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THE PORTFOLIO PROBLEM WITH PRESENT VALUE MODELLED BY A DISCRETE TRAPEZOIDAL FUZZY NUMBER[2]

A multi-asset portfolio in the case of its present value estimated by a discrete trapezoidal fuzzy number has been assessed. The benefits of owning a security have been evaluated according to an expected fuzzy discount factor. The ambiguity risk has been assessed by an energy measure and indistinctness risk has been evaluated by Kosko's entropy measure. The relationship between the expected fuzzy discount factor for a portfolio and the expected fuzzy discount factors for its components has been derived. An analogous relationship between the values of the energy measure has been presented. The model has been illustrated by means of a profound numerical case study.

Keywords: *portfolio, present value, discrete trapezoidal fuzzy number, discount factor*

1. Introduction

According to the uncertainty theory, as viewed by Mises [34] and Kaplan [19], any unknown future state of affairs is uncertain. This kind of the Mises–Kaplan uncertainty further referred to as uncertainty is a result of our lack of knowledge about the future state of affairs. Following [21, 22, 33, 26, 44, 4, 5, 2], we say that uncertainty may be modelled using probability theory if and only if we can point out a particular time in the future, at which the considered state of affairs will be already known. This postulate was formulated for the first time by Kolmogorov [21, 22]. Therefore, for convenience, we will use the term Kolmogorov's postulate to describe this condition.

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By a security we understand the right to receive a future financial revenue, payable at a certain maturity date. The value of this revenue is interpreted as an anticipated future value (*FV*) of the asset. Yet, in this case, we can point out the maturity date as a particular time in the future, at which the value of the considered income will already be known to the observer. It is worth to note that the *FV* is not burdened by Knight's uncertainty [20]. All these, together with Kolmogorov's postulate, lead to the conclusion that the *FV* is a random variable.

In [40], the present value (*PV*) is defined as the present equivalent of a cash flow to be obtained at given time in the present or future. Outside of bank account processes, the *PV* is defined by investors based on their subjective preferences, experience and other behavioural characteristics. These behavioural premises for deriving the *PV* mean that the value cannot be verified in the future. Together with Kolmogorov's postulate, it implies that the *PV* cannot be considered as a random variable. On the other hand, by its nature, any subjective evaluation is imprecise. Thus it is commonly accepted that the *PV* of a future cash flow is imprecise. One natural consequence of this approach is to estimate the *PV* using fuzzy numbers. Ward defined a fuzzy *PV* as a discounted fuzzy forecast of the value of a future cash flow [51]. Fuzzy numbers were introduced into financial arithmetic by Buckley [1]. As a result, Ward's definition was then further generalised by Greenhut et al. [10], Sheen [46] and Huang [15], who expanded Ward's definition to the case of a future cash flow given as a fuzzy variable. A more general definition of a fuzzy *PV* was proposed by Tsao [49], who assumed that future cash flow can be treated as a fuzzy probabilistic set. All these authors depict *PV* as a discounted, imprecisely estimated future cash flow. A different approach was given by Piasecki and Siwek [38, 40], who estimated the fuzzy *PV* by the current market value of the financial asset. Siwek described the fuzzy *PV* by a discrete fuzzy number [48]. Other authors [1, 13, 25, 27] have previously proved the usefulness of using triangular or trapezoidal fuzzy numbers as tools for application of fuzzy arithmetic to financial instruments.

The main tool for assessing any security is the return rate defined as a non-increasing function of the *PV* and a non-decreasing function of the *FV*. In [37], Piasecki has shown that if the *PV* of a security is a fuzzy real number, then its return rate is a fuzzy probabilistic set [14].

By a financial portfolio we understand an arbitrary, finite set of securities. Each component of the portfolio is called an asset. Any portfolio is also an authorisation to receive future financial revenue, payable by a certain maturity date. From this point of view, each portfolio has the same properties as a security. Therefore, each portfolio can be appraised in the same way as its components. Markowitz [32] presented a case of simple return rates where it was shown that the *PV* was a positive real number and the *FV* was a random variable with a normal distribution. By the method of mathematical deduction, he proved that the return rate from a portfolio is, in fact, a weighted arithmetic average of the return rates calculated for the components, with the weights corresponding to shares in the portfolio.

