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PROBABILITY OF RUIN FOR A DEPENDENT, TWO-DIMENSIONAL POISSON PROCESS

A two-dimensional, dependent Poisson risk process is investigated in the paper. Claims are divided into two classes. Within each class claims have the same distribution, but claims belonging to different classes can have different distributions and the corresponding counting processes can be dependent. This dependence is induced by a common factor. Three models of ruin and the probabilities of ruin are investigated. The influence of the degree of class dependence on the probability of ruin are studied for each model.

Keywords: *Poisson process, claim, model of ruin, simulation, phase-type distribution*

1. Introduction

The risk process

$$U(t) = u + ct - S(t),$$

where u is an initial surplus, c is a premium rate and $S(t) = \sum_{i=1}^{N(t)} X_i$ is an aggregate claim to time t, is investigated in the classical theory of ruin [4, 5, 6]. In that model, the claims $X_1, X_2, \ldots$ are assumed to be independent and identically distributed random variables with cumulative distribution function (cdf) F_X. Also, the counting process $N(t)$ is independent of the claims and it is a Poisson process with intensity λ. The relative security loading $\theta > 0$ satisfies the equation $c = (1 + \theta)\lambda m$, where $m = E(X_i)$.

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Such a model will be generalized in this paper. We will assume that the claims are divided into classes and claims belonging to the same class are identically distributed. Claims belonging to different classes may have different distributions and the counting processes from particular classes may be dependent. These are more realistic assumptions. They describe reality better than the classical model. Due to the weakness of the assumption that the risk process is homogeneous, dividing claims into classes and allowing dependence of the number of claims allows us to investigate more complicated and realistic actuarial problems. We can study situations in which a common, external factor, e.g. natural calamity, affects different risks. The model presented in this paper is based on the papers [7], [8].

The probability of ruin in such a generalized risk process is investigated in this paper. This probability is described by the formula

$$\psi(u) = P(T < \infty | \, U(0) = u),$$

where T is the ruin time:

$$T = \inf\{t: U(t) < 0\}.$$

When $c \le \lambda m$, that is $\theta \le 0$, ruin is certain, i.e. $\psi(u) = 1$ for any initial surplus. So, we assume in our investigations, that $c > \lambda m$.

We can derive the probability of ruin using the following random variable [4], [5]

$$L = \max_{t \ge 0}(S(t) - ct)$$

called the maximum aggregate loss. The following relation is satisfied

$$\psi(u) = P(L > u) = \overline{F_L}(u),$$

where $\overline{F_L}(u) = 1 - F_L(u)$ is the survival function for the random variable L. The maximum aggregate loss L has a compound geometric distribution:

$$L = V_1 + \ldots + V_K,$$

where the V_k are the amounts by which the k-th record low is less than the $(k-1)$-th one and K is the total number of records. The random variable K has a geometric distribution:

$$P(K = k) = (\psi(0))^k (1 - \psi(0)),$$

for $k = 0, 1, \ldots$, and $\psi(0) = \lambda m / c$. The record variables are identically distributed with cdf

$$F_V(y) = \frac{1}{m} \int_0^y \overline{F_X}(x) dx.$$

2. Two-dimensional, compound Poisson process

As mentioned in the introduction, the classical risk model is too simplified, unrealistic. Now let us weaken some assumptions of that model. We assume that there are two kinds of claims X_i and Y_i belonging to two classes. We will investigate a two--dimensional model [8]

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t - \begin{pmatrix} S_1(t) \\ S_2(t) \end{pmatrix},$$

where $S_1(t) = \sum_{i=1}^{N_1(t)} X_i$ and $S_2(t) = \sum_{i=1}^{N_2(t)} Y_i$. Let us assume that the numbers of claims $N_j(t), j = 1, 2$ are the sums of two processes:

$$N_1(t) = M_1(t) + M_0(t), \qquad N_2(t) = M_2(t) + M_0(t).$$

First, $M_j(t), j = 1, 2$, is an individual factor, characteristic to each class. This may reflect the impact of internal factors. The process $M_0(t)$ is common to both classes. We can interpret it as the impact of an external factor, which affects both classes. Such a factor may reflect the impact of the risk connected with natural disasters, such as: great fires, tornadoes, earthquakes or floods.

