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MEASURING CONFLICT AND POWER IN STRATEGIC SETTINGS

This is a quantitative approach to measuring conflict and power in strategic settings: non-cooperative games (with cardinal or ordinal utilities) and blockings (without any preference specification). A (0, 1)-ranged index is provided, taking its minimum on common interest games, and its maximum on a newly introduced class termed "full conflict" games.

Keywords: *strategic game, conflict, coalitional game, power index*

1. Introduction

In non-cooperative or strategic games there are $n \geq 2$ players, each of whom takes some action and everyone's utility depends on the n-tuple of actions taken, known as theaction profile [16], [20], [22]. In *common interest* strategic games there is at least one profile at which each player's utility is maximized [3], [6]. Conversely, there is conflict when for any profile at least one player strictly prefers another one. Hence, either there is common interest or else there is conflict. But how much conflict can there be? What maximum distance may separate a game from the common interest case? The motivation behind this paper is to provide quantitative answers to these and related questions.

In 2-player constant-sum games, not only there is no pair (or profile) of actions where both players attain their maximum payoff, but also whenever a player attains her maximum the other player attains her minimum. Hence, there is much conflict. Yet, n-player constant-sum games, $n > 2$, may display varying degrees of conflict *within coalitions*. For example, assume utilities only take non-negative values, and consider two cases: (a) for every action profile at most one player gets a strictly posi-

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tive utility (while all the others get 0 utility), and (b) for every profile half of the players get a strictly positive utility (and the others get 0). The former case displays more conflict than the latter.

It must be emphasized that strategic games are often regarded as situations where *there is no external mechanism available for the enforcement of agreements or commitments* [30], in which case no attention is placed on cooperative behavior. Conversely, this paper focuses on coordination within coalitions, because (as indexated by examples (a) and (b) above) there is conflict not only between individual players, but also between and within coalitions.

Technically, measuring conflict is an *aggregation* issue: each action profile corresponds to some n-tuple of utility values (although in general this is not a bijection), and if there are $\alpha \in \mathbb{N}$ distinct profiles (with[1] $\alpha \geq 4$), then $n\alpha$ real quantities must be aggregated into the sought *index*, which shall be a real number (possibly in the unit interval). In fact, the basic aggregation method proposed here disregards all those, often many, action profiles which are non-informative (or less informative) about conflict. In general, the focus has to be placed on those action profiles where different groups are recognized to pursue, through coordinated actions, conflicting goals. The worth of coordination within groups depends on how efficiently such goals can be pursued.

One way to observe different parties pursuing (possibly) conflicting goals is to just consider all two-party situations, that is to say every possible coalition opposed to its complement in pursuing their own goals, which is maximization of their members' utilities. In particular, if utility is transferable, then cooperation within coalitions aims to maximize the sum of its members' utilities, because such a sum can then be (internally) redistributed. Otherwise, cooperation aims to maximize the sum of members' *normalized* utilities, obtained as the ratio of utilities to their maximum, because there is no means of redistribution. Note that as long as conflict is measured as a distance from the common interest case, players' information may be ignored, at least in principle. In any case, everything about the game is here assumed to be common knowledge: everyone knows each player's utility for each action profile.

Although several possible extensions are proposed, the basic method used here for turning strategic games into coalitional ones relies upon the following simple (behavioral) idea: as long as no player takes an action, out-of-time bargaining takes place in order to reach overall coordination; as soon as any coalition takes a coordinated action, its complement also takes a coordinated action and if there is no n-tuple of actions maximizing each player's utility (in which case there is no conflict whatsoever), then coalitions are assumed to always exercise all the *power* they have. Power is commonly regarded as being the capability of sanctioning [6] and thus exercising power means retaliating. That is to say coalitions are assumed to always choose retaliation against their complement from among their best responses. Accordingly, the

[1] The *minimal* strategic settings, in this respect, are 2×2 (i.e. 2-player, 2-action) games.

worth of each coalition is obtained as the maximum, over all its group actions, of the (normalized) coalitional utility attained when its complement chooses retaliation from among its best responses.

The worth of coalitions is a fundamental concept in cooperative game theory, where coalitional games are defined by real-valued functions assigning a worth to each coalition [6], [20], [32], commonly interpreted as the total amount of TU (transferable utility) that members can earn without any help from non-members. In most conceivable applications a coalition's payoff is substantially affected by the actions of non-members [26]. From this perspective, strategic games precisely formalize those interactions where the worth of coalitions, however quantifiable, is explicitly modeled to depend on non-members' actions. Hence, the proposed method for mapping strategic games into coalitional ones, and associated issues, may also be interesting *per se*.

One way to obtain the sought index is to fix its behaviour for extreme cases in a desirable manner. One extreme case obviously corresponds to common interest games, where there is no conflict, as players all agree on any (possibly unique) action profile where everyone attains her maximum utility[2], so the index must take value 0. In any game, for each player the set of all action profiles can be partitioned into two blocks: one containing all those profiles where the player gets 0 utility, with the other containing all those giving her strictly positive utility. Now assume that there is no profile giving any two players strictly positive utilities. In this case, conflict is maximal, as at most one player gains from interaction. Additionally, at an intuitive level, if players' goals are pairwise mutually exclusive, then the measure of conflict should grow with the number n of such (mutually exclusive) goals. Hence, the index should attain its maximum for this latter case and such a maximum should approach unity as $n \to \infty$.

As long as retaliation plays a role, quantifying the worth of coalitions in line with the above argument inherently prevents separating issues concerning *conflict* from those concerning *power*. In fact, once the conceptual approach has been detailed for strategic games, it can also be applied to more abstract settings where preferences together with conflict on the one side, and actions together with power on the other, can be dealt with separately. This is achieved by introducing a (finite) number m of *outcomes*, over which each $i \in N$ has preferences in the form of a binary relation $\succsim_i$ (or collection of ordered pairs of outcomes), where N denotes the set of players. These relations are turned into families of $\succsim_i$-*consistent* permutations of outcomes, where (strictly) preferred outcomes have to appear before worse ones. If there is a permutation of outcomes which is $\succsim_i$-consistent for every $i \in N$, then there is no conflict. Developing this idea, a *distance* between permutations can be introduced and thus conflict within coalitions can be measured in terms of the distances between members' families of permutations.

[2] This means disregarding any *signaling problem* that players may have in coordinating [2].

On the other hand, abandoning preferences and reintroducing actions yields *game forms*, which are abstract settings perhaps even more suitable than games for measuring power. In fact, they are strategic games where utilities (or, more generally, players' preferences over outcomes) are not specified. Accordingly, the only information available is the *mechanism*, that is, how action profiles are mapped onto outcomes. The main tool for dealing with such a setting are *blockings* (or, dually, *effectivity functions*), specifying what (family of) outcomes each coalition can block. Without preferences, conflict is no longer measurable, but the power of coalitions still seems quantifiable, although exclusively on an enumerative basis. That is to say by counting the number of blocked outcomes. Using a suitable normalization, this also yields a [0, 1]-ranged (monotone) coalitional game and thus the whole former approach for strategic games can be reproduced with minor adjustments. The resulting index measuring power turns out to attain its maximum for *Maskin blockings*, where each (nonempty) coalition can block all subsets of possible outcomes that do not contain a fixed outcome, which is therefore most naturally interpreted as the *status quo*.

2. Preliminaries

Consider a finite set $N = \{1, ..., n\}$ of players and let $\mathbb{A}_i$ be the finite set of actions available to player $i \in N$, with cardinality $|\mathbb{A}_i| = \alpha_i \geq 2$. The n-product $\mathbb{A} = \mathbb{A}_1 \times ... \times \mathbb{A}_n$ contains all $\prod_{1 \leq i \leq n} \alpha_1 = \alpha$ distinct action profiles, a generic profile being denoted by $a = (a_1, ..., a_n)$. A *utility mapping* u: $\mathbb{A} \to \mathbb{R}_+^n$ quantifies as $u_i(a)$ the utility attained by $i \in N$ at action profile a. Denote $u(a) = (u_1(a), ..., u_n(a))$ for every $a \in \mathbb{A}$. Hence, u may be regarded as a point in $\mathbb{R}_+^{n\alpha}$. Utilities can be normalized by introducing $\psi_i(a) = \frac{u_i(a)}{\bigvee_{a' \in \mathbb{A}} u_i(a')}$ for all $i \in N$ and $a \in \mathbb{A}$, where $\vee$ and $\wedge$ denote the *max* and *min*, always used over finite sets of real quantities. In words, the utility attained by any player for any action profile is divided by the maximum utility that such a player attains over all the action profiles and therefore $\bigvee_{a \in \mathbb{A}} \psi_i(a) = 1$. Also, denote by $\varnothing \neq \mathbb{A}^i \subseteq \mathbb{A}$ the subset of action profiles for which i's utility is strictly positive, that is $\psi_i(a) > 0$ for all $a \in \mathbb{A}^i$ and $\psi_i(a') = 0$ for all $a' \in \mathbb{A} \backslash \mathbb{A}^i$. Notice that similar notation is used to denote two substantially different things: the sets $\mathbb{A}_i$ of individual actions and the sets $\mathbb{A}^i$ of action profiles where individual players get strictly positive utility ($i \in N$).

As formalized below, in common interest games there is some (at least one) $a \in \mathbb{A}$ such that $\psi_i(a) = 1$ for all $i \in N$. Let $\mathcal{G}_S^n$ denote the set of strategic games with n players, with generic element $(N, \mathbb{A}, u) = \Gamma \in \mathcal{G}_S^n$. For any $\Gamma \in \mathcal{G}_S^n$ and strictly positive real number t, set $(N, \mathbb{A}, tu) = t\Gamma \in \mathcal{G}_S^n$, where $tu \in \mathbb{R}_+^{n\alpha}$ is simply the utility mapping multiplied by t, that is to say $tu(a) = (tu_1(a), ..., tu_n(a))$ for every $a \in \mathbb{A}$. Measuring conflict in non-cooperative games formally means defining a mapping $\kappa : \mathcal{G}_S^n \rightarrow \mathbb{R}_+$ for all $n \geq 2$.

For $\varnothing \neq A \in 2^N = \{B\colon B \subseteq N\}$, let $\mathbb{A}_A = \underset{i \in A}{\times} \mathbb{A}_i$ denote the set of $|A|$-tuples of actions available to coalition $A = \{i_1, ..., i_{|A|}\}$. Each $|A|$-tuple $a_A = (a_{i_1}, ..., a_{i_{|A|}}) \in \mathbb{A}_A$ defines one action for each coalition member. For each $a \in \mathbb{A}$, let A's coalitional utility be quantified by $\psi_A(a) = \sum_{i \in A} \psi_i(a)$. Each coalition A aims to obtain $\underset{a \in \mathbb{A}}{\vee} \psi_A(a)$ while treating the $n - |A|$-tuple $(a_{j_1}, ..., a_{j_{n-|A|}})$ of actions taken by the players $j \in A^c = N \backslash A$ as given, where $A^c = \{j_1, ..., j_{n-|A|}\}$. For each $a_A \in \mathbb{A}_A$, consider the set $BR(a_A) \subseteq \mathbb{A}_{A^c}$ of A^c's *best responses* to a_A, defined by

$$\psi_{A^c}(a_{i_1}, ..., a_{i_{|A|}}, a_{j_1}, ..., a_{j_{n-|A|}}) = \underset{a'_{A^c} \in \mathbb{A}_{A^c}}{\vee} \psi_{A^c}(a_{i_1}, ..., a_{i_{|A|}}, a'_{j_1}, ..., a'_{j_{n-|A|}})$$

for all $(a_{j_1}, ..., a_{j_{n-|A|}}) \in BR(a_A)$.