Markowitz's work [32] became a starting point for further developments of the portfolio theory. One of the other factors is the theory of fuzzy sets, initiated by Zadeh [53]. Both financial theorists and practitioners noticed the problem of imprecision in the assessment of return rates and the problem of imprecise constraints. This led to the creation of many fuzzy models of portfolios. The monographies by Fang et al. [9] and Gupta et al. [12] are excellent sources of information about this topic. Research performed on such models is still ongoing, which can be seen in recent publications, e.g., [6, 11, 15–17, 28, 29, 35, 44, 52, 54]. A common feature connecting all of the abovementioned models is the use of membership functions of fuzzy sets as a substitute for probability distributions. This means that the randomness considered in these models is, in fact, replaced by imprecision. This kind of research paradigm was formulated by Kosko [24].

The papers by Piasecki and Siwek [36–38, 40–42, 47, 48] do not follow this trend in research because the membership functions in these models do not replace probability distributions, but only interact with them as distinct entities. This kind of model extension significantly enhances the possibilities of a reliable description of the return rate. Despite encompassing imprecise information in the assessment of the return rate, in the proposed fuzzy model the whole existing empirical knowledge about the probability distribution of the return rate can be used without any further amendments. This feature is highly beneficial, especially since it enables realistic applications of the model. Moreover, in such models randomness interacts with imprecision, which stands in agreement with the research paradigm formulated by Hiroto [14]. Nowadays, research is being developed based on both of the aforementioned paradigms. Sadly, the number of models analysing the interactions between randomness and imprecision is significantly lower. Most probably, this situation stems from the fact that the mathematical complexity of such models is far greater than the others. The only available research of this type in the field of quantified finance is given by the articles already mentioned in this paragraph, as well as some by other authors [15, [49]. It seems that the practical portfolio analysis has only been considered by Huang [15], Piasecki and Siwek [41–43] and Siwek [47, 48].

The most significant disadvantage of all of the fuzzy portfolio theory mentioned above (excluding [41–43, 47] and [48]) is defining the fuzzy return rate from a portfolio *ex cathedra* as a linear function of the return rates from the portfolio's components. The only justification of this state of affairs is the mechanical generalisation of Markowitz's model [32] to the fuzzy case. The proposed forms of linear functions, appointing to each component's return rate a portfolio return rate, is not justified by mathematical deduction. This highly undermines the reliability of any analysis performed.

Siwek [47] examined the case of a two-asset portfolio with fuzzy triangular *PV*s. As Markowitz [32], he assumes that the simple return rates have a normal distribution. It is proved there that the return rate from a portfolio is not a weighted average of the return rates from the portfolio's components with weights corresponding to their shares in the portfolio. Additionally, the forms of the energy and entropy measures for the portfolio's expected return rate derived in this research were highly complex, which made it difficult to continue researching the topic in the current form.

Piasecki and Siwek [41] suggested an alternative approach to solve the problem researched in [47]. An expected fuzzy discount factor was used for appraising a security using a triangular fuzzy *PV*. It was proved that the expected fuzzy discount factor for the portfolio is a uniquely determined linear combination of the expected fuzzy discount factors for the portfolios' components. Unfortunately, the entropy measure of an arbitrary triangular fuzzy number is constant, which makes it difficult to analyse the impact of diversification on the imprecision of the portfolio assessment. On the other hand, trapezoidal fuzzy numbers do not have this disadvantage. Therefore, in [43] an expected fuzzy discount factor was applied for securities with trapezoidal fuzzy *PV*s. The results obtained this way were, in fact, generalisations of the results obtained in [41].

On the other hand, security quotes are discrete. Therefore, the main goal of the article [42] was to characterise a two-asset portfolio with the components' *PV*s given as discrete triangular fuzzy numbers [50] and a simple return rate having a normal probability distribution.

The main purpose of this article is to generalise these results to the case where the *PV*s of portfolio assets are given by discrete trapezoidal fuzzy numbers. In addition, in our considerations, two-asset portfolios will be replaced by more general multiple asset portfolios. This article focuses on describing the imprecision risk for the portfolio rather than describing its uncertainty.

2. Elements of the theory of fuzzy numbers

By $F(R)$ we denote the family of all the fuzzy subsets of the real line R. Dubois and Prade [8] define a fuzzy number as a fuzzy subset $L \in F(R)$ with bounded support

$$S(L) = \{x \in R : \mu_L(x) > 0\} \tag{1}$$

and represented by its membership function $\mu_L \in [0; 1]^R$ satisfying the conditions:

$$\exists x \in S(L) : \mu_L(x) = 1 \tag{2}$$

$$\forall (x, y, z) \in (S(L))^3 : x \le y \le z \Rightarrow \mu_L(y) \ge \min\{\mu_L(x); \mu_L(z)\} \tag{3}$$

In the original work of Dubois and Prade, a fuzzy number is defined as a fuzzy set satisfying conditions (2) and (3), whose support is an interval on the real line, while the membership function is semi-continuous from above. Due to the need of introducing discrete fuzzy numbers, in this article a fuzzy number will be defined as above. We denote the set of all fuzzy numbers by the symbol F.

