsequent 10 rounds) randomly selected by a computer at the end of each session, with the pre-determined exchange rate in addition to the show-up fee. In this experiment, the exchange rate was 1 point = 1 JPY and the show-up fee was 1500 JPY.

At the beginning of each session, GBA (or VCG) is applied in the first 10 rounds with the general instruction. There is an intermission after the first 10 rounds, and then VCG (or GBA) applied in the second 10 rounds is explained. Before proceeding to the experiment, subjects play 1 round for practice to familiarize themselves with the software.

4.2. Results

A computerized laboratory experiment was conducted at the University of Tsukuba in Japan. We developed a software which uses Python CGI for the experiment. We had 4 sessions in February 2014 and 4 sessions in January 2016. Each session conducted in 2016 involved 8 groups of 3 subjects. At the beginning of each round, all subjects were randomly re-grouped into 8 groups by a computer. Subjects were informed of the fact that their two opponents were human bidders but were not informed of who were in the same group. Each session conducted in 2014 involved 8 groups of 1 subject as a human bidder and 2 machine bidders who were programmed as truth-telling bidders. Subjects were informed of the fact that their two opponents were machine bidders who obeyed a theory but were not informed of what theory was applied. At the beginning of each session, each subject was randomly assigned to one of the 8 groups by a computer. This assignment was fixed throughout that session, but this information was not revealed to the subjects.

Table 3. Features of the experimental sessions

Session No.	Machine bidders	Show-up fee (JPY)	Point-to--JPY ratio	# of subjects	Session date	Average point per subject
1	yes	1500	1.0	8	February13, 2014	513.75
2				8		609.63
3				8	February14, 2014	295.38
4				8		169.29
5	no			24	January 30, 2016	510.42
6				24		641.04
7				24	January 31, 2016	284.75
8				24		583.71

Subjects were recruited from all over the campus, and undergraduate students whose major was engineering were most populous among them. Once a subject participated in a session, he or she was prohibited to participate in any other sessions for this experiment. Upon arrival, they were provided with a written instruction, and then the experimenter read it aloud. (The instruction is available upon request. A part of the instruction is provided in the Appendix, where GBA and VCG are explained with examples for two bidders.) Subjects could ask questions regarding the instruction by raising their hand and the experimenter gave the answers to those questions privately. Any communication among subjects was strictly prohibited. Thus, their interactions were only through the information they enter in their computer screens. Each session lasted about 100 min including the instruction. There was no observation of bidding made after the time limit. Features of the experimental sessions are summarized in Table 3.

In Table 3, there were some outliers of unit bids in sessions 3, 4, and 7, which were extraordinarily higher than the corresponding unit valuation. Thus, the average points per subject were lower than those in the other sessions. We dropped extreme outliers in our regression analysis of subjects' bidding behaviour, the result of which is shown later in this section. The rate of efficiency in GBA (or in VCG) are defined by

$$\frac{\text{observed value of } (AP)_V}{\text{optimal value of } (AP)_V} \tag{14}$$

where the observed value of $(AP)_V$ is calculated with an allocation $\mathbf{x}$ observed when GBA (or VCG) is applied. The rate of the seller's revenue (profit) in GBA (or in VCG) is defined by

$$\frac{\text{observed total amount of payments}}{\text{optimal total amount of payments}} \tag{15}$$

The optimal total amount of payments is represented by $\sum_{j \in N} p_j$, where p_j is calculated for each $j \in N$ according to (6). Our main observation is then stated as follows.

Observation 1. In the subject experiment, there was a difference in efficiency rate but no significant difference in seller's revenue between GBA and VCG.

Tables 4 and 5 show the average rates of efficiency and seller's revenue (profit) observed in 2014 and 2016, respectively. Every subject bids in both GBA and VCG, although he or she is assigned to only one session. In analysis, the data should thus be merged in order to cancel the effect of the order of treatments on the results. We analysed the data taken from the last 5 rounds in each treatment, because we allowed subjects the opportunity to learn better bidding behaviour in GBA and VCG. Each treatment had 4 sessions, and there were 8 groups in each session, and thus the sample size is 160 for each treatment.

The *p*-values for the two-sided permutation test (perm) are reported under each panel that corresponds to the rates of efficiency and seller's revenue, respectively. The null hypotheses on the rates of efficiency were rejected at the 5% significance level with both data taken in 2014 and in 2016; the *p*-value is 0.0068 for the sessions conducted in 2014 and it is 0.0001 for the sessions conducted in 2016. The null hypotheses on the rates of seller's revenue, however, could not be rejected at the 5% significance level with both data taken in 2014 and in 2016; the p-value is 0.1703 for the sessions conducted in 2016 and it is 0.8471 for the sessions conducted in 2016[8].