An alternative model is obtained by introducing a set $\chi = \{x_1, ..., x_m\}$ of *outcomes* of the game, so that players' preferences may be, more generally, defined as a *binary relation* $\gtrsim_i$ for every $i \in N$. Technically, $\gtrsim_i \subseteq \chi \times \chi$ is a collection of ordered pairs of outcomes. As long as the binary relation is *complete* and *transitive* (or *rational* [16]), such an *ordinal approach does not prevent a player from having a utility function, but it somehow regards the specification of her utility function as being her own business* [9]. Yet, in order to measure conflict, how many alternatives are preferred w.r.t. (with respect to) one another must be quantifiable, requiring a cardinal approach. To this end, preferences $\gtrsim_i$ may be mapped into (non-empty) collections $\varnothing \neq \mathcal{S}_i \subseteq \mathcal{S}(m)$, where $\mathcal{S}(m)$ denotes the set of all $m!$ permutations $\pi : \{1, ..., m\} \rightarrow \{1, ..., m\}$. Introducing a *distance* $d(\mathcal{S}, \mathcal{S}')$ between collections $\mathcal{S}, \mathcal{S}' \subseteq \mathcal{S}(m)$ of permutations allows, even in this broader setting, to turn the situation into a coalitional game. As already outlined, turning the given strategic situation into a coalitional game is the general first step used in all scenarios where conflict is measured.

Finally, it may be assumed that the mechanism $\mathcal{M}:\mathbb{A} \to \chi$ maps action profiles onto outcomes. This setting is used when game forms enter the picture. Without preferences, one is lead to deal strictly with power, rather than conflict, as any sought quantification has to rely only on how efficaciously players and coalitions manage to force (any) outcomes. This is precisely the information formalized by blockings $b:2^N \to 2^{2^\chi}$, as $b(A) \in 2^{2^\chi}$ is the family of outcome subsets that coalition A may block [9] (see below).

Coalitional games with a set of players N are defined by *set functions* $v:2^N \to \mathbb{R}_+$, $v(\emptyset) = 0$. In cooperative game theory [21], [25], where such games play a central role, they are often assumed to be monotone, that is to say $v(A) \le v(B)$ for all $A \subseteq B \subseteq N$, and $v(A)$ is commonly interpreted as the worth of (cooperation within) coalition $A \in 2^N$ (see above). Let $\mathcal{G}_C^n$ denote the set of coalitional games with n players. The Shapley value [27] is the mapping $\phi^{Sh}: \mathcal{G}_C^n \to \mathbb{R}^n$ given, for $1 \le i \le n$, by

$$\phi_i^{Sh}(v) = \sum_{A \in 2^N : i \in A} \frac{v(A) - v(A \setminus i)}{n\binom{n-1}{|A|-1}}.$$

Another important *solution* (i.e. a semivalue) of coalitional games is the Banzhaf [4] value $\beta: \mathcal{G}_C^n \to \mathbb{R}^n$, which takes the form

$$\beta_i(v) = \sum_{A \in 2^N : i \in A} \frac{v(A) - v(A \setminus i)}{2^{n-1}} \quad \text{for } v \in \mathcal{G}_C^n \text{ and } 1 \le i \le n.$$

3. An index measuring conflict

Firstly consider non-transferable utility (or NTU) games $\Gamma \in \mathcal{G}_S^n$. As already mentioned, defining the desired mapping $\kappa: \mathcal{G}_S^n \to \mathbb{R}_+$ means solving an aggregation problem: once the action set $\mathbb{A}$ is given, any associated game Γ is defined by u alone and thus is, in fact, a point in $\mathbb{R}_+^{n\alpha}$. On the other hand, from a geometrical viewpoint coalitional games $v \in \mathcal{G}_C^n$ are points in $\mathbb{R}_+^{2^n-1}$. Accordingly, aggregation is performed in two steps: firstly turning any $\Gamma \in \mathcal{G}_S^n$ into $v_\Gamma \in \mathcal{G}_C^n$ and then mapping this latter game into the required real number, that is to say $\mathcal{G}_S^n \to \mathcal{G}_C^n \to \mathbb{R}_+$. Note that if $\alpha_i \geq 2$ for each $1 \leq i \leq n$, then $n\alpha = 2^n > 2^n - 1$, while $\alpha_i = n$, $1 \leq i \leq n$ yields $n\alpha = n^{n+1}$. Hence, depending on the number of action profiles, the first step may contribute to overall aggregation to varying degrees.

Definition 1. *For every* $\Gamma \in \mathcal{G}_S^n$, *define* $v_\Gamma \in \mathcal{G}_C^n$ *by* $v_\Gamma(\varnothing) = 0$, *as well as*

$$v_\Gamma(N) = \frac{\vee_{a \in \mathbb{A}} \psi_N(a)}{n},$$

and for $\varnothing \subset A \subset N$

$$v_\Gamma(A) = 1 \text{ if } v_\Gamma(N) = 1$$

and

$$v_\Gamma(A) = \frac{\bigvee_{a_A \in \mathbb{A}_A} \bigwedge_{a_{A^c} \in BR(a_A)} \psi_A(a_A, a_{A^c})}{|A|} \text{ if } v_\Gamma(N) < 1,$$

where $\psi_A(a_A, a_{A^c}) = \psi_A(a_{i_1}, \ldots, a_{i_{|A|}}, a_{j_1}, \ldots, a_{j_{n-|A|}})$ (*see above*).

Clearly, $v_\Gamma(N) \leq 1$ *for all* $\Gamma = (N, \mathbb{A}, u) \in \mathcal{G}_S^n$. *In particular*, $v_\Gamma(N) = 1$ *iff* (*if and only if*) Γ *is a* ***common interest game****, i.e. a game where there exists some (at least one) action profile maximizing each player's utility.*

According to this definition, the worth $v_\Gamma(A)$ of cooperation within coalition A in game Γ is quantified as follows. Firstly, if Γ is a common interest game, then the final outcome is assumed to be some socially optimal one, i.e. some action profile $a \in \mathbb{A}$ where $\psi_N(a) = n$. Under this assumption, in common interest games the worth of co-operation within coalitions simply equals their cardinality and thus the normalized worth equals 1. On the other hand, in games with conflict when any coalition $\varnothing, N \neq A \in 2^N$ evaluates which coordinated action to take, its complement A^c is assumed to also take a coordinated action and if there is no action profile maximizing each player's utility, then A^c chooses *retaliation among the best responses* to the coordinated action taken by A. Finally, if cooperation (i.e. coordination) is achieved within the grand coalition N, then some collective action $a \in \mathbb{A}$ maximizing $\psi_N(a)$ is taken.

Definition 2. $\Gamma \in \mathcal{G}_S^n$, *is a* ***full conflict game*** *if both the following hold*:

(i) $\mathbb{A}^i \cap \mathbb{A}^j = \varnothing$ *for all* $1 \leq i < j \leq n$,

(ii) *for all* $\varnothing \subset A \subset N$ *and all* $a_A \in \mathbb{A}_A$ *there is some* $a_{A^c} \in \mathbb{A}_{A^c}$ *such that* $(a_A, a_{A^c}) \notin \bigcup_{i \in A} \mathbb{A}^i$*, with* $\subset$ *denoting strict inclusion.*

(*This applies to both NTU and TU games as it does not involve utilities*.)

Clearly, $v_\Gamma(A) \in [0, 1]$ for all $A \in 2^N$. In particular, $v_\Gamma(A) \in \left[\frac{1}{n}, 1\right]$. Since there are $2^n - 1$ non-empty coalitions, the sought index $\kappa : \mathcal{G}_S^n \to [0, \kappa^*(n)]$, where $\kappa^*(n) = 1 - \frac{1}{n(2^n - 1)}$, depends simply on the average of the $2^n - 1$ values taken by v_Γ, i.e.

$$\kappa(\Gamma) = 1 - \frac{\sum_{\varnothing \neq A \in 2^N} v_\Gamma(A)}{2^n - 1}.$$

Claim 3. *The following two statements apply to all* $\Gamma \in \mathcal{G}_S^n$

(1) $\kappa(\Gamma) = 0$ *iff* Γ *is a common interest game,*

(2) $\kappa(\Gamma) = \kappa^*(n)$ *iff* Γ *is a full conflict game.*

Proof: Concerning (1), by construction $\kappa(\Gamma) = 0$ requires $v_\Gamma(A) = 1$ for every $\varnothing \neq A \in 2^N$ and therefore the desired conclusion follows straightforwardly from the construction of v_Γ itself according to Definition 1. As for (2), $\kappa(\Gamma) = \kappa^*(n)$ requires $v_\Gamma(A) = 0$ for all $N \neq A \in 2^N$ and $v_\Gamma(N) = \frac{1}{n}$, as $v_\Gamma(N) \geq \frac{1}{n}$. Full conflict games satisfy these conditions. Hence, it must be checked that non-full conflict games do not. If $\mathbb{A}^i \cap \mathbb{A}^j \neq \varnothing$ for some $i \in N$, $j \in N \backslash i$, then $v_\Gamma(A) > 0 < v_\Gamma(A^c)$ for all $A \in 2^N$ such that $i \in A$, $j \in A^c$. Hence, Condition (i) in Definition 3 above is a necessary one. Still, it is not sufficient, because it is also required that only coordination within the grand coalition N enables some player (i.e. a single one) to get a strictly positive (i.e. her maximum) utility. This is precisely what Condition (ii) yields. In fact, in view of (i), any coalition $\varnothing \subset A \subset N$ can attain a strictly positive utility for, at most, only one of its members. Furthermore, (i) also implies that any such coordination would also yield zero utility for all non-members $j \in A^c$. But then (ii) states that A^c is also capable, in turn, by coordinating all its members, to prevent any $i \in A$ from getting a strictly positive utility.

□

Note that $\mathbb{A}^i \cap \mathbb{A}^j = \varnothing$ for all $1 \leq i < j \leq n$ alone assures that at most one player may gain from interaction and thus that there is full conflict, in a broad sense. In fact, this is also sufficient (but not necessary) to ensure $v_\Gamma(N) = \frac{1}{n}$. But $v_\Gamma(A) = 0$ for all $N \neq A \in 2^N$ only occurs if, in addition, each $\varnothing \subset A \subset N$ is able, while coordinating a best response, to *fully retaliate* against its complement A^c. That is to say, every coalition must be able, by itself, to prevent any non-member from getting a strictly positive utility. As detailed in the sequel, this may also mean that every coalition (and thus singletons as well) has *maximum power*. In fact, the situation where no player gets a strictly positive utility may be seen as the *status quo*, which in full conflict games can be forced by any coalition (and thus by any player as well). Therefore, κ measures not only conflict, but also possibilities for retaliation or power. Also, the above definition of the coalitional game v_Γ relies upon the assumption that retaliation, or the full exercise of power, is a universal behavioral norm.