Arithmetic operations on fuzzy numbers are described by Dubois and Prade [7]. According to Zadeh's extension principle [53], the sum of two fuzzy numbers $K, L \in F$, represented by their corresponding membership functions $\mu_K, \mu_L \in [0;1]^R$ is the fuzzy number:

$$G = K \oplus L \tag{4}$$

described by its membership function $\mu_G \in [0;1]^R$ given as follows:

$$\mu_G(z) = \sup\{\mu_K(x) \wedge \mu_L(z-x) : x \in R\} \tag{5}$$

Analogously, the product of a real number $\gamma \neq 0$ and a fuzzy number $L \in F(R)$ represented by its membership function $\mu_L \in [0;1]^R$ is a fuzzy number:

$$H = \gamma \otimes L \tag{6}$$

described by its membership function $\mu_H \in [0;1]^R$ given by the formula:

$$\mu_H(z) = \mu_L \frac{z}{y} \tag{7}$$

Moreover, if $\gamma = 0$, then the product given by (6) is equal to zero.

For a given increasing sequence $\mathrm{Nod}(Y) = \{y_i\} \subset R$ of discretisation nodes, Voxman [50] defines a discrete fuzzy number $L \in F$ with support $S(L) \subset \mathrm{Nod}(Y)$. Particular attention will be paid to the arithmetical progression

$$\mathrm{Nod}(X) = \{x_i = \delta i;\ \delta \in R^+;\ i \in N\} \tag{8}$$

of discretisation nodes. Moreover, any fuzzy subset $A \in \mathcal{F}(\mathbb{R})$ with bounded support $S(A) \subset \mathrm{Nod}(X)$ will be evaluated using measure $m : F(R) \to R_0^+$ given in the following way

$$m(A) = \delta \sum_{x \in \mathrm{Nod}(X)} \mu_A(x) \tag{9}$$

Let $\text{Nod}(X)$ be a given fixed sequence of discretisation nodes. For each non-decreasing sequence $\{a, b, c, d\} \subset \text{Nod}(X)$ we define a discrete trapezoidal fuzzy number (*DTrFN*) $Tr(a, b, c, d)$ as a discrete fuzzy number with the following support

$$S(DTr(a, b, c, d)) = \ ]a, d[\cap \text{Nod}(X), \tag{10}$$

determined by its membership function $\mu_{DTr}(\cdot \mid a, b, c, d) \in [0; 1]^R$ in the following way

$$\forall x \in S(DTr(a, b, c, d)) : \mu_{DTr}(\cdot \mid a, b, c, d) = \begin{cases} \dfrac{x-a}{b-a} & \text{for} \quad a < x < b \\ 1 \quad \text{for} \quad b \le x \le c \\ \dfrac{x-c}{d-c} & \text{for} \quad c < x < d \end{cases} \tag{11}$$

Let us note that for any *DTrFN* we have

$$\begin{aligned} &\oplus_{i=1}^{n} DTr(a_i, b_i, c_i, d_i) \\ &\quad = DTr(a_1, b_1, c_1, d_1) \oplus DTr(a_2, b_2, c_2, d_2) \oplus \ldots \oplus DTr(a_n, b_n, c_n, d_n) \\ &\quad = DTr\left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i, \sum_{i=1}^{n} c_i, \sum_{i=1}^{n} d_i \right) \end{aligned} \tag{12}$$

$$\gamma \otimes DTr(a, b, c, d) = DTr(\gamma a, \gamma b, \gamma c, \gamma d) \tag{13}$$

$$\mu_{DTr}(\gamma x \mid a, b, c, d) = \mu_{DTr}\left(x \mid \frac{a}{\gamma}, \frac{b}{\gamma}, \frac{c}{\gamma}, \frac{d}{\gamma} \right) \tag{14}$$

where $\gamma \in R_0^+$.

Any fuzzy number holds information about the imprecise estimation of a given parameter. Considering the term imprecision, we can distinguish the ambiguity and indistinctness of information [19]. Ambiguity is interpreted as lack of a clear choice of any single alternative from a set of many. Indistinctness is interpreted as a lack of explicit distinction between alternatives. Any increase in the imprecision of information makes it less useful. Thus, there arises the problem of assessing imprecision.