Table 4. Average rates of efficiency
and seller's revenue (profit) in 2014

Parameter	Efficiency		Profit	
	GBA	VCG	GBA	VCG
Mean	0.9341	0.9626	1.0069	0.9828
St. dev.	0.0277	0.0345	0.0651	0.0402
p-Value (perm)	0.0068		0.1703	

Table 5. Average rates of efficiency
and seller's revenue (profit) in 2016

Parameter	Efficiency		Profit	
	GBA	VCG	GBA	VCG
Mean	0.9365	0.9737	0.9037	0.8978
St. dev.	0.0302	0.0220	0.0993	0.0775
p-Value (perm)	0.0001		0.8471	

As noted at the end of Subsection 2.1, the VCG mechanism, in theory, induces allocative efficiency by providing every bidder with an incentive to submit his or her valuations truthfully for each unit (i.e., truth-telling is a dominant strategy). The rates of efficiency shown in Tables 4 and 5, however, both suggest that human bidders should not necessarily report their true unit valuations of the item in VCG as well. Even in that case, the VCG mechanism could achieve high efficiency due to its algorithmic feature, when bidders would take on the behaviour of almost truth-telling.

On the other hand, the results of numerical experiment shown in Section 3 indicate that even under the assumption of truth-telling bidding, GBA should be inferior to VCG in terms of the efficiency rate measured by (15), because the approximation ratios were 0.622 to 0.915. If, between GBA and VCG, there is no significant difference in the number of almost truth-telling bids but there is clear difference in the number of almost efficiency, then we can infer that truth-telling does not deserve an essential factor that brings high efficiency.

[8]There was no significant difference in the rates of seller's revenue between GBA and VCG, even if the data are analysed in each session. As for the rate of efficiency, there was a significant difference between two algorithms only when VCG was applied first.

Thus, in order to confirm those feature of GBA and VCG in subject experiment, we counted the number of unit bids that satisfy

$$\frac{|\text{unit value} - \text{unit bid}|}{\text{unit value}} \leq 0.05 \tag{16}$$

and the number of efficiency rates each of which satisfies the efficiency rate ≥ 0.95.

We say that a unit bid obeys 95% truth-telling when it satisfies (16) and that an auction outcome is 95% efficient when the rate of efficiency not lower than 0.95. The next observation is confirmed in Table 6.

Table 6. Numbers of 95% truth-telling bidding and in 95% efficiency

Parameter	Truth-telling		Efficiency	
	GBA	VCG	GBA	VCG
Sessions 1–2	996 (196)	964 (164)	57	71
p-Value (Fisher)	0.1019		0.0095	
sessions 3–4	963 (163)	975 (175)	48	68
p-Value (Fisher)	0.5690		0.0007	
sessions 5–6	498	527	57	69
p-Value (Fisher)	0.2318		0.0325	
sessions 7–8	444	426	52	70
p-Value (Fisher)	0.4704		0.0014	

Observation 2. In the subject experiment, there was no significant difference in the number of bids that obey 95% truth-telling between GBA and VCG, whereas there was clear difference in the number of 95% efficiency between GBA and VCG.

Table 6 shows the numbers of 95% truth-telling unit bids and 95% efficiency observed in each pair of 2 sessions. The sample size is 1200 for 95% truth-telling unit bids and it is 80 for 95% efficiency. The sample size of human subjects' bids is 400 in each pair of sessions 1–2 and sessions 3–4. As noted, we would like to confirm here whether the VCG mechanism achieves high efficiency, when bidders take on the behaviour of 95% truth-telling. Thus, we took 800 truth-telling unit bids into account. (The numbers of human subjects' truth-telling unit bids are noted in the parentheses.)

The *p*-values for the two-sided Fisher exact test (Fisher) are reported under each panel that corresponds to truth-telling and efficiency. The null hypotheses on 95% truth-telling could not be rejected at the 5% significance level with both data taken in 2014 and in 2016. The null hypotheses on 95% efficiency were rejected at the 5% significance level with both data taken in 2014 and in 2016. Therefore, Observation 2 is consistent with the result in the numerical experiment.

Observation 2 also says that in the subject experiment there was no significant difference in the number of bids that obey "almost" truth-telling between GBA and VCG, although it is not clear in theory that bidders will truthfully tell their unit valuations by bidding in GBA.