Example 4. *Let i and j denote players, while* 0 *and* 1 *denote actions. For* $0 \le \delta, \delta'$, $\gamma, \gamma' \le 1$ *and* $\delta + \delta' \ge \gamma, \gamma'$, *the game* Γ *and index* $\kappa(\Gamma)$ *are defined as follows. When both players choose action* 0, *player i gets payoff* γ *while player j gets payoff* γ', *and analogously for the other three entries of this* 2×2-*matrix.*

Table 1

Payoff matrix for Γ

$(u_i(a_i, a_j), u_j(a_i, a_j))$	$a_j = 0$	$a_j = 1$
$a_i = 0$	(γ, γ')	$(0,1)$
$a_i = 1$	$(1, 0)$	(δ, δ')

- $\delta = \delta' = 1$ yields $\kappa(\Gamma) = 0$,
- $1 \le \delta + \delta' < 2$ yields $\kappa(\Gamma) = 1 - \dfrac{\delta+\delta'}{3} - \dfrac{\delta+\delta'}{6} = 1 - \dfrac{\delta+\delta'}{2}$
- $0 < \delta+\delta' < 1$ yields $\kappa(\Gamma) = 1 - \dfrac{\delta+\delta'}{3} - \dfrac{1}{6} = \dfrac{5}{6} - \dfrac{\delta+\delta'}{3}$,
- $0 = \delta+\delta'$ ($\Rightarrow 0 = \gamma = \gamma'$) yields $\kappa(\Gamma) = \dfrac{5}{6} = \kappa^*(2)$.

3.1. The transferable utility case

Minor adjustments allow us to extend the approach developed thus far to strategic TU games, where the utility of each player is measured in terms of some common unit, for example in monetary terms. Accordingly, now let $\Gamma \in \mathcal{G}_S^n$ denote a generic game of this form with n players and the definition of full conflict is maintained unchanged as it does not involve the transferability of utilities at all. For each $A \in 2^N$ and each $a \in \mathbb{A}$ let $u_A(a) = \sum_{i \in A} u_i(a)$.

Definition 5. *For every* $\Gamma \in \mathcal{G}_S^n$, *define* $w_\Gamma \in \mathcal{G}_C^n$ *by* $w_\Gamma(\varnothing) = 0$ *and* $w_\Gamma(N) = 1$. *Next, for* $\varnothing \subset A \subset N$, *set* $w_\Gamma(A) = 1$ *if* $\bigvee_{a \in \mathbb{A}} \psi_N(a) = n$ *and*

$$w_\Gamma(A) = \frac{\bigvee_{a_A \in \mathbb{A}_A} \bigwedge_{a_{A^c} \in BR(a_A)} u_A(a_A, a_{A^c})}{\bigvee_{a_A \in \mathbb{A}_A} u_A(a)} \text{ otherwise.}$$

Again w_Γ is [0, 1]-ranged, and the sought index η: $\mathcal{G}_S^n \to [0, \eta^*(n)]$ may be obtained simply by averaging the 2^n–1 values of w_Γ, i.e.

$$\eta(\Gamma) = 1 - \frac{\sum_{\varnothing \neq A \in 2^N} w_\Gamma(A)}{2^n - 1},$$

where $\eta^*(n) = 1 - \dfrac{1}{2^n - 1}$.

Claim 6. *The following two statements apply to all* $\Gamma \in \mathcal{G}_S^n$

(1) $\eta(\Gamma) = 0$ *iff* Γ *is a common interest game,*

(2) $\eta(\Gamma) = \eta^*(n)$ *iff* Γ *is a full conflict game.*

Proof: Concerning (1), $\eta(\Gamma) = 0$ requires $w_\Gamma(A) = 1$ for all $\varnothing \neq A \in 2^N$ and thus the desired conclusion follows straightforwardly from the definition of w_Γ. As for (2), $\eta(\Gamma) = \eta^*(n)$ requires $w_\Gamma(A) = 0$ for all $N \neq A \in 2^N$. As in the above proof of Claim 3, this is seen to hold for full conflict games, but not to hold for non-full conflict games. □

The worth of cooperation within coalitions in strategic TU games Γ is obtained, *mutatis mutandis*, as before for NTU ones. If Γ is a common interest game, then the final outcome is assumed to be some socially optimal one, as the chosen action profile $a \in \mathbb{A}$ is assumed to be such that $\psi_N(a) = n$. Under this assumption, in TU common interest games the worth of all $2^n - 1$ non-empty coalitions equals 1. Otherwise, the worth of coalition A is given by the ratio of the highest coalitional utility u_A attainable when its complement A^c chooses retaliation among its best responses, i.e. $\bigvee_{a_A \in \mathbb{A}_A} \bigwedge_{a_{A^c} \in BR(a_A)} u_A(a_A, a_{A^c})$, to the highest attainable coalitional utility u_A over all action profiles, i.e. $\bigvee_{a \in \mathbb{A}_A} u_A(a)$. Given this normalization, the worth $w_\Gamma(N)$ of the grand coalition N always equals unity, for any strategic TU game $\Gamma \in \mathcal{G}_S^n$.

If utility is transferable and both synergies and conflict are relevant, then (possibly new) distributional norms may well come about. In fact, in this case coordination may lead to substantial improvement w.r.t. non-cooperative outcomes and those who gain from coordination have the means (i.e. by transferring utility) for compensating those who are worse off at the (socially optimal) action profile implemented through coordination. In particular, if the level of conflict is high, then this latter fraction of players, and their loss w.r.t. their most favourable outcomes are important. Hence, $u_N^* \bigvee_{a \in \mathbb{A}_A} u_N(a)$ quantifies the maximum (gross) surplus attainable through cooperation. Therefore, $\eta(\Gamma)u_N^* \in \mathbb{R}_+$ also incidentally provides a measure of the degree to which the interaction at hand (i.e. Γ itself) requires, through bargaining, distributional norms.

For full conflict games, the NTU index always exceeds the TU one, as $\kappa^*(n) > \eta^*(n)$, $n \geq 2$. In practice, full conflict TU games formalize what in cooperative game theory is known as the *unanimity game* associating a worth of 1 to the grand coalition N and a worth of 0 to all other coalitions $A \neq N$. A reasonable way of dividing such a unitary worth attainable only through unanimous and overall cooperation is to give each player $\frac{1}{n}$. This is the *Shapley value* of the game. In this case, each player is sure of receiving a strictly positive amount, which is the same for all players, and thus should (reasonably) cooperate. Conversely, if utilities are incomparable, then in order to achieve overall cooperation, players might use some device to choose one number $i \in \{1, ..., n\}$ at random with uniform probability and then maximize i's utility, giving i one unit of *her* utility. Then, each player gets a random variable that takes value 1 with probability $\frac{1}{n}$ and value 0 with probability $1 - \frac{1}{n}$. Clearly, getting 1 with probability $\frac{1}{n}$ and 0 with probability $\frac{n-1}{n}$ is very much different (under risk aversion) than getting $\frac{1}{n}$ for sure. On an intuitive basis, this explains why $\kappa^*(n) > \eta^*(n)$, $n \geq 2$.

Remark 7. *Both κ and η are invariant w.r.t. linear transformations, that is to say $\kappa(\Gamma) = \kappa(t\Gamma)$, as well as $\eta(\Gamma) = \eta(t\Gamma)$ for all $\Gamma \in \mathcal{G}_S^n$ (whether NTU or TU) and all $t > 0$ (see above). This is an immediate consequence of the two different normalizations used in the definition of the coalitional games v_Γ and w_Γ.*

3.2. The power of players

Intuitively, the power of players $i \in N$ in strategic games $\Gamma \in \mathcal{G}_S^n$ might seem to depend mainly on the number $\alpha_i = |\mathbb{A}_i|$ of actions available to her. In fact, if a player has many (non-redundant) actions, then for any $n - 1$-tuple of actions taken by the others, she may choose from a large set of distinct outcomes. Still, such outcomes might only differ slightly, especially when considered from the viewpoint of other players $j \in N \backslash i$. Conversely, a player may have a very small set of (non-redundant) actions available to her, but nevertheless her choice could be significant for everybody, as she could prevent any player from getting a strictly positive utility (as in the full conflict case).

Claim 8. *The minimum number of action profiles in full conflict games with n players is* 2^n.

Proof: Note that 2 is the minimum number of actions that must be available to an individual to properly regard her as a player in a game. Otherwise, with just one action available to her, that individual would not interact at all. Hence, it suffices to check whether a full conflict game can be constructed in which all players have only two available actions, denoted 0 and 1. If $\mathbb{A}_i = \{0, 1\}$, $1 \le i \le n$, then the set of action profiles $\mathbb{A} = \underset{i \in N}{\times} \mathbb{A}_i = \{0,1\}^n$ is the set of vertices of the n-dimensional unit hypercube $[0, 1]^n$. These vertices or action profiles bijectively correspond to subsets $A \in 2^N$ via the indexator function χ_A:$N \to \{0, 1\}$ defined by $\chi_A(i) = 1$ if $i \in A$ and 0 if $i \notin A$. Now complete the construction by defining utilities according to $u_i(\chi_A) = 1$ if $A = \{i\}$ and 0 otherwise, noting that this is indeed a full conflict game.

□

Clearly, in full conflict games all the players have the same power (for which a measure is provided below). In fact, with very slight modifications the above construction enables us to see that players may be very powerful independently of the number of actions available to them and that, conversely, their power depends on their possibilities for retaliation. In particular, for every $i \in N$ expand A_i to $A_i = \{a_i^0, a_i^1, a_i^2, ..., a_i^{k_i}\}$, so that each player has her own (finite) number $k_i \ge 2$ of available actions. Next define utilities according to:

$$\text{if } a_j = a_j^0 \text{ for all } j \in N\backslash i, \text{ then } u_i(a_1, ..., a_n) = h \text{ such that } a_i = a_i^h,$$

where $h \in \{0, 1, ..., k_i\}$. Otherwise, $u_i(a_1, ..., a_n) = 0$. Even though players can choose from many more actions than before, and even get much greater utilities, this is still a full conflict game.

