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SOME NOTES ON THE PROPERTIES OF INCONSISTENCY INDICES IN PAIRWISE COMPARISONS

Pairwise comparisons are an important tool of modern (multiple criteria) decision making. Since human judgments are often inconsistent, many studies have focused on the means of expressing and measuring this inconsistency, and several inconsistency indices have been proposed as an alternative to Saaty's inconsistency index, *CI*, and consistency ratio, *CR*, for reciprocal pairwise comparison matrices. The aims of this paper are threefold: firstly, a row inconsistency index (*RIC*) is proposed and the properties of this index are examined. Secondly, a comparison of selected inconsistency indices for a corner pairwise comparison matrix is provided. Last, but not least, another axiom about the upper bound on the value of an inconsistency index is postulated, and a set of selected inconsistency indices is examined with respect to this axiom. Numerical examples complete the paper.

Keywords: *pairwise comparisons, AHP, inconsistency, inconsistency index, axioms of inconsistency*

1. Introduction

Pairwise comparisons are a well-known tool for decision making with their history dating back to the early works of Lull and Condorcet, together with Thurstone's *A Law of Comparative Judgments* [29]. They enable us to compare two objects, usually alternatives, qualities or features, at the same time. Pairwise comparisons are especially useful when the number of objects to be compared is large, as they reduce the complexity of a problem and aid avoiding cognitive overload, as according to Miller [20], humans are only able to compare 7 objects at one time. Also, from pairwise comparisons, a priority vector (a vector of the weights of compared objects) can be derived via some well-known method such as the eigenvalue method or the geometric mean method (the least logarithmic squares method), for the former see, e.g., Saaty [24–27], for the latter see,

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e.g., Chandran et al. [9]. In particular, the analytic hierarchy process (AHP) is perhaps the best known application of pairwise comparisons, successful applications of the AHP can be found, e.g., in [30].

Apart from the problem of deriving a priority vector, another important issue is the problem of the inconsistency of pairwise comparisons, which has been the subject of studies on a (numerical) comparison of existing inconsistency indices and efforts to define an axiomatization of their properties. For example, when comparing three alternatives, A, B and C, a decision maker has to perform three pairwise comparisons: A with B, A with C and B with C. If A is judged to be twice as good as B and B is judged to be 3 times as good as C, then A should be 6 times as good as C. Although this task seems simple, decision makers are seldom consistent, and their judgments are "erroneous" to a certain degree.

To measure inconsistency in pairwise comparisons, several inconsistency indices (see Section 2) have been proposed since 1977 [24, 12, 14, 21, 18, 6]. However, until recently, the properties of these indices and their mutual similarity have only rarely been studied [5]. Furthermore, with the growing number of different indices, the problem of what conditions a suitable index should satisfy from a mathematical point of view has emerged, leading to a collection of studies focusing on the axiomatization of the properties of inconsistency indices (cf., e.g. [16, 22, 6, 7]). Nevertheless, the problem of axiomatization has not yet been solved, as new indices have emerged, in particular, extensions of pairwise comparisons to fuzzy sets and linearly ordered Abelian groups (cf., e.g., [8, 23].

The aims of this paper are as follows: A new row inconsistency index (*RIC*) has been defined and its properties in terms of satisfying (or, rather, not satisfying) a selected set of six axioms have been discussed. This index is based on a dot product of the row vectors of a pairwise comparison matrix. A "cosine" approach was followed in the "cosine optimisation" of Kou and Lin [18], but their inconsistency index, *CCI*, was defined to attain the value of 1 for a fully consistent set of pairwise comparisons, unlike all the other indices, which assign the value 0 to fully consistent cases. Also, the authors provided only one property of their *CCI* index (namely that $0 \leq CCI \leq 1$). Last, but most importantly, although both the *CCI* and *RIC* indices use the cosine function, they are defined differently, and also attain different numerical values. The *CCI* is computed from a normalised pairwise comparison matrix via an objective function for a given optimization model (see [18], p. 227), while the *RIC* index utilises only (and directly) the row vectors of a given pairwise comparison matrix.

An "upper bound axiom" for inconsistency indicators has been postulated and a set of selected inconsistency indicators has been examined with respect to this axiom. A comparison of selected inconsistency indices has been provided for a corner pairwise comparison matrix. Also, several numerical examples have been provided for better understanding.