Let us assume that the processes $M_j(t), j = 0, 1, 2$, are independent Poisson processes with intensities λ_i. But, the common factor $M_0(t)$ implies that the aggregated processes $N_j(t), j = 1, 2$, are dependent. We also assume that the claims X_i and Y_i are independent, they are independent of $M_j(t)$ and identically distributed in each class. If $c_1 > (\lambda_1 + \lambda_0)m_X$ and $c_2 > (\lambda_2 + \lambda_0)m_Y$, where $m_X = E(X_i)$ and $m_Y = E(Y_i)$, then ruin is uncertain. So, we make such an assumption in our paper.

Let $T_i = \inf\{t: U_i(t) < 0\}$ and $\psi_i(u) = P(T_i < \infty | U_i(0) = u)$ be the ruin time and probability of ruin in class i, $i = 1, 2$. We will study three models of ruin in this paper. In the first model, the ruin time is described by formula [2], [8]:

$$T_{\text{or}} = \inf\{t: U_1(t) < 0 \text{ or } U_2(t) < 0\} = \min\{T_1, T_2\}.$$

This is the time to the first ruin. We will investigate the probability of the following event:

$$\psi_{\text{or}}(u_1, u_2) = P(T_{\text{or}} < \infty | U_1(0) = u_1, U_2(0) = u_2).$$

The ruin time in the second model is described in the following way [2]:

$$T_{\text{and}} = \inf\{t: U_1(t) < 0 \text{ and } U_2(t) < 0\} = \max\{T_1, T_2\}.$$

We can interpret this as the first moment of ruin in both classes. In this case, the probability of ruin is equal to

$$\psi_{\text{and}}(u_1, u_2) = P(T_{\text{and}} < \infty | U_1(0) = u_1, U_2(0) = u_2)$$

$$= P(T_1 < \infty, T_2 < \infty | U_1(0) = u_1, U_2(0) = u_2). \quad (1)$$

We will also study the sum of both risk processes [7], the joint capital (value of internal assets) for the two classes:

$$U(t) = U_1(t) + U_2(t) = u + ct - S(t),$$

where $u = u_1 + u_2$, $c = c_1 + c_2$ and $S(t) = S_1(t) + S_2(t)$. The ruin time T_s and probability of ruin $\psi_s(u)$ are described in the conventional way:

$$T_s = \inf\{t: U(t) < 0\},$$

$$\psi_s(u) = P(T_s < \infty | U(0) = u).$$

We will investigate these three models connected with a two-dimensional, dependent Poisson process deeply in the next part of the paper. We will mainly study the impact of the degree of dependence of the numbers of claims, i.e. processes $N_1(t)$ and $N_2(t)$, on the probability of ruin. The common process $M_0(t)$ reflects the external factor and the degree of dependence of the two processes is determined by the relative value of its intensity λ_0 to the intensities λ_i, $i = 1, 2$, of the internal factors .

3. Time to first ruin and ruins in both classes

The time to the first ruin T_{or} is not greater than the time to ruin in both classes T_{and}. Hence, the probability of ruin ψ_{or} in the first model is not smaller than the probability of ruin ψ_{and} in the second model. We can treat the first model as a conservative approach, as a warning system, which alarms against potential threats [2].

Now, we evaluate the probability of ruin in both classes $\psi_{\text{and}}(u_1, u_2)$. When $\lambda_0 = 0$, the processes $N_1(t)$ and $N_2(t)$ are independent and

$$P(T_1 < \infty, T_2 < \infty) = P(T_1 < \infty)P(T_2 < \infty).$$

In this case, the probability of ruin is equal to

$$\psi^I_{\text{and}}(u_1, u_2) = P(T_1 < \infty, T_2 < \infty | U_1(0) = u_1, U_2(0) = u_2) = \psi_1(u_1)\psi_2(u_2).$$

Yuen, Guo and Wu showed in [8] that the random variables T_1 and T_2 are associated, i.e. $\text{Cov}(f(T_1), g(T_2)) \geq 0$ is valid for all non-decreasing functions f and g. This fact implies the following inequalities:

$$P(T_1 < \infty)P(T_2 < \infty) \leq P(T_1 < \infty, T_2 < \infty) \leq \min\{P(T_1 < \infty), P(T_2 < \infty)\},$$

i.e. the following lower and upper bounds on the probability of ruin hold

$$\psi_1(u_1)\,\psi_2(u_2) \le \psi_{\text{and}}(u_1, u_2) \le \min\{\psi_1(u_1),\ \psi_2(u_2)\}.$$

Let us investigate the probability of at least one ruin $\psi_{\text{or}}(u_1, u_2)$. Using the basic properties of probability, we obtain:

$$P(\min\{T_1, T_2\} < \infty) = P(T_1 < \infty) + P(T_2 < \infty) - P(T_1 < \infty, T_2 < \infty).$$