A power index for strategic games is a mapping $\phi: \mathcal{G}_S^n \to \mathbb{R}_+^n$ such that $\varphi_i(\Gamma)$ measures the power of player $i \in N$ in the game Γ. In particular, for strategic TU games $\Gamma \in \mathcal{G}_S^n$ consider $\varphi(\Gamma) = (\varphi_1(\Gamma), ..., \varphi_n(\Gamma))$ defined for $1 \le i \le n$ by $\varphi_i(\Gamma) = \eta(\Gamma)\varphi_i^{Sh}(w_\Gamma)$, where $\varphi^{Sh}(w_\Gamma) = (\varphi_1^{Sh}(w_\Gamma), ..., \varphi_n^{Sh}(w_\Gamma))$ is the Shapley value of the coalitional game w_Γ. Hence, φ is fully characterized by its *efficiency*

$$\sum_{i \in N} \varphi_i(\Gamma) = \eta(\Gamma) \sum_{i \in N} \varphi_i^{Sh}(w_\Gamma) = \eta(\Gamma) w_\Gamma(N) = \eta(\Gamma),$$

and *balanced contributions* [19]

$$\phi_i(\Gamma)-\phi_i(\Gamma_{N\setminus j})=\phi_j(\Gamma)-\phi_j(\Gamma_{N\setminus i}) \text{ for all } 1\leq i\leq n,$$

where

$$\phi_i(\Gamma_{N\setminus j})=\eta(\Gamma)\sum_{A\in 2^N:i\in A}\frac{w_\Gamma(A\setminus j)-w_\Gamma(A\setminus\{i,j\})}{n\binom{n-1}{|A|-1}} \text{ for all } 1\leq i<j\leq n.$$

In fact, this is simply one of the existing characterizations of the Shapley value applied to the coalitional game which assigns to each coalition A a worth of $\eta(\Gamma)w_\Gamma(A)$. Also, note that

$$\phi_i(\Gamma)-\phi_i(\Gamma_{N\setminus j})=\phi_j(\Gamma)-\phi_j(\Gamma_{N\setminus j})=$$

$$\eta(\Gamma)\sum_{A\in 2^N:\{i,j\}\subseteq A}\frac{w_\Gamma(A)-w_\Gamma(A\setminus i)-w_\Gamma(A\setminus j)+w_\Gamma(A\setminus\{i,j\})}{n\binom{n-1}{|A|-1}}.$$

In cooperative game theory, the above balanced contribution condition reads as follows: the difference between i's share when j cooperates and i's share when j does not cooperate equals the difference between j's share when i cooperates and j's share when i does not cooperate. Here, in addition, if players do not cooperate within any coalition A, then they are actually committed to some (coordinated within A^c) best response-retaliation against A. Thus, how much i and j can retaliate against one another can be measured.

Finally, efficiency assures that $\varphi^{Sh}(w_\Gamma)$ constitutes a distributional norm. That is to say, if overall cooperation is attained and u_N^* is the total amount of TU produced (see above), then $\varphi_i^{Sh}(w_\Gamma)u_N^*$ may reasonably be i's share. Accordingly, whenever a player i has a lot of power (i.e. is capable of relevant retaliation against many coalitions $A\subseteq N\setminus i$), that player gets a large share, independently of her TU contribution, even when $u_i(a)<u_j(a)$ for all $a\in\mathbb{A}$ and $j\in N\setminus i$.

4. Preferences: outcomes without actions

Thus far, the word *outcome* has been used as a synonym of *n-tuple of utilities*. In fact, different action profiles $a, a'\in\mathbb{A}$ may well yield the same n-vector of utilities $(u_1(a), ..., u_n(a))=(u_1(a'), ..., u_n(a'))$. This leads us to introduce a set $\chi=\{x_1, ..., x_m\}$ of outcomes, which can be seen either as m distinct possible n-vectors of utilities resulting from interaction, or else from a more abstract viewpoint, allowing us to deviate from strategic games in two directions:

– firstly, ignoring actions and focusing exclusively on preferences over outcomes, one may address the issue of measuring conflict in a more direct manner;

– secondly, abandoning preferences and looking only at what outcomes coalitions can block, power (rather than conflict) becomes more clearly quantifiable.

As already outlined, players' preferences $\gtrsim_i$, $i \in N$ take the form of collections $\gtrsim_i \subseteq \chi \times \chi$ of ordered pairs of outcomes, where $x_h \gtrsim_i x_k$ (or, equivalently, $(x_h, x_k) \in \gtrsim_i$)) reads as follows: *player i weakly prefers x_h over x_k.*

Definition 9. *For any preference relation $\gtrsim_i$, the collection $\mathcal{S}_i$ of $\gtrsim_i$-consistent permutations consists of all $\pi \in \mathcal{S}(m)$ satisfying $\pi(h) < \pi(k)$ for all $x_h, x_k \in \chi$ such that $x_h \gtrsim_i x_k \not\gtrsim x_h$ (or, equivalently, such that $(x_h, x_k) \in \gtrsim_i \not\ni (x_k, x_h)$).*

Typically, preferences $\gtrsim_i$, $i \in N$ may be required to satisfy:

completeness: for all $x_h, x_k \in \chi$, either $x_h \gtrsim_i x_k$ or $x_k \gtrsim_i x_h$ or both;

transitivity: for all $x_h, x_k, x_l \in \chi$, if $x_h \gtrsim_i x_k$ and $x_k \gtrsim_i x_l$, then $x_h \gtrsim_i x_l$.

antisymmetry: for all $x_h, x_k \in \chi$, if $x_h \gtrsim_i x_k$ and $x_k \gtrsim_i x_h$, then $h = k$.

Complete and transitive preferences $\gtrsim_i$ bijectively correspond to *ordered partitions* $P^i = \{A_1^i, \ldots, A_{q_i}^i\} \subset 2^\chi$ of outcomes, where a partition of a set $\mathcal{S}$ is a collection of non-empty and pairwise disjoint subsets of $\mathcal{S}$ – the blocks of the partition – whose union yields $\mathcal{S}$. That is to say, $A_1^i \cap A_{l'}^i = \varnothing$ for all $1 \leq l < l' \leq q_i$, as well as $\bigcup_{1 \leq l \leq q_i} A_l^i = \chi$, with $A_1^i \neq \varnothing$, $1 \leq l \leq q_i$. In addition, the partition is ordered as for any $x_h \in A_{h'}^i$ and $x_k \in A_{k'}^i$, with $1 \leq h', k' \leq q_i$,

$$\text{if } h' < k' \text{ then } x_h \gtrsim_i x_k \not\gtrsim_i x_h \Rightarrow x_h >_i x_k \text{ or } \textit{strict preference},$$

$$\text{if } h' = k' \text{ then } x_h \gtrsim_i x_k \gtrsim_i x_h \Rightarrow x_h \sim_i x_k \text{ or } \textit{indifference},$$

In words, there is indifference between outcomes in the same block, and strict preference between outcomes in different blocks, with preferred outcomes appearing before worse ones in the blocks' natural ordering $1 \leq \ldots \leq q_i$. Hence, if $\gtrsim_i$ is a complete and transitive preference, then the family $\mathcal{S}_i \subseteq \mathcal{S}(m)$ of $\gtrsim_i$-admissible permutations contains all $\prod_{1 \leq k \leq q_i} |A_k^i|!$ permutations π such that

$$\text{for all } x_h, x_k \in \chi, \text{ if } x_h \in A_{h'}^i, x_k \in A_{k'}^i \text{ and } h' < k' \text{ then } \pi(h) < \pi(k).$$

In particular, if $\gtrsim_i$ is complete, transitive and antisymmetric, then the ordered partition $P^i = \{A_1^i, ..., A_{q_i}^i\}$ corresponding to it consists of $q_i = m$ blocks, and thus there is only one P^i-admissible permutation, i.e. $|\mathcal{S}_i| = 1$. In fact, in many *social choice mechanisms* players are required to submit precisely one permutation of outcomes (or alternatives) [9]. Accordingly, consider the generalization where players may submit *any* non-empty family $\mathcal{S}_i$ of permutations. In this way, traditional well-behaved (i.e. complete, transitive and possibly antisymmetric) preferences take the form of *permutation groups* [1], and at the same time generic preferences are also representable as (non-empty) families of permutations, which fail to be groups. In fact, the set $\mathcal{S}_i$ of $\gtrsim_i$-consistent permutations is well-defined and non-empty for any collection $\gtrsim_i \subseteq \chi \times \chi$ of ordered pairs of outcomes (in particular, $\gtrsim_i = \varnothing \Rightarrow \mathcal{S}_i = \mathcal{S}(m) \Leftarrow \gtrsim_i \chi \times \chi$).

Permutations $\pi \in \mathcal{S}(m)$ are integer-valued m-vectors whose entries $\pi(k)$ specify the position of outcome $x_k \in \chi$ in π for $1 \leq k \leq m$. Accordingly, let $d(\pi, \sigma)$ be a measure of the distance between any $\pi, \sigma \in \mathcal{S}(m)$. For example, in terms of the ℓ_2 norm, $d(\pi, \sigma) = \left(\sum_{1\leq k\leq m}(\pi(k)-\sigma(k))^2\right)^{1/2}$. Any chosen distance can be used to measure conflict within coalitions. For simplicity, first consider a pair $\{i, j\} \in 2^N$ where the members i, j have preferences represented by collections $\varnothing \neq \mathcal{S}_i, \mathcal{S}_j$ of permutations. A measure of conflict (of interest) between i and j is $\bigwedge_{(\pi,\sigma)\in\mathcal{S}_i\times\mathcal{S}_j} d(\pi, \sigma)$. In fact, if $\mathcal{S}_i \cap \mathcal{S}_j \neq \varnothing$, then there is no conflict between these two players. Extending this reasoning to arbitrary coalitions, for $\pi \in \mathcal{S}(m)$ and $\varnothing \neq \mathcal{S}_i \subseteq \mathcal{S}(m)$, let $d(\pi, \sigma) = \bigwedge_{\sigma\in\mathcal{S}}$, so that conflict within coalitions may be quantified by the coalitional game v, where

$$v(A) = \frac{1}{1 + \bigwedge_{\pi\in S(m)} \sum_{i\in A} \frac{d(\pi, S_i)}{|A|}} = \frac{|A|}{|A| + \bigwedge_{\pi\in S(m)} \sum_{i\in A} d(\pi, S_i)} (\varnothing \neq A \in 2^N).$$

To see why this game is useful, consider the case where preferences are such that $\varnothing \neq \bigcap_{i\in N} \mathcal{S}_i$. To put it differently, there is at least one permutation of outcomes which meets each player's preferences. Such a situation is sometimes said to display *pure common interest* [6]. Here, $v(A) = 1$ for all $\varnothing \neq A \in 2^N$ iff there is pure common interest. Otherwise, $v(A)$ takes (strictly) smaller values the more coalition members $i \in A$ have conflicting preferences over outcomes (and thus monotonicity clearly does not hold). In particular, $v(A) \in (0, 1]$ for all $\varnothing \neq A \in 2^N$. The *upper bound* on such a measure of conflict within any $A = \{i_1, ..., i_{|A|}\} \in 2^N$, i.e. $\bigvee_{\pi_{i_1},...,\pi_{i_{|A|}} \in \mathcal{S}(m)} \bigwedge_{\pi \in \mathcal{S}(m)} \sum_{1 \le j \le |A|} d(\pi, \mathcal{S}_{i_j})$, depends on m and $|A|$, as well as, of course, on the chosen distance or metric $d(\cdot,\cdot)$. Determining this bound for a given $d(\cdot,\cdot)$ is not a problem addressed here.