The paper is organised as follows: in Section 1 preliminaries are provided, section 2 describes some chosen inconsistency indices and section 3 presents some inconsistency axioms, in section 4 a new inconsistency index is introduced and its properties are proved, in section 5 selected indices are examined with respect to an additional axiom, and section 6 includes a numerical example. The article ends with some conclusions.

2. Preliminaries

For simplicity, but without loss of generality, let us consider pairwise comparisons of alternatives.

Let X be a given set of n alternatives to be compared. Let a_{ij} denote the decision maker's preference for the i-th alternative over the j-th alternative. Also, we set $a_{ij} > 0$; $\forall i, j \in \{1, 2, ..., n\}$.

Pairwise comparisons are called reciprocal, if the following property is satisfied:

$$a_{ij} = \frac{1}{a_{ji}}, \ \forall i, j \in \{1, 2, ..., n\} \tag{1}$$

Property (1) is usually strictly required for pairwise comparisons. All pairwise comparisons can be arranged into a square $n \times n$ matrix, $A(a_{ij})$, called a pairwise comparison matrix (PCM):

$$A_{n\times n}(a_{ij}) = \begin{pmatrix} 1 & a_{12} & ... & a_{1n} \\ a_{21} & 1 & ... & a_{2n} \\ ... & ... & 1 & ... \\ a_{n1} & a_{n2} & ... & 1 \end{pmatrix}$$

Further, pairwise comparisons (a pairwise comparison matrix) are called consistent, if the following property is satisfied:

$$a_{ij}a_{jk} = a_{ik}; \ \forall i, j, k \tag{2}$$

The matrix A is consistent, if and only if the priority vector (vector of weights) $w = (w_1, ..., w_n)$ satisfies the following relation:

$$a_{ij} = \frac{w_i}{w_j}, \quad \forall i, j$$

A priority vector w can be derived via Saaty's eigenvalue method (EM) [24]:

$$Aw = \lambda_{\max} w \tag{3}$$

where $\lambda_{\max}$ is the largest (positive) eigenvalue of A. The existence of the largest (positive) eigenvalue $\lambda_{\max}$ of the matrix A is guaranteed by the Perron–Frobenius theorem [25]. Usually, the vector w is normalised so that $\|w\| = 1$.

Also, the geometric mean method (the least squares method) can be used to obtain w:

$$w_i = \frac{\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}} \tag{4}$$

Both methods yield the same result when the matrix A is consistent. Otherwise, the priority vectors differ slightly, see the comparative study of Ishizaka and Lusti [13].

For further considerations, it should be noted that if the matrix A is reciprocal and consistent, then $\lambda_{\max} = n$ and rank(A) = 1, which means that all of the rows (columns) of A differ only by a multiplicative constant.

Definition 1. Let M_R denote the set of all matrices $A_{n\times n}(a_{ij})$, $a_{ij} > 0$; $\forall i, j \in \{1, 2, ..., n\}$, satisfying (1), and let M_C denote the set of all matrices $A_{n\times n}(a_{ij})$, $a_{ij} > 0$; $\forall i, j \in \{1, 2, ..., n\}$ satisfying (2).

By the definition above, M_R is the set of all reciprocal pairwise comparison matrices with positive elements of order n, and M_C is the set of all consistent pairwise comparison matrices with positive elements of order n.

Remark 1. As the matrix A with elements $a_{ij} = 1, \ \forall i, j$ belongs to both M_R and M_C, both sets are non-empty, and clearly: $M_C \subset M_R$. Furthermore, consistency (2) implies reciprocity (1), but not vice versa.

Definition 2. A corner pairwise comparison matrix CPC of order n is defined as follows:

$$CPC_{n\times n} = \begin{pmatrix} 1 & 1 & \dots & 1 & x \\ 1 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 & 1 \\ 1/x & 1 & \dots & 1 & 1 \end{pmatrix}, \; x > 0$$

A corner pairwise comparison matrix is a special (simplified) case of a reciprocal matrix which is consistent if and only if $x = 1$.