Finally, we report the regression results on the subjects' bidding behaviour. For each bidder, each unit valuation is drawn independently of the other unit valuations. We thus analyse the data unit by unit. If the absolute value of a unit valuation minus a unit bid falls within the top 5% of all those absolute values, we then dropped the data as an outlier for our regression analysis. Tables 7 and 8 show the regression results with the data taken in 2014 and 2016, respectively.

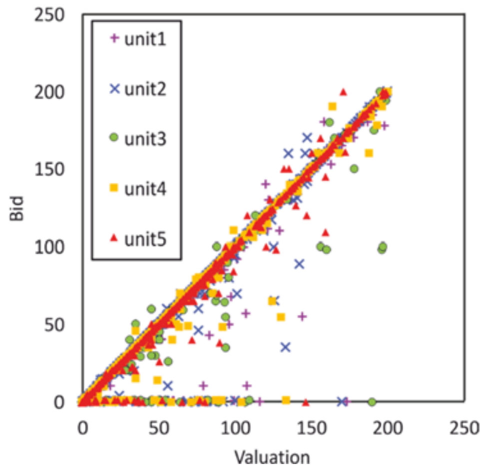

Fig. 4. GBA in 2014

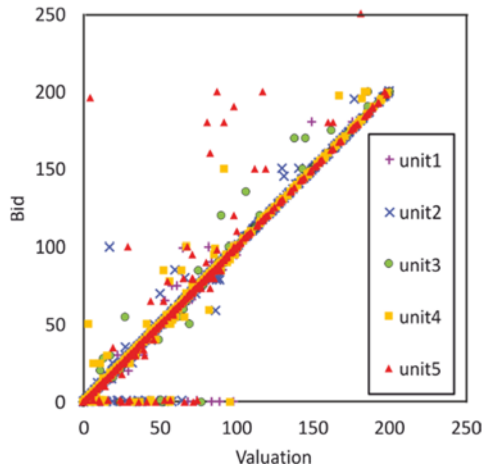

Fig. 5. VCG in 2014

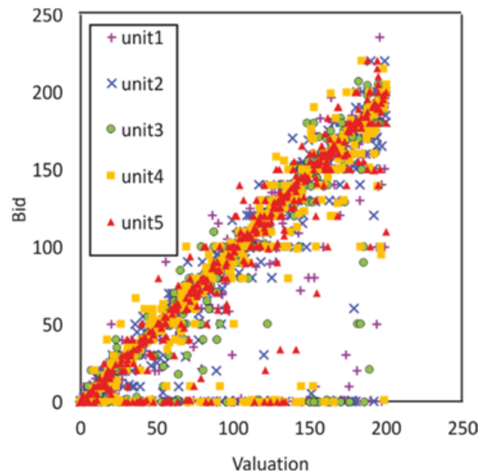

Fig. 6. GBA in 2016

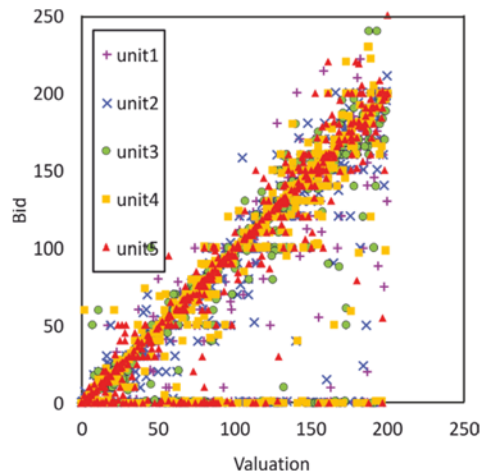

Fig. 7. VCG in 2016

The *p*-values for the two-sided *t*-test and coefficients of determination (*R*-squared) are also reported corresponding to the estimated coefficients of constants (constant) and unit valuations (valuation). Figures 4–7 depict unit valuations and unit bids observed in 2014 and 2016, respectively. The coefficients on valuations were less than one and they are statistically significant, except in the session for VCG conducted in 2014.