Following the same route as before for strategic games, the sought index φ: $\times_{i \in N} \mathcal{S}_i \to [0, 1)$, mapping n-profiles $\mathcal{S}_1, ..., \mathcal{S}_n \subseteq \mathcal{S}(m)$ of preferences over m outcomes into a measure of conflict, can thus take the following form: 1 minus the average of the $2^n - 1$ values taken by v for non-empty coalitions:

$$\varphi(\mathcal{S}_1, ..., \mathcal{S}_n) = 1 - \frac{\sum_{\emptyset \neq A \in 2^N} v(A)}{2^n - 1}.$$

5. Blockings: outcomes without preferences

In game forms one basically has all the ingredients of strategic games, apart from preferences. Hence, outcomes are induced by action profiles $a \in \mathbb{A}$ via a *mechanism* $\mathcal{M}$:$\mathbb{A} \to \chi$, which is here assumed to be subjective, i.e. $\cup_{a \in \mathbb{A}} \mathcal{M}(a) = \chi$, but not necessarily bijective. Hence, given N and χ, a mechanism or game form is defined by a pair $(\mathbb{A}, \mathcal{M})$. These situations are traditionally studied by means of *effectivity functions* $e : 2^N \to 2^{2^\chi}$ [17] or, equivalently, by means of *blockings* $b : 2^N \to 2^{2^\chi}$ [9]. The interpretation is the following: for all $Y \in 2^\chi$, $A \in 2^N$, if $Y \in e(A)$, then coalition A is *effective* on Y, that is to say A is capable of forcing the outcome to be in Y. Equivalently, if $\chi \backslash Y = Y^c \in b(A)$, then A *blocks* Y^c, that is to say A is capable of preventing the outcome from being in Y^c. Hence, $b(A) = \{Y \in 2^\chi: Y^c \in e(A)\}$ and $e(A) = \{Y \in 2^\chi: Y^c \in b(A)\}$ for all $A \in 2^N$. Given a mechanism $(\mathbb{A}, \mathcal{M})$, the associated blocking $b_{\mathcal{M}}$:$2^N \to 2^{2^\chi}$ is defined by: $Y \in b_{\mathcal{M}}(A)$ if there exists some (at least one) $a_A(Y^c) \in \mathbb{A}_A$ such that $\mathcal{M}(a_A(Y^c), a_{A^c}) \in Y^c$ for all $a_{A^c} \in \mathbb{A}_{A^c}$ (this is sometimes called an *alpha-blocking*, while a *beta-blocking* $\tilde{b}_{\mathcal{M}}$ is defined by: $Y \in \tilde{b}_{\mathcal{M}}(A)$ if for every $a_{A^c} \in \mathbb{A}_{A^c}$ there exists $a_A(a_{A^c}) \in \mathbb{A}_A$ such that $\mathcal{M}(a_A(a_{A^c}), a_{A^c}) \notin Y$; in general, $\tilde{b}_{\mathcal{M}}$ fails to satisfy

C2 below). Any blocking $b_{\mathcal{M}}$ generated in this way (with $\mathcal{M}$ being surjective) satisfies the following conditions:

C.1: $Y \in b(A), B \supseteq A, Z \subseteq Y \Rightarrow Z \in b(B)$,

C.2: $Y \in b(A), Z \in b(B), A \cap B = \varnothing \Rightarrow (Y \cup Z) \in b(A \cup B)$,

C.3: $\{\chi\} \notin b(A) \ni \varnothing$ for all $A \in 2^N$, $\varnothing = b(\varnothing)$, $2^\chi \setminus \{\chi\} = b(N)$

Given N and χ, define any $b{:}2^N \to 2^{2^\chi}$ satisfying C.1-C.3 to be a blocking.

Remark 10. *If b is a blocking, then $b(A)$ is an ideal, in poset $(2^\chi, \supseteq)$, for all $A \in 2^N$. That is to say there is a family $\mathcal{AK}_A = \{Y_1, ..., Y_k\} \subset 2^\chi$ such that (i) if $Z \subseteq Y_j \in \mathcal{AK}_A$, then $Z \in b(A)$, and (ii) $Y_i \not\subseteq Y_j \not\subseteq Y_i$, $1 \le i < j \le k$ (i.e. $\mathcal{AK}_A$ is an antichain). To see this, simply let $B = A$ in C.1.*

Given N and χ, let $\mathcal{B}^n_m$ denote the family of all blockings $b{:}2^N \to 2^{2^\chi}$ with $n \ge 2$ players and $m \ge 2$ outcomes. In fact, $(\mathcal{B}^n_m, \supseteq)$ is a poset, where for any $b, b' \in \mathcal{B}^n_m$ the partial order is: $b \supseteq b' \Leftrightarrow b(A) \supseteq b'(A)$ for all $A \in 2^N$. There exists a unique minimal element, the bottom one $b_\perp \in \mathcal{B}^n_m$, defined by $b_\perp(\mathrm{A}) = \varnothing$ for all $A \in 2^N$, $A \ne \mathrm{N}$. Conversely, the collection of maximal elements contains all those $b \in \mathcal{B}^n_m$ satisfying $Y \notin b(A) \Rightarrow Y^c \in b(A^c)$ for all $A \in 2^N$, $Y \in 2^\chi$ [9, proposition 1.5.13, p. 35].

Without players' preferences we cannot define any measure of conflict, best responses or retaliation. Still, if $b(A) \subseteq b(B)$, then quite safely[3] one can say that coalition B has at least the same power as coalition A in blocking b. On this ground, any coalition $\varnothing \subset A \subset N$ has maximum power if $b(A) = 2^\chi \setminus \{\chi\} = b(N)$ and minimum power if $b(A) = \varnothing = b(\varnothing)$. For every $b \in \mathcal{B}^n_m$, define the coalitional game $v_b \in \mathcal{G}^n_S$ by $v_b(A) = \dfrac{|b(A)|}{|b(N)|}$ for all $A \in 2^N$, where $|b(N)| = 2^m - 1$ (and $v_b(\varnothing) = 0$). As before, $v_b(A) \in [0, 1]$ for every $A \in 2^N$ and $v_b(N) = 1$. In fact, developing the above argument, v_b quantifies the power of coalitions: if $v_b(A) \le v_b(B)$, then either $b(A) \subseteq b(B)$, or else $b(A) \not\subseteq b(B) \not\subseteq b(A)$ but $|\mathcal{AK}_A| \le |\mathcal{AK}_B|$ and/or $\mathcal{AK}_B$ contains larger subsets $Y_1, ..., Y_k$ than $\mathcal{AK}_A$ (see above). Accordingly, the index $\xi : \mathcal{B}^n_m \to \left[\dfrac{1}{2^n - 1}, \xi^*(n, m)\right]$ defined by

$$\xi(b) = \frac{\sum_{\varnothing \ne A \in 2^N} v_b(A)}{2^n - 1}$$

[3] For any blocking b there exists an associated *canonical mechanism* $(\mathbb{A}, \mathcal{M}_b)$ such that $b = b_{\mathcal{M}_b}$ and, most importantly, $\mathbb{A}_i = 2^\chi$ for all $i \in N$ (see [9, p. 34]).

is the average over all coalitions of their power. In fact, $\xi(b) = \frac{1}{2^n - 1}$ iff $v_b(A) = 0$ for all $A \neq N$, i.e. if no $A \neq N$ can force the outcome to belong to any proper subset of the whole outcome set χ. In terms of mechanisms $\mathcal{M}$, for every $|A|$-tuple $\sigma_A \in \mathbb{A}_A$ of actions taken by members $i \in A$ and for every outcome $x \in \chi$, there is a $|A^c|$-tuple $a_{A^c} \in \mathbb{A}_A$ of actions that non-members $j \in A^c$ may take such that $\mathcal{M}(a_A, a_{A^c}) = x$. Hence, the lower bound $\frac{1}{2^n - 1} = \xi(b_\perp)$ is attained for the bottom element $b_\perp$ of $\mathcal{B}^n_m$.

Claim 11. *For any n, $m \geq 2$, the upper bound $\xi^*(n, m) = \bigvee_{b \in \mathcal{B}^n_m} \sum_{\emptyset \neq A \in 2^N} \frac{v_b(A)}{2^n - 1}$ is given by*

$$\xi^*(n,m) = \frac{1}{2^n - 1}\left[1 + (2^n - 2)\frac{2^{m-1}}{2^m - 1}\right] = \frac{2^{n+m-1} - 1}{(2^n - 1)(2^m - 1)}.$$

Proof: By definition, if $\xi(\hat{b}) = \bigvee_{b \in \mathcal{B}^n_m} \sum_{\emptyset \neq A \in 2^N} v_b(A)$, then $\hat{b}$ must be some (but not any) maximal element of the poset ($\mathcal{B}^n_m$, $\supseteq$). To check the features of $\hat{b}$, in view of C.3 above, focus has to be placed on all $2^{n-1} - 1$ sums of the form $v_b(A) = v_b(A^c)$ with $\emptyset \subset A \subset N$. If b is a (i.e. any) maximal element, then for all $A \in 2^N$, $A \neq N$ and all $\chi \neq Y \in 2^\chi$ (at least) one of the following holds: (i) $Y \in b(A)$ and $Y^c \notin b(A^c)$, (ii) $Y^c \in b(A)$ and $Y \notin b(A^c)$, , (iii) $Y \notin b(A)$, and $Y^c \in b(A^c)$, or (iv) $Y^c \notin b(A)$ and $Y \in b(A^c)$ (see above). In fact, these cases are not mutually exclusive. In particular, both (i) and (iv) or, alternatively, both (ii) and (iii) may occur simultaneously. Now let $B = A^c$ in C.2 above, and note that C.3 requires $\{\chi\} \notin b(A)$, $A \in 2^N$. Hence, if $\emptyset \subset A \subset N$, then $v_b(A) + v_b(A^c)$ is maximized when $b(A) = b(A^c) = 2^{\chi\backslash x}$ for some single outcome $x \in \chi$, in which case $v_b(A) = v_b(A^c) = 2\frac{2^{m-1}}{2^m - 1} = \frac{2^m}{2^m - 1}$. To see whether this sum may attain its maximum on *all* pairs $\{A, A^c\}$, $\emptyset \subset A \subset N$, simply set $\hat{b}(A) = \hat{b}(A^c) = 2^{\chi \backslash x}$ with fixed $x \in \chi$ for *all* $\emptyset \subset A \subset N$. It may be verified that $\hat{b}$ does satisfy C.1–C.3 and thus constitutes a blocking. Thus, $\xi(v_{\hat{b}}) = \xi^*(n,m)$.

A blocking b such that there is some outcome $x \in \chi$ for which $b(A) = 2^{\chi\backslash x}$ for every $\emptyset \subset A \subset N$ is a *Maskin blocking* [9, pp. 115, 167]. That is to say, every non-empty coalition can block any subset $Y \subset \chi$ of outcomes such that $x \notin Y$. Thus, x may be interpreted as the *status quo*, such that if the (grand) consensus is not achieved, then x is triggered by some constant mechanism. Clearly, if b is a Maskin blocking, then

$\xi(b) = \xi^*(n,m)$. To see that the converse is also true (i.e. $\xi(b) = \xi^*(n,m)$ only if b is a Maskin blocking), note that $\xi(b) = \xi^*(n,m)$ requires that for every $\emptyset \subset A \subset N$ there is some $x \in \chi$ such that $b(A) = b(A^c) = 2^{\chi \backslash x}$. Now let $\emptyset \subset A$, $B \subset N$, with $A \neq B$, so that exactly one of the following holds: $A \cap B = \emptyset$ or $A^c \cap B = \emptyset$ or $A^c \cap B^c = \emptyset$ or $A \cap B^c = \emptyset$. Without loss of generality, suppose $A \cap B = \emptyset$. Assume $b(A) = b(A^c) = 2^{\chi \backslash x}$, as well as $b(B) = b(B^c) = 2^{\chi \backslash x}$, with $y \neq x$. By C.2, $((\chi \backslash x) \cup (\chi \backslash y))\ ]\{\chi\} \in b(A \cup B)$, contradicting C3.