3. Inconsistency indices

This section introduces and discusses several inconsistency indices for a pairwise comparison matrix, starting with the oldest one, Saaty's consistency index *CI* [24–27]:

$$CI = \frac{\lambda_{\max} - n}{n-1} \tag{5}$$

According to Saaty, a PCM with a $CI \le 0.10$ is sufficiently consistent, and the EM method (3) can be employed. However, the *CI* tends to grow with *n*, hence Saaty introduced a more suitable measure of inconsistency, the consistency ratio *CR* [26–27]):

$$CR = \frac{CI}{RI} \tag{6}$$

RI in (6) denotes random inconsistency, that is the average of the *CI* of random matrices (generated using the Monte Carlo method) of order *n*. For the values of *RI* see, e.g., [2].

Again, a PCM with a *CR* less than or equal to 0.10 is sufficiently consistent for the eigenvalue method. Nevertheless, this threshold of 0.10 has been criticised by some authors [11, 15]. Notably, the *RI* was found to converge to the value 1.58 with increasing *n*. This fact led to some criticism of the *CR* (and *CI*), as random inconsistency should be increasing in *n* (the larger a random matrix, the more "mess" it contains).

Let $A(a_{ij}) \in M_R$ and let $\overline{A}(\overline{a}_{ij})$ be the normalised matrix obtained from *A* by dividing each column by the sum of all the elements in that column. Further, let $\overline{w}$ be the normalized priority vector (the vector of weights) obtained from *A* using the EM or the geometric mean method. Then the *GWI* inconsistency index [12] is defined as:

$$GWI = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\overline{a}_{ij} - \overline{w}_i\right| \tag{7}$$

Let $A(a_{ij}) \in M_R$. Then the *PLI* inconsistency index [21] is defined as follows:

$$PLI = \frac{6}{n(n-1)(n-2)}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2\right) \tag{8}$$

The geometric consistency index *GCI* [1] is defined as follows:

$$GCI(A) = \frac{2}{(n-1)(n-2)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\ln a_{ij}\frac{w_j}{w_i}\right)^2 \tag{9}$$

where the priority vector *w* is obtained by the geometric mean method.

Let $T(n)$ be the set of all ordered triples ("triads") (a_{ij}, a_{jk}, a_{ik}) satisfying (2) for $\forall i, j, k \in \{1, 2, ..., n\}$. Then Koczkodaj's index *KII* [14] is defined as follows:

$$KII = \max_{\mathrm{T}(n)}\left(\min\left(\left|1 - \frac{a_{ik}}{a_{ij}a_{jk}}\right|, \left|1 - \frac{a_{ij}a_{jk}}{a_{ik}}\right|\right)\right) \tag{10}$$

For other inconsistency indices, see, e.g., [4, 28, 23, 22].

The majority of such indices describe the central tendency (mean) of inconsistent judgments. On the other hand, Koczkodaj's inconsistency index aims to express the largest inconsistency present in a given PCM. It should be noted that indices such as *PLI* or *GWI* can be easily modified to also return the maximum inconsistency, by substituting the averaging operators by the maximum operator. Therefore, indices may be divided into two groups:

- mean-based (*CI*, *CR*, *GWI*, *PLI*, *GCI*),
- extreme-based (*KII*).

One advantage of the mean-based indices is that they take into account every pair-wise comparison (and their changes). However, they do not provide (explicitly) information about extremes. On the other hand, extreme-based indices express inconsistency in terms of the most inconsistent judgment, which can be useful when the most "erroneous" comparison is to be found and revised. One disadvantage of this class of indices is that all changes besides the most inconsistent comparison (a triad) are neglected. This may lead to the idea of a new (compromise) family of indices of the following form:

$$IC_{\text{compr}} = \lambda IC_{\text{mean}} + (1-\lambda) IC_{\text{extreme}}, \lambda \in [0,1]$$

where a decision maker decides how he/she wants to capture both features by selecting an appropriate value of λ. Here, $\lambda=0$ yields an extreme index, and $\lambda=1$ a central based index.

4. Axioms for inconsistency indices

An inconsistency index should have some reasonable properties. Koczkodaj and Szwarc [16] introduced three axioms an inconsistency index must satisfy, while Brunelli and Fedrizzi [6] postulated five axioms. Two axioms are common to both papers.

Below, the five axioms of Brunelli and Fedrizzi [6] are described with slight modifications, along with the newly formulated axiom 6.

Definition 3. An inconsistency index (*ICI*) is a real-valued function:

$$ICI\text{: } A \in M_R \to [0, \infty[$$

or alternatively,

$$ICI\text{: } A \in M_R \to [0, 1]$$

A strong case for the latter "normalisation" of *ICI* values was provided in [17].