An appropriate tool for measuring the ambiguity of a fuzzy subset $A \in F(R)$ is the energy measure $d : F(R) \to R_0^+$, defined by de Luca and Termini [31] for an arbitrary discrete fuzzy number $L \in F$ as follows:

$$d(L) = m(L) \tag{15}$$

An appropriate tool for measuring indistinctness is the entropy measure, also proposed by de Luca and Termini [30]. The entropy measure $e: F(R) \rightarrow R_0^+$ will be described as in [23]. For an arbitrary discrete fuzzy number $L \in F$ we have:

$$e(L) = \frac{m(L \cap L^C)}{m((L \cup L^C) \cap S(L))} \tag{16}$$

where the symbol L^C denotes the complement of the fuzzy subset representing the fuzzy number $L \in F$. Let us note that for an arbitrary *DTrFN* $DTr(a, b, c, d)$ we have:

$$d(DTr(a, b, c, d)) = \frac{1}{2}\left(d + c - a - b\right) \tag{17}$$

$$e\left(DTr(a, b, c, d)\right) = \frac{b - a + d - c}{-a - 3b + 3d + c} \tag{18}$$

3. Discount factor for a security

All considerations in this and the following chapter will be based on a fixed maturity time $t > 0$. We will use the simple return rate r_t defined by the equation:

$$r_t = \frac{V_t - V_0}{V_0} \tag{19}$$

where: V_t is a *FV* described by the random variable $\tilde{V}_t : \Omega \rightarrow R$, V_0 is a *PV* assessed precisely or imprecisely.

For any elementary state $\omega \in \Omega$ of the financial market, the variable *FV* is described by the relationship

$$\tilde{V}_t(\omega) = \breve{C}(1 + \tilde{r}_t(\omega)) \tag{20}$$

where the simple return rate $\tilde{r}_t : \Omega \rightarrow R$ is determined on the assumption that the *PV* is equal to the market price $\breve{C}$. It is obvious that the return rate $\tilde{r}_t$ is a random variable with probability distribution described by its cumulative distribution function $F_r : R \rightarrow [0; 1]$. As Markowitz [32], we assume that the return rate $\tilde{r}_t$ has a normal probability distribution $N(\ddot{r}, \sigma)$. For the purposes of evaluation, we define the sequence

$$\text{Nod}\,(PV) = \{x_i \in R : x_i = 0.01i;\ i \in N\} \tag{21}$$

since quotations of securities are given with the accuracy of 0.01 PLN. We additionally assume that the *PV* is imprecisely estimated by *DTrFN*

$$PV = DTr(\breve{C}_{\min}, \breve{C}_*, \breve{C}^*, \breve{C}_{\max}) \tag{22}$$

with membership function $\mu_{PV}(\cdot\,|\,\breve{C}_{\min}, \breve{C}_*, \breve{C}^*, \breve{C}_{\max}) \in [0; 1]^R$. The *PV*'s parameters are interpreted as follows:

- $\breve{C}$ is the market price,
- $\breve{C}_{\min} \in\,]0, \breve{C}]$ is the maximal lower bound on the *PV*,
- $\breve{C}_{\max} \in [\breve{C}, +\infty[$ is the minimal upper bound on the *PV*,
- $\breve{C}_* \in [\breve{C}_{\min}, \breve{C}]$ is the minimal upper assessment of prices visibly lower than the market price C,
- $\breve{C}^* \in [\breve{C}, \breve{C}_{\max}]$ is the maximal lower assessment of prices visibly higher than the market price $\breve{C}$. A method of determining the parameters $\breve{C}_{\min}, \breve{C}_{\max}$ is given in [40].

According to Zadeh's Extension Principle, the simple return rate calculated for the *PV* assessed according to this method is a fuzzy probabilistic set represented by its membership function $\tilde{\rho} \in [0; 1]^{R\times\Omega}$, which is given by

$$\tilde{\rho}(r, \omega) = \sup\left\{ \mu_{PV}\left(x\,|\,\breve{C}_{\min}, \breve{C}_*, \breve{C}^*, \breve{C}_{\max}\right);\ x = \frac{\tilde{V}_t(\omega)}{1+r},\ x \in R \right\}$$

$$= \mu_{PV}\left(\frac{\tilde{V}_t(\omega)}{1+r}\,|\,\breve{C}_{\min}, \breve{C}_*, \breve{C}^*, \breve{C}_{\max}\right) = \mu_{PV}\left(\frac{\breve{C}\cdot(1+\tilde{r}_t(\omega))}{1+r}\,|\,\breve{C}_{\min}, \breve{C}_*, \breve{C}^*, \breve{C}_{\max}\right) \tag{23}$$