□

Example 12. Let i and j denote players, while 0 and 1 denote actions. The outcome set is $\chi = \{x_1, x_2\}$. Mechanisms $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{M}''$: $\{0,1\}^2 \rightarrow \chi$ are defined by means of the following three matrices. Looking at the top-left entry, with mechanism $\mathcal{M}$, when both players choose action $a_i = a_j = 0$ the resulting outcome is x_1. The remaining entries are defined analogously.

Table 2

Mechanisms $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{M}''$

$\mathcal{M}$	$a_j = 0$	$a_j = 1$
$a_i = 0$	$\mathcal{M}(0, 0) = x_1$	$\mathcal{M}(0, 1) = x_1$
$a_i = 1$	$\mathcal{M}(1, 0) = x_1$	$\mathcal{M}(1, 1) = x_2$
$\mathcal{M}'$	$a_j = 0$	$a_j = 1$
$a_i = 0$	$\mathcal{M}'(0, 0) = x_1$	$\mathcal{M}'(0, 1) = x_2$
$a_i = 1$	$\mathcal{M}'(1, 0) = x_2$	$\mathcal{M}'(1, 1) = x_1$
$\mathcal{M}''$	$a_j = 0$	$a_j = 1$
$a_i = 0$	$\mathcal{M}''(0, 0) = x_1$	$\mathcal{M}''(0, 1) = x_2$
$a_i = 1$	$\mathcal{M}''(1, 0) = x_1$	$\mathcal{M}''(1, 1) = x_1$

- *Under the first mechanism $\mathcal{M}$, by choosing a suitable response both players are able to force outcome x_1 or, equivalently, to block outcome x_2, i.e. $b_{\mathcal{M}}(\{i\}) = b_{\mathcal{M}}(\{j\}) = \{\emptyset, \{x_2\}\}$.*
- *Under the second mechanism $\mathcal{M}'$, no player can block (nor force, of course) any outcome, i.e. $b_{\mathcal{M}'}(\{i\}) = b_{\mathcal{M}'}(\{j\}) = \emptyset$.*
- *Under the third mechanism $\mathcal{M}''$, while player i again cannot block any outcome, player j can block both outcomes x_1 and x_2, i.e. $b_{\mathcal{M}''}(\{i\}) = \emptyset$ while $b_{\mathcal{M}''}(\{j\}) = \{\emptyset, \{x_1\}, \{x_2\}\} = 2^{\chi} \backslash \{\chi\}$.*

Accordingly,

$$\xi(b_{\mathcal{M}}) = \frac{v_{b_{\mathcal{M}}}(N) + v_{b_{\mathcal{M}}}(\{1\}) + v_{b_{\mathcal{M}}}(\{2\})}{3} = \frac{1}{3}\left(1 + \frac{2}{3} + \frac{2}{3}\right) = \frac{7}{9} = \xi^*(2,2)\,,$$

$$\xi(b_{\mathcal{M}'}) = \frac{v_{b_{\mathcal{M}'}}(N) + v_{b_{\mathcal{M}'}}(\{1\}) + v_{b_{\mathcal{M}'}}(\{2\})}{3} = \frac{1}{3}(1+0+0) = \frac{1}{3} = \frac{1}{2^2-1}\,,$$

$$\xi(b_{\mathcal{M}''}) = \frac{v_{b_{\mathcal{M}''}}(N) + v_{b_{\mathcal{M}''}}(\{1\}) + v_{b_{\mathcal{M}''}}(\{2\})}{3} = \frac{1}{3}(1+0+1) = \frac{2}{3}\,.$$

As for strategic games, turning blockings b (or, equivalently, game forms) into coalitional games v_b enables us to measure players' power via some solution of v_b. In particular, the Shapley value may be used again. In cooperative game theory situations such as those formalized by mechanisms $\mathcal{M}'$ and $\mathcal{M}''$ are *simple games* (i.e. a peculiar family of coalitional games), in which power is also traditionally measured using the Banzhaf index [4].

5.1. Blockings and simple games

A simple game is a $\{0, 1\}$-ranged, monotone coalitional game $v \in \mathcal{G}_C^n$ such that $v(N) = 1$. These games are often associated with voting situations: those coalitions $A \in 2^N$ such that $v(A) = 1$ are said to be *winning*, while if $v(A) = 0$, then A is said to be *losing*. In particular, in $\mathcal{M}'$ above the only winning coalition is N, while in $\mathcal{M}''$ coalitions N and $\{2\}$ are winning while $\{1\}$ is losing. In fact, blockings generalize simple games, as these latter (injectively) correspond to blockings. Formally, a simple game may be regarded as a family $W \subset 2^N$ of winning coalitions, with $N \in W$, satisfying

(I) $A \subseteq B, A \in W \Rightarrow B \in W$ for all $A, B \in 2^N$,

(II) $A \in W \Rightarrow A^c \notin W$.

In fact, although simple games need not, in general, satisfy (II), as soon as they are intended to model voting situations such a condition seems rather appropriate. Accordingly, define a blocking b_W by $b_W(A) = 2^\chi \backslash \{\chi\}$ if $A \in W$ and $b_W(A) = \varnothing$ if $A \notin W$ [9, example 1.5.7, p. 32]. Note that the coalitional game v_{b_W} defined above, i.e. $v_{b_W}(A) = \frac{|b_W(A)|}{2^m - 1}$ for all $A \in 2^N$, is $\{0, 1\}$-ranged, monotone and satisfies $v_{b_W}(N) = 1$. In other words, it is simple. In this respect, blockings may be regarded as a generalization of simple games: for any blocking b, the coalitional game v_b defined above is $[0, 1]$-ranged and monotone, with $v_b(N) = 1$. It becomes $\{0, 1\}$-ranged as soon as

$b = b_W$ for some family W of winning coalitions satisfying (I) and (II) above. Also, the maximum of

$$\xi(b_W) = \frac{\sum_{\varnothing \neq A \in 2^N} v_{b_W}(A)}{2^n - 1}$$

over all such conceivable families $W \subset 2^N$ of winning coalitions is attained when $A \notin W \Rightarrow A^c \in W$ for all $A \in 2^N$, in which case $\xi(b_W) = \dfrac{2^{n-1}}{2^n - 1}$. Furthermore,

$$\frac{2^{n-1}}{2^n - 1} < \frac{2^{n+m-1}}{(2^n - 1)(2^m - 1)} = \xi^*(n,m) \text{ for all } n,m \geq 2 .$$

In simple games v the power of players is often measured using the *Banzhaf index* $\beta(v) \in \mathbb{R}^n_+$ (see Section 2). Accordingly, as a measure of the overall power characterizing game v one may focus on the ℓ_2 norm of $\beta(v)$ or, equivalently, on $\| \beta(v) \|_2^2 = \sum_{1 \leq i \leq n} (\beta_i(v))^2$. In fact, $\| \beta(v) \|_2^2 \leq 1$. Also, if ω_v denotes the number of winning coalitions in v, then $\| \beta(v) \|_2^2 \leq 2 \min \left\{ \dfrac{\omega_v}{2^n}, \dfrac{2^n - \omega_v}{2^n} \right\}$. In particular, $\| \beta(v) \|_2^2 = 1$ iff v is dictatorial, i.e. iff there is some $i \in N$ such that every $A \in 2^N$ is winning if $i \in A$ and losing if $i \notin A$. On the other hand, $\| \beta(v) \|_2^2 = 2 \min \left\{ \dfrac{\omega_v}{2^n}, \dfrac{2^n - \omega_v}{2^n} \right\}$ iff v is either dictatorial or else a *unanimity game* u_A such that $|A| = 2$, where $u_A(B) = 1$ if $B \supseteq A$ and 0 otherwise [14].

Alternatively, the ℓ_1 norm $\| \beta(v) \|_2^2 = \sum_{1 \leq i \leq n} \beta_i(v)^2$ attains its maximum, *over simple games*, when v is the majority game v_M, defined by $v_M(A) = 1$ if $|A| \geq \left\lfloor \dfrac{n}{2} \right\rfloor + 1$ and 0 otherwise, in which case $\| \beta(v_M) \|_1 = \dfrac{n}{2^{n-1}} \begin{pmatrix} n-1 \\ \left\lfloor \frac{n}{2} \right\rfloor \end{pmatrix}$, and thus $\|\beta(v_M)\| \approx \sqrt{2n/\pi}$, with $\pi = 3.14$.... This latter approximation is obtained by applying *Stirling's formula* [13, p. 112] $n! = \sqrt{2\pi n} \left(\dfrac{n}{e} \right)^n \left(1 + O\left(\dfrac{1}{n} \right) \right)$ to [10, Theorem 2, p. 106]. Now, given that any blocking b may be turned into a coalitional game v_b whose range is a finite set of rational numbers $\dfrac{q}{2^m - 1}, 0 \leq q \leq 2^m - 1$, where q is an integer, it seems interesting to check whether the upper bound on $\|\beta(v)\|_1$ obtained in [10] for simple n-player games v also applies to $\|\beta(v_b)\|_1$ for $b \in \mathcal{B}_m^n$, $m \geq 2$. This is addressed in the appendix.

6. Aggregation: remarks and developments

Approaching the measurement of conflict and power in strategic games as an issue of aggregation leads us to emphasize, once again, that utilities (whether transferable or not) are normalized. This means that for any $\Gamma \in \mathcal{G}_S^n$ the amount of conflict is the same for all games $t\Gamma$ obtained by multiplying each of the players' utilities by some $t > 0$. Technically, this defines the *cone* spanned by Γ, and the proposed indexes κ, η are constant on such a cone for all $\Gamma \in \mathcal{G}_S^n$. This is an important and desirable feature. Consider a two-player game for simplicity: conflict essentially depends on the difference, considered for each of the players in turn, between a player's maximum utility over all pairs of actions and her maximum utility over all pairs of actions furnishing the other player with her maximum utility. For any two games in which these two differences are the same, the measure of conflict must also be the same. This is precisely what the chosen normalization yields.

It is also worth noticing that the three coalitional games v_Γ, w_Γ and v_b defined above for NTU or TU games Γ and for blockings b are monotone. This is obvious for v_b, as any coalition A can block no less than any sub-coalition $B \subseteq A$ (in view of C.1). Concerning v_Γ and w_Γ, monotonicity results from the fact that players' utilities take only positive values. This, in turn, relies upon the idea that there are no true gains or losses, but only different (positive) utility levels. Conceptually, whether gains and losses exist in NTU strategic games seems debatable. In fact, the use of bipolar scales, i.e. with positive and negative payoffs, leads us to conceive of zero as denoting *neutral* satisfaction. Now, if a player has an available action which guarantees a certain NTU independently of the actions taken by the others, then this could be the sought zero utility level or neutral satisfaction *for this player*. Otherwise, where to place the zero would be unclear. Anyway, apart from this, from a technical perspective the case where utilities take negative values may be handled via very minor adjustments: very simply, utilities have to be initially re-scaled so that the minimum of a player's utility over all action profiles corresponds to zero utility for her. Of course, these re-scaled utilities are positive-valued, and therefore the whole approach proposed here applies.