Axiom 1. $A \in M_C \Leftrightarrow ICI(A) = 0.$ Axiom 1 states that consistent pairwise comparison matrices are identified by a unique real value of inconsistency. Usually, this value is set to 0, as zero inconsistency means consistency.

Axiom 2. Let *P* denote a permutation matrix of order *n* and let $A_{n \times n} \in M_R$. Then for all *P* and all *A*′, such that $A' = P \cdot A \cdot P^T$, the following condition holds:

$$ICI(A') = ICI(A), \ \forall P$$

Axiom 2 states that the inconsistency index is invariant under the permutation of alternatives. In other words, changing the order of the alternatives should not result in a change in an inconsistency index.

Axiom 3. Let f be a continuous transformation: $f : a_{ij} \to a_{ij}^b;\ \forall i,\ j;\ b \in R,\ b > 1.$ Let the matrix with elements a_{ij}^b be denoted A^b. Then:

$$ICI(A^b) \geq ICI(A)$$

Axiom 3 deals with the monotonicity of the intensity of preference: if preferences are intensified, then an inconsistency index cannot return a lower value.

Axiom 4. Let $A \in M_C$ and let at least one $a_{ij} \neq 1$ for $i \neq j$. Let A^{δ} denote a matrix obtained from A by substituting the element $a_{ij} \neq 1$ (and also $a_{ji} \neq 1$) by the element a_{ij}^{δ}, where $\delta \in R,\ \delta > 0,\ \delta \neq 1$. Then $ICI(A^{\delta})$ is a non-decreasing function for $\delta > 1$ and a non-increasing function for $\delta < 1$.

Axiom 4 requires monotonicity based on a single comparison: the larger the change in a given entry a_{ij} (and a_{ji} respectively) from a consistent matrix, the more inconsistent the resulting matrix and, hence, the greater the value of an inconsistency index.

Axiom 5. An inconsistency index is a continuous function of its entries.

Axiom 5 ensures that there are no "jumps" or other discontinuities in the *ICI* values. Whether the presented set of five axioms is the best possible (optimal) choice is certainly open to discussion.

One feature not captured by the five axioms given by Brunelli and Fedrizzi [6], or axioms given by Brunelli [7], is the problem of the existence of an upper bound on the value of an inconsistency index. If an inconsistency index is not bounded from above, it might be problematic to interpret its values (what information does a decision maker obtain when the value of such an *ICI* is, for example, equal to 984 669?).

Therefore, it seems natural to require that an *ICI* possesses an upper bound. The need for an axiom on the existence of an upper bound was expressed, e.g., in [17].

Axiom 6. An *ICI* is bounded from above if and only if:

$$\exists K \in R;\ ICI(A) \leq K,\ \forall A \in M_R$$

Table 1 summarises which axioms are (not) satisfied by indices (5), (7–11). This is based on Brunelli and Fedrizzi [6], but has been extended to encompass the *KII* and *RIC* indices (introduced later), as well as axiom 6.

Table 1. Satisfaction of the axioms by inconsistency indices

Index/Axiom	A1	A2	A3	A4	A5	A6
CI	Y	Y	Y	Y	Y	N
GWI	Y	Y	N	?	Y	Y
GCI	Y	Y	Y	Y	Y	N
PLI	Y	Y	Y	Y	Y	N
RIC	Y	Y	N	Y	Y	Y
KII	Y	Y	Y	Y	Y	Y

Y indicates that an index satisfies an axiom, N that an index does not it and that the result is open. Source: modified from Brunelli and Fedrizzi [6] by the author.

5. A row inconsistency index for a pairwise comparison matrix

The concept of this measure of inconsistency of a pairwise comparison matrix (*PCM*) comes from a geometrical point of view: the rows of a *PCM* can be considered as vectors in *n*-dimensional Euclidean space. If a *PCM* is fully consistent, then the *PCM*'s rows are collinear (they differ only by a multiplicative constant). When an inconsistency appears, the rows will not be collinear any more, and their "deviation" from collinearity can be expressed by the cosines of the angles between each pair of row vectors.

Again, we set $a_{ij} \in (0, \infty)$.