The membership function $\rho \in [0; 1]^R$ of the expected return rate is calculated in the following way:

$$\rho(r) = \int_{-\infty}^{+\infty} \mu_{PV}\left(\breve{C}\,\frac{1+y}{1+r}\,|\,\breve{C}_{\min}, \breve{C}_*, \breve{C}^*, \breve{C}_{\max}\right) dF_r(y)$$

$$= \mu_{PV}\left(\breve{C}\,\frac{1+\overline{r}}{1+r}\,|\,\breve{C}_{\min}, \breve{C}_*, \breve{C}^*, \breve{C}_{\max}\right) \tag{24}$$

This fuzzy expected return is a discrete fuzzy number with the following support

$$S(\Re) \subset \mathrm{Nod}(\Re) = \left\{ r_i \in R : x_i = \frac{1+\overline{r}}{1+r_i} ;\, x_i \in \mathrm{Nod}(PV);\, i \in N \right\}$$
$$= \left\{ r_i \in R : r_i = 100 \cdot \frac{1+\overline{r}}{i} - 1;\, i \in N \right\} \tag{25}$$

It is very easy to see that the expected return rate obtained above is not a *DTrFN*. Therefore, we shall consider the expected discount factor $\overline{v}$ defined by the relation:

$$\overline{v} = \frac{1}{1+\overline{r}} \tag{26}$$

Thus, if the expected return rate $\Re \in F$ is a discrete fuzzy number with support $S(\Re)$ determined by (24), then the expected discount factor is a discrete fuzzy number $V \in F$ with support $S(V)$ fulfilling the condition

$$S(V) \subset \mathrm{Nod}(V) = \left\{ v_i \in R : v_i = \frac{1}{1+r_i} ;\, r_i \in \mathrm{Nod}(\Re);\, i \in N \right\}$$
$$= \left\{ v_i \in R : v_i = 0.01 \cdot \overline{v} \cdot i;\, i \in N \right\} \tag{27}$$

In agreement with (25), the membership function $\delta \in [0;1]^R$ of the discount factor is given by the relation:

$$\delta(v) = \delta\left(\frac{1}{1+r} \right) = \rho(r) = \rho\left(\frac{1}{v} - 1 \right) \tag{28}$$

Combining (14), (24) and (28) we get:

$$\delta(v) = \mu_{PV}\left(\breve{C}\, \frac{1+\overline{r}}{1+\frac{1}{v}-1} \,|\, \breve{C}_{\min}, \breve{C}_*, \breve{C}^*, \breve{C}_{\max} \right)$$
$$= \mu_{PV}\left(\breve{C}\, \frac{1}{\overline{v}}\, v \,|\, \breve{C}_{\min}, \breve{C}_*, \breve{C}^*, \breve{C}_{\max} \right)$$
$$= \mu_{PV}\left(v \,|\, \frac{\breve{C}_{\min}}{\breve{C}}\, \overline{v}, \frac{\breve{C}_*}{\breve{C}}\, \overline{v}, \frac{\breve{C}^*}{\breve{C}}\, \overline{v}, \frac{\breve{C}_{\max}}{\breve{C}}\, \overline{v} \right) \tag{29}$$

where $\overline{v}$ is the discount factor determined using the expected return rate $\overline{r}$. It is easy to see that the discount factor $V \in F$ defined above is a *DTrFN* given by the formula

$$V = DTr\left(\frac{\breve{C}_{\min}}{\breve{C}} \overline{v}, \frac{\breve{C}_{*}}{\breve{C}} \overline{v}, \frac{\breve{C}^{*}}{\breve{C}} \overline{v}, \frac{\breve{C}_{\max}}{\breve{C}} \overline{v} \right) \tag{30}$$

The increase in the ambiguity of the expected discount factor $V \in F$ leads to an increase in the number of alternative investment recommendations. This implies an increase in the risk of making a financial decision that will be burdened *ex post* by lost profit. This kind of risk is called ambiguity risk. The ambiguity risk burdening the expected discount factor *V* is evaluated using the energy measure *d*(*V*). According to (17), this equals:

$$d(V) = \frac{\overline{v}}{2\breve{C}} \left(\breve{C}_{\max} + \breve{C}^{*} - \breve{C}_{\min} - \breve{C}_{*} \right) \tag{31}$$