Table 7. Results of regression analysis for 2014

Parameter	GBA				
# of units	1	2	3	4	5
Constant	–2.5966	–3.2360	–0.3177	–1.8116	–2.0598
p-Value	0.0370	0.0210	0.7930	0.0790	0.0140
Valuation	0.9832	0.9850	0.9500	0.9850	0.9982
p-Value	<0:0001	<0:0001	<0:0001	<0:0001	<0:0001
R-squared	0.9140	0.8980	0.9050	0.9350	0.9590
Parameter	VCG				
# of units	1	2	3	4	5
Constant	–1.6190	–1.0880	–1.1676	–0.9239	–0.3216
p-Value	0.0270	0.0450	0.0190	0.1440	0.7950
Valuation	1.0029	1.0069	1.0114	1.0085	1.0233
p-Value	<0:0001	<0:0001	<0:0001	<0:0001	<0:0001
R-squared	0.9660	0.9800	0.9840	0.9710	0.9030

Table 8. Results of regression analysis for 2016

Parameter	GBA				
# of units	1	2	3	4	5
Constant	–1.1523	–3.6660	–2.8788	–4.0178	–6.9879
p-Value	0.6980	0.1620	0.3110	0.1490	0.0000
Valuation	0.8618	0.9063	0.8848	0.9154	0.9761
p-Value	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001
R-squared	0.7060	0.7720	0.7220	0.7580	0.8700
Parameter	VCG				
# of units	1	2	3	4	5
Constant	0.7320	–6.3762	–6.5492	–5.3207	–6.4748
p-Value	0.8420	0.0440	0.0400	0.0960	0.0050
Valuation	0.7857	0.9030	0.9219	0.9173	0.9693
p-Value	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001
R-squared	0.5590	0.6940	0.7130	0.7060	0.8170

In Table 8, the estimated coefficients of valuations are all smaller than 1, which implies underbidding both in GBA and VCG. In Table 7, however, the estimated coefficients of valuations are all larger than 1 in the case of VCG, which implies overbidding in the environment where human subjects bid against truth-telling machine bidders.

Observation 3. In the subject experiment, subjects would underbid, except in the sessions for VCG in which human bidders bid against truth-telling machine bidders. Subjects would overbid under VCG, when human bidders compete with truth-telling machines.

There is little literature on experiments for VCG, and thus we have not yet found the essential fact that induces subjects to overbid when human bidders compete with machine bidders. As for individual bidding behaviour, GBA and VCG show a sharp contrast when a human bidder competes against machine bidders; underbidding was observed in GBA, while overbidding was observed in VCG.

5. Final remarks

There is little literature on subject experiments which investigated how approximation algorithms of the VCG mechanism work in the multi-unit non-reverse auctions to which Kothari et al. [8] referred. To begin with, the experimental results on how VCG works themselves are still rare. Kagel and Levin [5], for instance, studied subjects' bidding behaviour in multi-unit auctions, but they imposed a uniform price on all units of the item[9]. We thus carefully prepared for the experimental design. It was assumed that for all bidders, each unit valuation is drawn independently of the other unit valuations. As mentioned in Section 1, we found that it was better for us to do so from a result of a preliminary experiment.

The preliminary experiment was conducted also at the University of Tsukuba. We had 4 sessions on February 13 and 14 in 2015. Each session consisted of 20 rounds in total, and 2 sessions were paired; in a session unit valuations were randomly drawn independently of the other units in the first 10 rounds and they were aligned for five units in the monotone non-increasing order (descending order in case of no tie in unit valuations) after independent random draws in the second 10 rounds, while the order of those displays was reversed in another session. The other part of the experimental design was completely the same as the one described in this paper. In the data taken from last 5 rounds in each treatment, the average rate of seller's revenue was 0.977 with standard deviation of 0.0132 when unit valuations were drawn in the random order, whereas it was 1.0564 with standard deviation of 0.258 when they were drawn in the monotone non-increasing order. Truly, the average rate of seller's revenue was slightly higher when unit valuations were aligned in the monotone non-increasing order after independent random draws, but the standard deviation was remarkably higher than that in the case of random draws. The credibility of the experimental observations depends on low

[9]Kagel and Levin [7] presented a comprehensive survey of experimental results in various auctions and mechanisms. Dobzinski and Nisan [3] showed the latest theory in multi-unit auctions.

standard error of the results. We thus in this time chose an experimental environment where each unit valuation is drawn independently of the other unit valuations.

At the end, we will leave two remarks for future investigation. Chen and Takeuchi [2] reported underbidding in VCG, although they studied combinatorial auctions. Kagel et al. [6] conducted an experiment in which a human bidder with at demand for two units competes against machine bidders each demanding a single unit, and they reported overbidding of each human bidder for both units. It is interesting that observation 3 is similar to these results, although the direct comparison to them is not appropriate. The other remark is on observation 2; not only in VCG but also even in GBA, the number of 95% truth-telling unit bids in the environment of a human bidder and two machine bidders is about twice as many as the one in the environment of all human bidders, although it is not clear in theory whether each bidder has an incentive to submit his or her true unit valuations in GBA. Thus, it is an open question to identify some reason why subjects learned such a bidding behaviour.