Let $r_i = (a_{i1}, ..., a_{in})$ and $r_j = (a_{j1}, ..., a_{jn})$ be row vectors of a pairwise comparison matrix $A \in M_R$ of order *n*. Then the inconsistence index based on the rows of *A* is given as:

Let $A_{n\times n}(a_{ij}) \in M_C$. Then the row inconsistency index *RIC* is given as follows:

$$RIC = 1 - \frac{2\sum_{i=1}^{n-1}\sum_{j>i}^{n}\cos\varphi_{ij}}{n(n-1)} \tag{11}$$

where $\cos\varphi_{ij} = \frac{r_i \cdot r_j}{\|r_i\| \cdot \|r_j\|}$, and $r_i \cdot r_j$ denotes the dot product of r_i and r_j.

The *RIC* index is a "geometrically" based inconsistency index, which is equal to 1 minus the arithmetic mean of the cosines of the angles between each pair of row vectors of a given PCM. Also, the values of this index are conveniently bounded in the interval $[0, 1]$ (which will be proved below), where the larger the value of the *RIC*, the greater the inconsistency is. It should be noted that although the definition of the *RIC* is based on the row vectors, the use of the column vectors of the matrix *A* would be feasible as well. Now, some properties of the *RIC* index will be discussed.

Remark 2. Since $a_{ij} > 0, \forall i, j,$ the angle φ_{ij} satisfies $0 \le \varphi_{ij} < 90°$ for all r_i and r_j, and thus $0 \le \cos\varphi_{ij} < 1$.

Proposition 1. $A \in M_C \Leftrightarrow RIC = 0$.

Proof. When *A* is consistent, its rows are collinear, thus $\varphi_{ij} = 0, \forall i, j$ and $\cos\varphi_{ij} = 1, \forall i, j$, which yields *ICI* = 0.

If *RIC* = 0, then $\frac{2\sum_{i=1}^{n-1}\sum_{j>i}^{n}\cos\varphi_{ij}}{n(n-1)} = 1$. Since the codomain of cosine is $[-1,1]$ and the number of cosine terms in the numerator is $\binom{n}{2}$, we immediately get $\cos\varphi_{ij} = 1, \forall i, j > i.$ Hence, the row vectors r_i are collinear, rank (*A*) = 1 and $A \in M_C$.

Proposition 2. $RIC > 0 \Leftrightarrow A$ is not consistent.

Proof. Assume that $RIC > 0,$ then at least one $\cos\varphi_{ij} < 1,$ which means that at least one $\varphi_{ij} > 0.$ Therefore, there are at least two row vectors that are non-collinear. Hence, rank (*A*) ≥ 2 and *A* is not consistent.

Now assume that *A* is not consistent. Thus rank $(A) \ge 2,$ so at least two row vectors are not collinear and at least one $\varphi_{ij} > 0.$ Therefore, at least one element $\cos\varphi_{ij} < 1,$ which yields $RIC > 0.$

Proposition 3. $0 \le RIC \le 1.$

Proof. Since all the elements of the matrix *A* are positive, all the row vectors *r* lie in the first orthant of the space R^n. Hence, from (11) we get $\cos\varphi_{ij} \ge 0, \forall i, j > i,$ but also $\cos\varphi_{ij} \le 1, \forall i, j > i.$ Therefore, $0 \le \frac{2\sum_{i=1}^{n-1}\sum_{j>i}^{n}\cos\varphi_{ij}}{n(n-1)} \le 1,$ and finally

$$0 \le RIC = 1 - \frac{2\sum_{i=1}^{n-1}\sum_{j>i}^{n}\cos\varphi_{ij}}{n(n-1)} \le 1$$

Proposition 4. *RIC* satisfies Axiom 1.

Proof. This follows directly from Proposition 1.

Proposition 5. *RIC* satisfies Axiom 2.

Proof. This statement is obvious, as the average value of $\cos\varphi_{ij}$ in (11) is computed from all the pairs of rows regardless of their order (all rows are treated equally).

Proposition 6. *RIC* does not satisfy Axiom 3.