An increase in the indistinctness of the factor *v* means that the boundaries distinguishing recommended alternatives become blurred. This results in an increase in the risk of choosing a not recommended decision. This kind of risk is called indistinctness risk. The indistinctness risk burdening the expected discount factor $V \in F$ is evaluated using the entropy measure $e(V)$. According to (18), this equals

$$e(V) = \frac{\breve{C}_{*} + \breve{C}_{\max} - \breve{C}^{*} - \breve{C}_{\min}}{-3\breve{C}_{*} + 3\breve{C}_{\max} + \breve{C}^{*} - \breve{C}_{\min}} \tag{32}$$

Let us note that we have

$$e(V) = e(PV) \tag{33}$$

Together, the ambiguity risk and vagueness risk will be referred to as imprecision risk.

In each of the considered cases, the return rate is a function of the *FV*, which is uncertain by its nature, as mentioned in the Introduction. This uncertainty stems from an investor's lack of knowledge about future states of affairs. This lack of knowledge implies that no investor is sure of their future profits or losses. An increase in uncertainty can result in a greater risk of making a wrong financial decision. This type of risk is called uncertainty risk. The properties of such risk are discussed in a rich body of literature. In this paper, we evaluate the uncertainty risk using the variance σ^2 of the return rate.

The formal simplicity of the obtained description of an expected discount factor encourages its further application as a tool for analysing portfolios. The criterion of maximising the expected return rate can then be substituted by the criterion of minimising the expected discount factor. In the case when both parameters have non-fuzzy values, these criteria are equivalent.

4. Portfolio

In accordance with the tentative definition given in the Introduction, any financial portfolio is an arbitrary, finite set of securities called portfolio's assets. Each of these assets is characterised by its assessed *PV* and anticipated return rate. Let us consider the case of a portfolio π consisting of the financial assets Y_i $(i = 1, 2..., n)$.

The *PV* of the asset Y_i is estimated by the *DTrFN* $DTr(\breve{C}_{\min}^{(i)}, \breve{C}_{*}^{(i)}, \breve{C}^{(i)*}, \breve{C}_{\max}^{(i)})$, whose parameters are given as follows:

$\breve{C}_{\min}^{(i)}$ is the market price,

$\breve{C}_{\min}^{(i)} \in]0, \breve{C}^{(i)}]$ is the maximal lower bound on the *PV*,

$\breve{C}_{\max}^{(i)} \in [\breve{C}^{(i)}, +\infty[$ is the minimal upper bound on the *PV*,

$\breve{C}_{*}^{(i)} \in [\breve{C}_{\min}^{(i)}, \breve{C}^{(i)}]$ is the minimal upper assessment of prices visibly lower than the market price $\breve{C}_{\min}^{(i)}$,

$\breve{C}^{(i)*} \in [\breve{C}^{(i)}, \breve{C}_{\max}^{(i)}]$ is the maximal lower assessment of prices visibly higher than the market price $\breve{C}_{\min}^{(i)}$.

We assume that for each security Y_i we know the simple return rate $\tilde{r}_t^i : \Omega \to R$ defined by (19) for the *PV* equal to the market price $\breve{C}_{\min}^{(i)}$. As Markowitz [32], we assume that the *n*-dimensional variable $(\tilde{r}_t^1, \tilde{r}_t^2, ..., \tilde{r}_t^n)^T$ has a multivariate normal distribution $N(\bar{r}, \Sigma)$, where $\bar{r} = (\bar{r}_1, \bar{r}_2, ..., \bar{r}_n)^T$. We define the expected discount factor of the asset Y_i as follows:

$$V^{(i)} = DTr\left(\frac{\breve{C}_{\min}^{(i)}}{\breve{C}^{(i)}} \cdot \bar{v}_i, \frac{\breve{C}_{*}^{(i)}}{\breve{C}^{(i)}} \cdot \bar{v}_i, \frac{\breve{C}^{(i)*}}{\breve{C}^{(i)}} \cdot \bar{v}_i, \frac{\breve{C}_{\max}^{(i)}}{\breve{C}^{(i)}} \cdot \bar{v}_i \right), \tag{34}$$

where $\bar{v}_i$ is the expected discount factor defined by (26) with the use of the expected return rate $\bar{r}_i$. According to (31), the energy measure of $V^{(i)}$ is given by

$$d(V^{(i)}) = \frac{\overline{v}_i}{2\breve{C}^{(i)}} \left(\breve{C}^{(i)}_{\max} + \breve{C}^{(i)*} - \breve{C}^{(i)}_{\min} - \breve{C}^{(i)}_{*} \right) \tag{35}$$

and from (32), the entropy measure of the discount factor can be calculated as

$$e(V^{(i)}) = \frac{\breve{C}^{(i)}_{*} + \breve{C}^{(i)}_{\max} - \breve{C}^{(i)*} - \breve{C}^{(i)}_{\min}}{-3\breve{C}^{(i)}_{*} + 3\breve{C}^{(i)}_{\max} + \breve{C}^{(i)*} - \breve{C}^{(i)}_{\min}} \tag{36}$$