Thus the probability of ruin for independent processes is equal to

$$\begin{aligned}\psi^{I}_{\text{or}}(u_1,u_2) &= P(\min\{T_1, T_2\} < \infty |\ U_1(0) = u_1, U_2(0) = u_2)\\ &= \psi_1(u_1) + \psi_2(u_2) - \psi_1(u_1)\,\psi_2(u_2)\end{aligned}$$

and lower and upper bounds on the probability of ruin are described by the inequalities

$$\max\{\psi_1(u_1),\ \psi_2(u_2)\} \le \psi_{\text{or}}(u_1, u_2) \le \psi_1(u_1) + \psi_2(u_2) - \psi_1(u_1)\,\psi_2(u_2),$$

since the relation $a + b - \min\{a, b\} = \max\{a, b\}$ is satisfied for every a, b. Moreover, we obtain

$$\psi_{\text{or}}(u_1, u_2) = \psi_1(u_1) + \psi_2(u_2) - \psi_{\text{and}}(u_1, u_2). \tag{2}$$

Now, we will study the impact of the degree of dependence of the counting processes $N_1(t)$, $N_2(t)$ on the probability of ruin. To this end, we will investigate the following two-dimensional Poisson processes:

$$\begin{pmatrix}U_1(t)\\ U_2(t)\end{pmatrix} = \begin{pmatrix}u_1\\ u_2\end{pmatrix} + \begin{pmatrix}c_1\\ c_2\end{pmatrix}t - \begin{pmatrix}S_1(t)\\ S_2(t)\end{pmatrix} \qquad \begin{pmatrix}U_1'(t)\\ U_2'(t)\end{pmatrix} = \begin{pmatrix}u_1\\ u_2\end{pmatrix} + \begin{pmatrix}c_1\\ c_2\end{pmatrix}t - \begin{pmatrix}S_1'(t)\\ S_2'(t)\end{pmatrix},$$

where $S_1(t) = \sum_{i=1}^{N_1(t)} X_i$, $S_2(t) = \sum_{i=1}^{N_2(t)} Y_i$, $S_1'(t) = \sum_{i=1}^{N_1'(t)} X_i'$ and $S_2'(t) = \sum_{i=1}^{N_2'(t)} Y_i'$. Moreover,

$$N_1(t) = M_1(t) + M_0(t), \quad N_2(t) = M_2(t) + M_0(t)$$

$$N_1'(t) = M_1'(t) + M_0'(t), \quad N_2'(t) = M_2'(t) + M_0'(t),$$

where $M_i(t)$ and $M_i'(t)$ $i = 0, 1, 2$, are Poisson processes with intensities λ_i and λ_i'. We assume that the claims X_i, X_i' and Y_i, Y_i' and processes $N_i(t)$, $N_i'(t)$ are identically distributed for each i, so we obtain $\lambda_i + \lambda_0 = \lambda_i' + \lambda_0' = \nu_i$. The intensities λ_0 and λ_0' reflect the degree of dependence. For instance, when $\lambda_0 = 0$, the processes are independent. In this case, the probability of ruin ψ^{I}_{and} is equal to

$$\psi^{I}_{\text{and}}(u_1,u_2) = \psi_1(u_1)\,\psi_2(u_2)$$

in this situation. This is the "best" case. In every case where the individual probabilities of ruin ψ_i are the same, the following inequality holds: $\psi^I_{\text{and}}(u_1,u_2) \le \psi_{\text{and}}(u_1,u_2)$. However, for the probability of at least one ruin, ψ^I_{or}, we have $\psi^I_{\text{or}}(u_1,u_2) \ge \psi_{\text{or}}(u_1,u_2)$ ψ^I_{or}. A generalization of this fact is presented in Theorem 1.

Theorem 1

a) If $\lambda_0 \ge \lambda'_0$, then $\psi_{\text{and}}(u_1, u_2) \ge \psi'_{\text{and}}(u_1,u_2)$.

b) If $\lambda_0 \ge \lambda'_0$, then $\psi_{\text{or}}(u_1, u_2) \le \psi'_{\text{or}}(u_1,u_2)$ [8].

Proof: These follow directly from (1), (2) and the inequality

$$P(T_1 < \infty, T_2 < \infty) \ \ge\ P(T'_1 < \infty, T'_2 < \infty)\,,$$

which was proved by Yuena, Guo and Wu in [8, Th.3].