Proof. (By counterexample): Let $A \in M_R$ be the pairwise comparison matrix given as follows:

$$A = \begin{pmatrix} 1 & 0.1 & 0.15 \\ 10 & 1 & 0.3 \\ 6.6666 & 3.3333 & 1 \end{pmatrix}, \quad RIC(A) = 0.047$$

Let B be the pairwise comparison matrix obtained from the matrix A by squaring each entry of the matrix A:

$$B = A^{(b=2)} = \begin{pmatrix} 1 & 0.01 & 0.0225 \\ 100 & 1 & 0.09 \\ 44.4444 & 11.1111 & 1 \end{pmatrix}, \quad RIC(B) = 0.018.$$

Thus, although the preferences expressed in the matrix B are “stronger”, the inconsistency index *RIC* returns a smaller value.

Proposition 7. *RIC* satisfies axiom 4.

Proof. *RIC* is increasing in the angle φ between any two row vectors of A (all else fixed). The greater the change in one element (and its reciprocal) from a consistent matrix, the larger the angle φ between this row and the other (collinear) rows. Hence the *RIC* is larger as well.

Proposition 8. *RIC* satisfies Axiom 5.

Proof. The proof is obvious as the cosine function in (11) is continuous.

Proposition 9. *RIC* satisfies Axiom 6.

Proof. $RIC<1$ follows directly from Proposition 3.

6. Satisfaction of axiom 6 for a set of selected indices

In this section satisfaction of Axiom 6 is discussed for selected inconsistency indices, namely for *CI*, *CR*, *GWI*, *PLI, GCI* and *KII*.

Proposition 10. *CI* does not satisfy axiom 6.

Proof. Consider a corner pairwise comparison matrix of order *n*. The characteristic polynomial of the matrix is: $\lambda_{\max}^3 - n\lambda_{\max}^2 = (n-2)(x+x^{-1}-2)$ [16].

Since $n \in N$ and $\lambda_{\max} \geq n,$ we have:

$$\lambda_{\max}^3 > \lambda_{\max}^3 - n\lambda_{\max}^2 = (n-2)(x+x^{-1}-2)$$

and

$$\lambda_{\max} > \left[(n-2)(x+x^{-1}-2)\right]^{1/3}$$

From the last inequality, for *n* fixed and we obtain $\lambda_{\max} \to \infty$, therefore *CI* is not bounded from above.

As for the consistency ratio, the question of whether it is bounded from above or not cannot be answered without knowledge of the precise asymptotic behaviour of *RI*, which remains unknown, although, for example, Alonso and Lamata [2] provide a linear estimate of *RI* for increasing *n*.

Proposition 11. *GWI* satisfies axiom 6.

Proof. In the definition of *GWI* $\bar{A}(\bar{a}_{ij})$ is a normalized matrix where the sum of entries in each column is equal to 1. Let $K = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{a}_{ij}$. Certainly, $K > GWI$, because $\forall i \in \{1, 2, ..., n\}$. Now, *K* is the mean of *n* column sums, where each sum is equal to one. Hence, *K* = 1 for all *n*. Therefore, *K* provides an upper bound for *GWI* as required.

Proposition 12. *PLI* does not satisfy axiom 6.

Proof. (By counterexample): Consider a corner pairwise comparison matrix of order $n = 3$. Then, $PLI = x + 1/x$, which is not bounded from above.

Proposition 13. *GCI* does not satisfy axiom 6.

Proof. (by counterexample): Consider a corner pairwise comparison matrix of order $n = 3$. In this case we get: $GCI = \ln^2(w_2/w_1) + (xw_3w_1) + \ln^2(w_3/w_2)$. The first and third terms are constant, but the middle one is logarithmically increasing, and because the logarithmic function is not bounded from above, neither is *GCI*.

Proposition 14. *KII* satisfies Axiom 6.

Proof. This is obvious from the definition of *KII*.

7. Numerical example. A corner pairwise comparison matrix

To numerically illustrate the behaviour of selected indices, the corner pairwise comparison matrix *C* of order $n = 3$ (see below) will be examined. In the matrix *C* all of the entries, except for $a_{1n} \equiv x$ and $a_{n1} \equiv 1/x$, are set to 1, and $x \in R,\ x \geq 1$.

$$C = \begin{pmatrix} 1 & 1 & x \\ 1 & 1 & 1 \\ 1/x & 1 & 1 \end{pmatrix}$$

The dependence of selected inconsistency indices on *x* is shown in Table 2, Figs. 1 and 2. It should be noted that in this case $PLI = x + 1/x$.