The market value $\breve{C}^{(\pi)}$ of the portfolio π is equal to

$$\breve{C}^{(\pi)} = \sum_{i=1}^{n} \breve{C}^{(i)}. \tag{37}$$

The share p_i of the asset Y_i in the portfolio π is given by

$$p_i = \frac{\breve{C}^{(i)}}{\breve{C}^{(\pi)}} \tag{38}$$

According to (11), the present value of the portfolio is also a discrete trapezoidal fuzzy number

$$PV^{(\pi)} = DTr\left(\sum_{i=1}^{n} \breve{C}^{(i)}_{\min}, \sum_{i=1}^{n} \breve{C}^{(i)}_{*}, \sum_{i=1}^{n} \breve{C}^{(i)*}, \sum_{i=1}^{n} \breve{C}^{(i)}_{\max} \right) = DTr(\breve{C}^{(\pi)}_{\min}, \breve{C}^{(\pi)}_{*}, \breve{C}^{(\pi)*}, \breve{C}^{(\pi)}_{\max}) \tag{39}$$

Piasecki, Siwek [41] proved that the expected discount factor $V^{(\pi)} \in F$ of portfolio π is given by the formula

$$V^{(\pi)} = \left(\sum_{i=1}^{n} \frac{p_i}{v_i} \right)^{-1} \otimes \left(\oplus_{i=1}^{n} \frac{p_i}{v_i} \otimes V^{(i)} \right) \tag{40}$$

Moreover, the energy measure of the expected discount factor $V^{(\pi)}$ is the following linear combination of energy measures calculated for each of the component assets:

$$d(V^{(\pi)}) = \left(\sum_{i=1}^{n} \frac{p_i}{\overline{v}_i} \right)^{-1} \sum_{i=1}^{n} \frac{p_i}{\overline{v}_i} d(V^{(i)}) \tag{41}$$

The relation above suggests that the energy of the fuzzy expected discount factor of a portfolio π is, in fact, a linear combination of the weighted energies of the factors calculated for its components. The weights calculated for the assets Y_i are increasing in their shares in the portfolio and decreasing in the value of their discount factor $\overline{v}_i$. This fact leads to the conclusion that when trying to minimise the ambiguity risk of a portfolio, one should focus on minimising the ambiguity of the component assets which are characterised by the highest expected return rates. On the other hand, the shares of an asset in the whole portfolio are, according to this theory, appointed *post factum,* by gathering the information available on said assets. Condition (40) shows that, in the case considered, diversification in the portfolio only "averages" the risk of ambiguity.

According to (32), the entropy measure of the expected discount factor is equal to

$$e(D^{(\pi)}) = \frac{\breve{C}_*^{(\pi)} + \breve{C}_{\max}^{(\pi)} - \breve{C}^{(\pi)*} - \breve{C}_{\min}^{(\pi)}}{-3\breve{C}_*^{(\pi)} + 3\breve{C}_{\max}^{(\pi)} + \breve{C}^{(\pi)*} - \breve{C}_{\min}^{(\pi)}} \tag{42}$$

The entropy measure $e(D^{(\pi)})$ cannot be calculated in an analogous way to the portfolio energy measure $d(D^{(\pi)})$ using the linear combination (41). The variance of the portfolio return rate is calculated using

$$\sigma^2 = \overline{p}^T \Sigma \, \overline{p} \tag{43}$$

By constructing a portfolio which minimises the variance, Markowitz proved that portfolio diversification can "minimise" the uncertainty risk.