Another key fact in terms of aggregation is that, by construction, only $2^n - 1$ (at most) action profiles (i.e one for each non-empty coalition) are used to define the mapping $\mathcal{G}_S^n \to \mathcal{G}_C^n$. Indeed, the coalitional games v_Γ and w_Γ quantify the maximum (normalized) coalitional utility that coalitions attain when their complement plays best response-retaliation. Accordingly, a coalition A may have a high worth simply because the actions and preferences happen to be such that its complement A^c, by choosing best response-retaliation, allows A to get a good outcome. To put it differently, the possibility that A^c deviates from best responses so as to inflict stronger retaliation upon A is disregarded. Nevertheless, A^c could deviate from best responses without doing it on purpose. More precisely, if $|A^c| >> |A|$, then deviation could simply be due to lack of coordination. This leads us to conceive of a more sophisticated model, briefly summarized hereafter.

If the worth of each coalition was to be determined by considering more than just one action profile, then such a worth would have to be placed between the maximum and the minimum associated coalitional utility over all action profiles. For the sake of concreteness, focusing on v_Γ (the same applies, *mutatis mutandis*, to w_Γ), consider the following variations of definition 1 above:
for $\varnothing \subset A \subset N$, if $v_\Gamma(N) \neq 1$, if, then

$$\overline{v}_\Gamma(A) = \frac{\bigvee_{a \in \mathbb{A}} \psi_A(a)}{|A|} \quad \text{as well as} \quad \underline{v}_\Gamma(A) = \frac{\bigvee_{a_A \in \mathbb{A}_A} \bigvee_{a_{A^c} \in \mathbb{A}_{A^c}} \psi_A(a_A, a_{A^c})}{|A|}.$$

In words, $\overline{v}_\Gamma(A)$ is the maximum normalized coalitional utility that A can attain over all action profiles. It equals 1 when there is no internal conflict within A, i.e. when there is some n-tuple of actions at which all A's members attain their maximum utility. Behaviorally, this worth is obtained under the assumption that the complement A^c is fully conciliating, although this is likely to occur only in common interest games, where $v_\Gamma(A) = 1 = v_\Gamma(A^c)$. Conversely, $\underline{v}_\Gamma(A)$ quantifies the worth of A under the assumption that its complement always chooses full retaliation (thereby deviating, in general, from best responses). Now, obtaining a unique real number (for each coalition A) for all the values $\frac{\psi_A(a)}{|A|}$ associated with action profiles $a \in \mathbb{A}$ such that $\frac{\psi_A(a)}{|A|} \in [\underline{v}_\Gamma(A), \overline{v}_\Gamma(A)]$ is an subtle aggregation issue which can only be dealt with by making precise strategic (i.e. behavioral) assumptions.

From this perspective, Definition 1 takes a short-cut by selecting only one such action profile, precisely one (i.e. any) where the complement plays a best response, but still retaliates as much as possible. In other words, v_Γ is determined by considering a unique value between $\underline{v}_\Gamma(A)$ and $\overline{v}_\Gamma(A)$. Whether this is reasonable or not depends on what one has in mind to model (using the game Γ) and, given this, on n itself, because achieving coordination is harder the greater the number of players. In any case, in order to aggregate in a more comprehensive manner, for any coalition one may formalize *beliefs* over all the possible behaviors of the complement and next derive the associated worth as an *expectation* w.r.t. such beliefs. For example, A may believe that a *partition* of A^c does form in response to A's coordinated group action. In order to handle this, let $\mathcal{P}^{A^c}$ denote the set (i.e. lattice [1]) of partitions of A^c for all $\varnothing \subset A \subset N$. The resulting setting is one where for each coalition A there is a distinct strategic game for each partition $\mathcal{P}^{A^c} \in \mathcal{P}^{A^c}$, with $|P^{Ac}| + 1$ players (i.e. A and each of P^{Ac}'s blocks). Also,the behavior or choice of a coordinated group action (such as best response-retaliation above) for every block $B \in P^{Ac}$ must be specified (with subsets $B \subseteq A^c$ possibly displaying different behavior when considered as blocks of different partitions of A^c). Hence, for every A there is a unique worth of A under each partition of Ac. In cooperative game theory this is called a *game in partition function form* [18]. For each A, all the values that the worth of A may take should finally be aggregated into a unique, comprehensive (expected) worth of A. This means computing the expectation of a random variable taking its values over partitions of a finite set [24]. Once the idea of aggregating over ordered structures has been conceived, it can also be observed that coalitional games are themselves functions taking values on a *distributive atomic lattice* $(2^N, \cap, \cup)$. Apart from the averaging adopted here, there exists a variety of further techniques for aggregating such functions [23].

7. Conclusions

Conflict is easily observed to affect a wide range of human relations, at very different levels: between single individuals, as well as between entire (groups of) countries. Several studies from psychology, sociology, anthropology and politics [5], [7], [8], [11], [12], [15], [28], [29], [31] deal with conflict resulting in wars, as well as conflict due to ethnic or other socio-economic causes. It is commonly noted how game theory provides very useful analytical tools. In fact, strategic games constitute precisely defined abstract interactive situations where conflict appears in a most explicit manner. Yet, most attention is devoted to conflict resolution and/or evolution, rather than to measuring the level of conflict in itself. This paper provides a new, strictly game-theoretical and quantitative approach to this latter issue. The general idea is to address the issue in different settings by firstly turning a strategic game into a coalitional game, on the basis of which the sought index measuring conflict is defined and, possibly, players' power shares derived according to the Shapley and Banzhaf solutions. Hence, the whole procedure is treated mainly in terms of a two-step aggregation. In particular, the first step, i.e. turning strategic settings into coalitional games, seems novel and thereby deserves investigation in terms of an axiomatic characterization. Clearly, this is not addressed here: apart from the fact that the proposed mappings are invariant w.r.t. linear transformations of the given strategic games, no further information in the form of axioms is provided. Also, when studying other more abstract settings, with preferences and without actions or with actions and without preferences, providing a comprehensive axiomatic treatment would surely be hard. Anyway, when seeking a characterization for mappings $\mathcal{G}_S^n \xrightarrow{v_\Gamma} \mathcal{G}_C^n$ that turn strategic games Γ into coalitional ones v_Γ, the following two questions seem significant, and will be considered in future work: (1) which actions (such as *dominated* ones [16]) may be deleted from Γ so as to have a *reduced* game Γ′ such that $v_\Gamma = v_{\Gamma'}$? (2) which strategic games Γ are mapped into *additive* coalitional ones v_Γ, i.e. satisfying $v_\Gamma(A) = \sum_{i \in A} v_\Gamma(\{i\})$ for all $A \in 2^N$?

Appendix

This final section is devoted to showing that the upper bound on the ℓ_1-norm $\|\beta(v)\|_1$ for the Banzhaf index for simple games v provided in [10] also applies to games induced by blockings v_b with $b \in \mathcal{B}_m^n$. Formally,

$$\bigvee_{b \in B_m^n} \|\beta(v_b)\|_1 = \bigvee_{b \in B_m^n} \sum_{1 \le 1 i \le n} \beta_1(v_b) = \|\beta(v_M)\|_1 = \frac{n}{2^{n-1}} \binom{n-1}{\left\lfloor \frac{n}{2} \right\rfloor},$$

where the majority voting game v_M is defined by $v_M(A) = 1$ if $|A| \ge \left\lfloor \frac{n}{2} \right\rfloor + 1$ and 0 otherwise for each $A \in 2^N$ (see above). Consider the class $\mathcal{V}$ of *monotone* games v:$2^N \to [0,1]$ such that $v(\varnothing) = 0$, $v(N) = 1$, noting that $v_b \in \mathcal{V}$ for all b $\in \mathcal{B}_m^n$. Any $v \in \mathcal{V}$ is said to be *symmetric* if there is some γ: $\{0, 1, ..., n\} \to [0, 1]$ satisfying $v(A) = \Gamma(|A|)$ for all $A \in 2^N$. The majority voting game is both simple and symmetric.

Claim 13. *For every game* $v \in \mathcal{V}$*, the symmetric game* $\gamma_v \in \mathcal{V}$ *defined by*

$$\gamma_v(k) = \sum_{A \in 2^B : |A| = k} \frac{v(A)}{\binom{n}{k}}, 0 \le k \le n$$

satisfies $\|\beta(v)\|_1 = \|\beta(\gamma_v)\|_1$.

Proof: The number of pairs $\{A, B\} \in 2^N \times 2^N$ such that $|A| = k$ and $A \supset B$, $|A| = |B| + 1$ is $\binom{n}{k}k$, with $0 \le k \le n$. Accordingly,

$$2^{n-1}\,\|\beta(\gamma)\|_1 = \sum_{1\le k\le n}[\gamma(k)-\gamma(k-1)]\binom{n}{k}k$$

for any symmetric γ. Now let $\Delta v(k)$, $1 \le k \le n$ denote the sum of all the differences $v(A) - v(B)$ such that $|A| = k$ and $A \supset B$, $|A| = |B| + 1$, i.e.

$$\Delta v(k) = k\sum_{A\in 2^N:|A|=k} v(A) - (n-k+1)]\sum_{B\in 2^N:|B|=k-1} v(B)\,.$$

Lastly, define γ_v recursively by $\gamma_v(k) = \dfrac{\Delta v(k)}{\binom{n}{k}k} + \gamma_v(k-1)$. By construction,

$$\Delta v(k) = \Delta\gamma_v(k) = [\gamma_v(k)-\gamma_v(k-1)]\binom{n}{k}k \quad \text{for all} \quad 1 \le k \le n,$$

and therefore

$$\|\beta(\gamma)\|_1 = \sum_{1\le k\le n}\frac{\Delta v(k)}{2^{n-1}} = \sum_{1\le k\le n}\frac{\Delta\gamma_v(k)}{2^{n-1}} = \|\beta(\gamma_v)\|_1\,.$$

To see that $\gamma_v(k) = \sum_{A\in 2^N:|A|=k}\dfrac{v(A)}{\binom{n}{k}}$ for $k > 1$ (when $k = 1$ this is easily checked), use induction:

$$\gamma_v(k) = \sum_{A\in 2^N:|A|=k}\frac{v(A)}{\binom{n}{k}} - \sum_{B\in 2^N:|B|=k-1}\frac{v(B)(n-k+1)}{\binom{n}{k}k} + \frac{\Delta v(k-1)}{\binom{n}{k}(k-1)} + \gamma_v(k-2)$$

$$= \sum_{A\in 2^N:|A|=k}\frac{v(A)}{\binom{n}{k}} + \sum_{B\in 2^N:|B|=k-1} v(B)\left[\frac{1}{\binom{n}{k-1}} - \frac{n-k+1}{\binom{n}{k}k}\right] + \sum_{D\in 2^N:|D|=k-2} v(D)\left[\frac{1}{\binom{n}{k-2}} - \frac{n-k+2}{\binom{n}{k}(k-1)}\right],$$

where the second (i.e. last) equality results precisely from the induction assumption. Both terms within square parentheses equal 0, as $\dfrac{n-k+1}{k} = \dfrac{\binom{n}{k}}{\binom{n}{k-1}}$ for all $1 \le k \le n$.