As can be seen from this comparison, *RIC* is the least rapidly increasing index in *x*, while *PLI* is the most rapidly increasing one. *GCI*, *PLI*, and *CI* grow almost linearly. The *KII* index grows rapidly for small *x* (lower than 5), and then levels off.

Table 2. Comparison of inconsistency indices for various values of *x*

x	1	2	3	4	5	6	7	8	9	10	100
RIC	0	0.0474	0.1011	0.1391	0.1658	0.1853	0.1999	0.2113	0.2204	0.2279	0.292
CI	0	0.027	0.068	0.109	0.147	0.184	0.218	0.25	0.28	0.309	1.428
GWI	0	0.1595	0.2509	0.3113	0.3547	0.3875	0.4134	0.4344	0.4518	0.4666	0.6492
PLI	0	0.5	1.3333	2.25	3.2	4.1667	5.1429	6.125	7.1111	8.1	98.1
KII	0	0.5	0.667	0.75	0.8	0.8333	0.857	0.875	0.889	0.9	0.99
GCI	0	0.1602	0.4023	0.6406	0.8634	1.07	1.2622	1.4414	1.6093	1.7676	7.0692

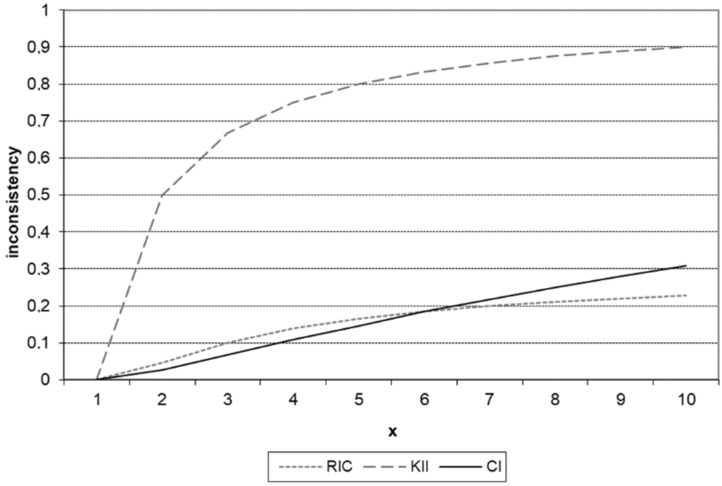

Fig. 1. A graphical comparison of the inconsistency indices *RIC*, *KII* and *CI*

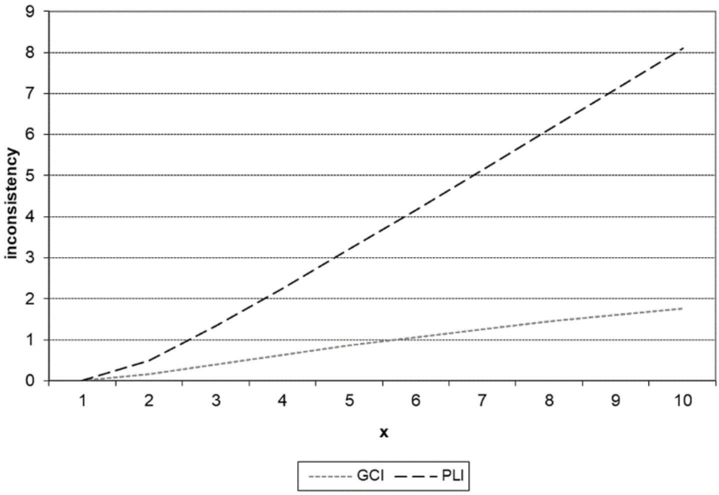

Fig. 2. A graphical comparison of the inconsistency indices *GCI* and *PLI*

8. Conclusions

The goal of this paper was to introduce the row inconsistency index (*RIC*) and to study its properties. Other aims of the paper included examining the existence of an upper bound on several inconsistency indices, and a comparison of these indices' behaviour for a corner pairwise comparison matrix.

Further study may take a broader scope concerning the "quality" (consistency) of pairwise comparisons, a look beyond inconsistency indices in the form they are studied today. As indicated by several recent studies [3] or [19], even low values of inconsistency indices cannot guarantee satisfaction of several natural properties, such as preservation of the order of preference. Therefore, to evaluate (in)consistency in pairwise comparisons, some other approach might be necessary.

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