5. Case study

In order to evaluate the PV, we will use a sequence $\mathrm{Nod}(PV)$ of discretization nodes. The portfolio π consists of two financial assets, Y_1 and Y_2. The vector $(\tilde{r}_t^1, \tilde{r}_t^2)^T$ of their anticipated simple return rates has the following two-dimensional normal distribution:

$$N\left((0.25, 0.5)^T, \begin{bmatrix} 0.5 & -0.1 \\ -0.1 & 0.4 \end{bmatrix} \right)$$

The current market price for a unit of the security Y_1 is equal to $\breve{C}^{(1)} = 90$. The *PV* for a unit of the security Y_1 is given by the *DTrFN* $DTr(50; 80; 100; 110)$. First, we derive the expected discount factor $V^{(1)} \in F$. According to (30), we have:

$$V^{(1)} = DTr\left(\frac{1}{1+0.25}\frac{50}{90}; \frac{1}{1+0.25}\frac{80}{90}; \frac{1}{1+0.25}\frac{100}{90}; \frac{1}{1+0.25}\frac{110}{90}\right)$$

$$= DTr(0.444; 0.711; 0.888; 0.978)$$

Using (31), we can calculate the energy measure for this factor:

$$d(V^{(1)}) = \frac{0.80}{2 \cdot 90}(110 - 50) = 0.267$$

The current market price for a unit of the security Y_2 is equal to $\breve{C}^{(2)} = 96$. The *PV* of a unit of the security Y_2 is given by the *DTrFN* $DTr(90; 91; 120; 144)$. Next, we find the expected discount factor $V^{(2)} \in F$. According to (30), we have:

$$V^{(2)} = DTr\left(\frac{1}{1+0.5}\frac{90}{96}; \frac{1}{1+0.5}\frac{91}{96}; \frac{1}{1+0.5}\frac{120}{96}; \frac{1}{1+0.5}\frac{144}{96}\right)$$

$$= DTr(0.625; 0.632; 0.883; 1.000)$$

Using (31), we can calculate the energy measure for this factor:

$$d(V^{(2)}) = \frac{0.667}{2 \cdot 96}(144 - 90) = 0.188.$$

Let the share of the asset Y_i in the portfolio π be equal to p_i. Then, according to (40), the expected discount factor $V^{(\pi)} \in F$ of the portfolio π can be calculated in the following way:

$$V^{(\pi)} = \left(\frac{p_1}{0.8} + \frac{p_2}{0.667}\right)^{-1}$$

$$\otimes \left(\frac{p_1}{0.8} \otimes DTr(0.444; 0.711; 0.888; 0.0978) \oplus \frac{p_2}{0.667} \otimes DTr(0.625; 0.632; 0.883; 1\right)$$

$$= \frac{0.667 p_1 \otimes DTr(0.444; 0.711; 0.888; 0.0978)}{0.667 p_1 + 0.8 p_2}$$

$$\frac{\oplus 0.8 p_2 \otimes 0.DTr(0.625; 0.632; 0.883; 1)}{0.667 p_1 + 0.8 p_2}$$

$$= \frac{p_1 \otimes DTr(0.2961; 0.4742; 0.5923; 0.6523)}{0.667 p_1 + 0.8 p_2}$$

$$\frac{\oplus p_2 \otimes 0.DTr(0.5; 0.5056; 0.7064; 8)}{0.667 p_1 + 0.8 p_2}$$

$$\frac{\oplus\ 0.8 p_2 \otimes 0.DTr(0.625; 0.632; 0.883; 1)}{0.667 p_1 + 0.8 p_2}$$

We see that the expected fuzzy discount factor for the portfolio can be expressed as a combination of an asset's shares and its expected fuzzy discount factors. In an analogous way, the ambiguity risk may be evaluated, since the energy measure for this factor is given by (41) as follows:

$$d(V^{(\pi)}) = \left(\frac{p_1}{0.8} + \frac{p_2}{0.667} \right)^{-1} \left(\frac{p_1}{0.8} 0.267 + \frac{p_2}{0.667} 0.188 \right)$$

$$= \frac{0.667 p_1 \times 0.267 + 0.8 p_2 \times 0.188}{0.667 p_1 + 0.8 p_2}$$

$$= \frac{0.178 p_1 + 0.15 p_2}{0.667 p_1 + 0.8 p_2}$$

The last two equations can be applied to define a portfolio optimization problem to be solved using mathematical programming.

6. Summary

This research indicates that there exist effective methods of managing the imprecision risk of portfolios, which have their source in approximating the *PV*s of the component assets of a portfolio. The main focus of this research involves multi-asset portfolios consisting of instruments whose *PV*s are derived in the form of discrete trapezoidal fuzzy numbers. For this case, we have shown that it is possible to build a model of a portfolio that minimizes the expected fuzzy discount factor, while simultaneously controlling the risk.

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