Acknowledgements

The authors thank Maiko Shigeno, Tatsuyoshi Saijo, Yoshitsugu Yamamoto, participants in seminars at DC2012, Universit d'Aix-Marseille, and SING13, and two anonymous referees for their valuable comments and suggestions, and Toru Suzuki, Hiroshi Tanaka, and Yosuke Watanabe for their excellent research assistance. This research is supported by JSPS Grant-in-Aid for Young Scientists (B) 26870200 and Grant--in-Aid for Scientific Research (B) 15H02972 (Takahashi) and Japan Center for Economic Research (Watanabe).

Appendix. Examples in the instruction

In the instruction, we explained GBA and VCG with the following examples. Subjects were announced that three units of an object were available in those examples, although they were asked to bit for five units in the sessions.

GBA. Allocation problem

Item allocation problem: 5 steps in total. Unit valuations and bids are given as below.

1. Find the highest unit bid. Give "tentatively" the unit to the highest unit bidder. In the example shown in Table 1A, the highest unit bid is 75 cast by bidder 1 for 1 unit.

Table 1A. GBA

		1	2	3
Bidder 1	valuation	80×1	60×2	55×3
	bid	75×1	55×2	40×3
Bidder 2	valuation	40×1	70×2	65×3
	bid	40×1	63×2	65×3

2. Update the other unit bids of the highest unit bidder in the following way.

• updated unit bid for 2 units $= \dfrac{55\times 2-75\times 1}{2-1}=35,$

• updated unit bid for 3 units $= \dfrac{40\times 3-75\times 1}{3-1}=22.5.$

As for 1 unit, according to this update formula, the numerator is zero (75×1 – 75×1) but the denominator is also zero (1 – 1). The GBA thus leaves it blank for updated unit bid for 1 unit. (See Table 2A.)

3. Find the highest (updated) unit bid. Give tentatively the corresponding unit to the highest (updated) unit bidder. This bidder is also called a "tentative winner".

4. If all units are just assigned, the assignment is then implemented. If some units are not assigned, go to step 2. If the number of units is less than the sum of assigned units (there is the "excess demand"), then go to step 5.

The highest (updated) unit bid is 65 cast by bidder 2 for 3 units. In the first round, bidder 1 was assigned 1 unit as a tentative winner, and thus there is the excess demand. Thus, go to step 5.

Table 2A. Bidder 1's updated unit bids

		1	2	3
Bidder 1	valuation	80×1	60×2	55×3
	bid		35×2	22.5×3
Bidder 2	valuation	40×1	70×2	65×3
	bid	40×1	63×2	65×3

5. Choose such an allocation that maximizes the total amount of bids among the allocations of tentative winners.

• 1 unit to bidder 1 and 2 units to bidder 2. Total amount of bids = 75×1 + 63×2 = 201.

• 0 unit to bidder 1 and 3 units to bidder 2. Total amount of bids = 65×3 = 195.

Choose allocation 1.

GBA. Payment determination

Payment of bidder *i* (winner) = total amount of bids in the auction that excludes bidder *i* (65×3 for bidder 2, 40×3 for bidder 1 in allocation 1) – total amount of bids in the original auction (201 in allocation 1) + bidder *i*'s bid for the unit assigned to (75×1 for bidder 1, 63×2 for bidder 2 in allocation 1);

• payment of bidder 1 = (65×3) – 201 + (75×1) = 69,

• payment of bidder 2 = (40×3) – 201 + (63×2) = 45.

Exact VCG. Allocation problem

Choose such an allocation that maximizes the total amount of bids among all possible allocations;

(0, 0): 0, (1, 1): 70×1 + 40×1 = 110 (1, 0): 70×1 = 70, (2, 0): 55×2 = 110, (3, 0): 50×3 = 150, (0, 1): 40×1 = 40, (0, 2): 60×2 = 120: (0, 3): 65×3 = 195, (1, 2): 70×1 + 60×2 = 190, (2,1): 55×2 + 40×1 = 150.

Table 3A. Exact VCG

		1	2	3
Bidder 1	Valuation	80×1	60×2	55×3
	Bid		35×2	22.5×3
Bidder 2	Valuation	40×1	70×2	65×3
	Bid	40×1	63×2	65×3

Choose (0, 3). The total amount of bids is 195, which is less than the value GBA gives, i.e., 201.

Exact VCG. Payment determination

The payment is determined in the same way as shown in GBA.

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Received November 12, 2018
Accepted April 13, 2018