□

This result implies that searching for the upper bound over the set of all games is equivalent to searching over the set of all symmetric games. Hence, finding $\bigvee_{b\in\mathcal{B}^n_m} \|\beta(v_b)\|_1$ is equivalent to solving the following problem:

$$\bigvee_{\Delta^\gamma(1),\ldots,\Delta^\gamma(n)\geq 0}\left[n\sum_{1\leq k\leq n}\frac{k\binom{n}{k}}{n2^{n-1}}\Delta^\gamma(k)\right] \quad \text{subject to} \quad \sum_{1\leq k\leq n}\Delta^\gamma(k)=1,$$

where

$$\Delta^\gamma(k)=\gamma_v(k)-\gamma(k-1)=\frac{\Delta\gamma(k)}{\binom{n}{k}k},$$

i.e. maximize $n <a, \Delta^\gamma>$ subject to the constraint $\sum_{1\leq k\leq n}\Delta^\gamma(k)=1$, where $a_k = \frac{k\binom{n}{k}}{n2^{n-1}}$, $1 \leq k \leq n$ and $<\cdot\,, \cdot>$ denotes the scalar product (between n-vectors), that is to say $<a, \Delta^\gamma> = \sum_{1\leq k\leq n}a_k\Delta^\gamma(k)$. Note that

$$\sum_{1\leq k\leq n}a_k=\sum_{1\leq k\leq n}\frac{k\binom{n}{k}}{n2^{n-1}}=\sum_{1\leq k\leq n}\frac{n\binom{n-1}{k-1}}{n2^{n-1}}=\sum_{0\leq k\leq n-1}\frac{\binom{n-1}{k}}{2^{n-1}}=1.$$

Therefore, $a \in \Delta_n$, where $\Delta_n = \{p\Delta_n = \{p \in \mathbb{R}^n_+ : \sum_{1\leq k\leq n} p_k = 1$ is the $n - 1$-dimensional simplex and thus the problem reduces to maximizing the scalar product $<a, p>$, where $p \in \Delta_n$ and $a \in \Delta_n$ is given. Define $K^* \subseteq \{1, ..., n\}$ by: $k \in K^*$ iff $a_k \geq a_h$ for all $1 \leq h \leq n$. In words, K^* is the set of indexes of a's maximal coordinates. Inspection reveals that the set of all $p \in \Delta_n$ maximizing $<a, p>$ is the set of all *convex combinations* of the $|K^*|$ *extreme points* $p^k \in \Delta_n$ each defined by $p^k_h = 1$ if $h = k \in K^*$ and 0 otherwise, for $1 \leq h \leq n$. In particular, it follows from $k\binom{n}{k}=k\binom{n-1}{k-1}$ that a's maximal coordinates are those whose indexes $k \in K^*$ satisfy $k - 1 = \left\lfloor\frac{n-1}{2}\right\rfloor$ or $k - 1 = \left\lceil\frac{n}{2}\right\rceil$ (or both), i.e. $K^* = \left\{\left\lfloor\frac{n-1}{2}\right\rfloor+1, \left\lceil\frac{n-1}{2}\right\rceil+1\right\}$. Hence, if n is odd, then of course

$\left\lfloor \frac{n-1}{2} \right\rfloor = \left\lceil \frac{n-1}{2} \right\rceil = \frac{n-1}{2}$ and therefore $K^* = \left\{\frac{n+1}{2}\right\}$, while if n is even, then $K^* = \left\{\frac{n}{2}, \frac{n}{2}+1\right\}$. This proves that $\bigvee_{b \in \mathcal{B}_m^n} \|\beta(v_b)\|_1 = \|\beta(v_M)\|_1$. Also, if n is even, then the upper bound $\frac{n}{2^{n-1}}\binom{n-1}{\left\lfloor \frac{n}{2} \right\rfloor}$ is attained not only when $v_b = v_M$ (i.e. when $|b(A)| = 2^m - 1$ if $|A| \geq \frac{n}{2}+1$ and 0 otherwise), but also when

$$|b(A)| = 0 \text{ if } |A| < \frac{n}{2},$$

$$|b(A)| = q \text{ if } |A| = \frac{n}{2},$$

and

$$|b(A)| = 2^m - 1 \text{ if } |A| > \frac{n}{2}$$

for any integer q such that $1 \leq q \leq 2^m - 1$.

References

[1] AIGNER M., *Combinatorial Theory*, Springer-Verlag, 1979.

[2] ANDERLINI L., *Communication, computability and common interest games*, Games and Economic Behavior, 1999, 27, 1–37.

[3] AUMANN R., SORIN S., *Cooperation and bounded recall*, Games and Economic Behavior, 1989, 1(1), 5–39.

[4] BANZHAF J., *Weighted voting doesn't work: a mathematical analysis*, Rutgers Law Review, 1965, 19, 317–343.

[5] BORNSTEIN G., *Intergroup conflict: individual, group and collective interests*, Technical Report 297, The Hebrew University of Jerusalem, Center for the Study of Rationality, 2002.

[6] BOWLES S., *Microeconomic Theory: Behavior, Institutions and Evolution*, Princeton University Press, 2003.

[7] BRAMS S., *A generic negotiation game*, Journal of Theoretical Politics, 2002, 4(1), 53–66.

[8] BRAMS S., KILGOUR D., *Winding down if preemption or escalation occurs: A game-theoretic analysis*, Journal of Conflict Resolution, 1987, 31(4), 547–572.

[9] DANILOV V., SOTSKOV A., *Social Choice Mechanisms*. Springer, 2002.

[10] DUBEY P., SHAPLEY L.S., *Mathematical properties of the Banzhaf Power index*, Mathematics of Operations Research, 1979, 4(2), 99–131.

[11] ESTEBAN J., SCHNEIDER G., *Polarization and conflict: Theoretical and empirical numbers*, Journal of Peace Research, 2008, 45(2), 131–141.
[12] FALK A., FEHR E., FISCHBACHER U., *Driving forces behind informal sanctions*, Econometrica, 2005, 73(6), 2017–2030.
[13] GRAHAM R., KNUTH D., PATASHNIK O., *Concrete Mathematics – A Foundation for Computer Science*. Addison-Wesley, 1994.
[14] HOLZMAN R., LEHRER E., LINIAL N., *Some bounds for the Banzhaf index*, Mathematics of Operations Research, 1988, 13(2), 358–63.
[15] KESTEN C.G., *Game theory, simulated interaction, and unaided judgement for forecasting decisions in conflicts: Further evidence*, International Journal of Forecasting, 2005, 21, 463–472.
[16] MAS-COLELL A., WHINSTON M.D., GREEN J.R., *Microeconomic Theory*. Oxford University Press, 1995.
[17] MOULIN H., PELEG B., *Cores of effectivity functions and implementation theory*, Journal of Mathematical Economics, 1982, 10, 115–145.
[18] MYERSON R., *Values of games in partition function form*, International Journal of Game Theory, 1977, 6, 23–31.
[19] MYERSON R., *Conference structures and fair allocation rules*, International Journal of Game Theory, 1980, 9, 169–182.
[20] MYERSON R.B., *Game Theory: Analysis of Conflict*, Harvard University Press, 1997.
[21] OWEN G., *Game Theory* (3rd edition), Academic Press, 1995.
[22] PETROSJAN L.A., ZENKEVICH N.A., *Game Theory – Series on Optimization, 3*, World Scientific Publishing, 1996.
[23] ROSSI G., *Extending the choquet integral*, *EUSFLAT 2007 Proceedings*, 2007, Vol. 1, 273–280.
[24] ROSSI G., *Aggregation of information functions – extended abstract*, In FSTA (Fuzzy Sets Theory and Applications) 2008 Proceedings, 2008.
[25] ROTH A., *The Shapley value – Essays in honor of Lloyd S. Shapley*, Cambridge University Press, 1988.
[26] SANDHOLM T., LARSON K., ANDERSSON M., SHEHORY O., TOHM`E F., *Coalition structure generation with worst case guarrantees*, Artificial Intelligence, 1999, 111, 209–238.
[27] SHAPLEY L.S., *A value for n-person games*, [in:] *Contributions to the Theory of Games*, H. Kuhn, A.W. Tucker (eds.), Vol. 2, 307–317, Princeton University Press, 1953.
[28] SIGMUND K., *Punish or perish? Retaliation and Collaboration among humans*, Technical Report IR-07-054, International Institute for Applied Systems Analysis, 2007.
[29] TAN J., ZIZZO D., *Groups, cooperation and conflict in games*, The Journal of Socio-Economic, 2008, 37, 1–17.
[30] VAN DAMME E., *Strategic equilibrium*, [in:] *Handbook of Game Theory with Economic Applications*, R. Aumann, S. Hart (eds.), Vol. 2, 1521–1589, Elsevier, 2002.
[31] VAN HASSEN M., STOKMAN S., VAN OOSTEN R., *Conflict measures In cooperative exchange models of collective decision-making*, Rationality and Society, 2003, 15(1), 85–112.
[32] WEBER R., *Games in coalitional form*, [in:] *Handbook of Game Theory with Economic Applications*, R. Aumann, S. Hart (eds.), Vol. 2, 1285–1303, Elsevier, 2002.

O pomiarze konfliktu i siły w strategicznych konfiguracjach

Zgoda na koordynacje strategii w grach strategicznych owocuje tym, że koalicje mogą podejmować grupowe akcje oraz mogą być traktowane jak samoistni gracze, dla których opozycją jest ich dopełnienie.

Konflikt pojawia się wtedy, kiedy dla każdego rezultatu istnieje co najmniej jeden gracz, który ściśle preferuje inny wynik. W pracy zaproponowano indeks pomiaru tego konfliktu. Ogólna idea zawiera się w dwóch krokach: w pierwszym należy przekształcić strategiczną strukturę w grę koalicyjną przyjmującą wartości z przedziału jednostkowego, której wartości wyznaczają znormalizowaną wartość koordynacji wewnątrz koalicji, oraz w drugim kroku, na zagregowaniu po niepustych koalicjach wprowadzonego pomiaru konfliktu w jeden indeks także z przedziału jednostkowego. W grach strategicznych działania grupy koalicji prowadzą z założenia do maksymalizacji znormalizowanych użyteczności jej członków, podczas gdy ich dopełnienia wybierają odwet wśród najlepszych możliwych odpowiedzi. Została także dokonana charakteryzacja gry *pełnego konfliktu*, dla której proponowany indeks przyjmuje wartość maksymalną. Analizując możliwości odwetu (jednocześnie działań i preferencji), faktycznie mieszamy *siłę* i *konflikt*. W rzeczywistości akcje mogą być ignorowane, a cała uwaga może być skierowana wyłącznie na preferencje, w pracy modelowane jako rodzina permutacji wyników. Konflikt (a nie siła) pomiędzy koalicjami jest mierzony w kategoriach odległości pomiędzy rodzinami permutacji ich członków. Tak jest w szczególności w grach, gdzie zdolność do *blokowania* wyznacza, jaki wynik dana koalicja może blokować. Wybrane podejście na bazie wartości kardynalnych powoduje, że siła (a nie konflikt) osiąga swoje maksimum na blokach Maskina, gdzie osiągane jest *status quo* wyniku.

Słowa kluczowe: *gra strategiczna, konflikt, gra koalicyjna, indeks siły*