From this theorem, we obtain that a greater intensity λ_0, i.e. greater degree of dependence, gives us a greater (smaller) probability of ruin in both classes (at least one ruin). When $\lambda_0 = \min\{v_1, v_2\}$, i.e. when λ_0 takes its greatest potential value, we obtain the smallest (greatest) dependence between the processes. One process depends only on $M_0(t)$. We have the „worst" ("best") situation, the probability of ruin in both classes $\psi^d_{\text{and}}(u_1,u_2)$ (at least one ruin $\psi^d_{\text{or}}(u_1,u_2)$) is the greatest (smallest). The "best" ("worst") situation occurs when the processes are independent, as we obtain the following relations:

$$\psi^d_{\text{and}}(u_1,u_2) \ge \psi_{\text{and}}(u_1,u_2) \ge \psi^I_{\text{and}}(u_1,u_2)\,, \quad \psi^d_{\text{or}}(u_1,u_2) \le \psi_{\text{or}}(u_1,u_2) \le \psi^I_{\text{or}}(u_1,u_2)\,.$$

Example 1. Let $v_1 = v_2 = 1$ and $c_1 = c_2 = 1.1$. Moreover, we assume that the random variables X_i and Y_i have the same exponential distribution: Exp(1). Tables 1 and 2 contain the probabilities of ruin ψ_{or} and ψ_{and} for intensity $\lambda_0 = 0$, which represents independence, and for λ_0 equal to 0.5 and 1, when we have strict dependence. These probabilities were calculated using simulations with an initial surplus $u_1 = u_2 = u$ equal to 0, 5, 10, 15, 20, 25, 30, 35 and 40. Figures 1 and 2 contain the graphs of the functions ψ_{or} and ψ_{and}.

Table 1. The probability of ruin ψ_{or}

ψ_{or}	U								
λ_0	0	5	10	15	20	25	30	35	40
0	0.9917	0.8211	0.5984	0.4109	0.2734	0.1786	0.1154	0.0740	0.0473
0.5	0.9852	0.7908	0.5757	0.3961	0.2659	0.1680	0.1135	0.0669	0.0473
1	0.9776	0.7417	0.5269	0.3712	0.2314	0.1463	0.1021	0.0636	0.0473

Table 2. The probability of ruin ψ_{and}

ψ_{and}	u								
λ_0	0	5	10	15	20	25	30	35	40
0	0.8264	0.3330	0.1341	0.0540	0.0218	0.0088	0.0035	0.0014	0.0006
0.5	0.8330	0.3633	0.1569	0.0688	0.0293	0.0193	0.0054	0.0086	0.0006
1	0.8406	0.4124	0.2056	0.0938	0.0638	0.0411	0.0168	0.0119	0.0006

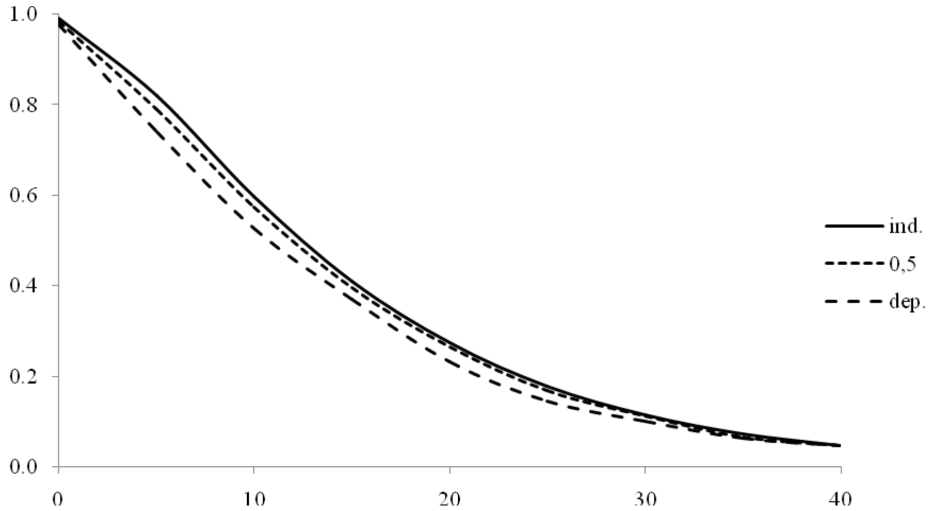

Fig. 1. Graph of the function ψ_{or}

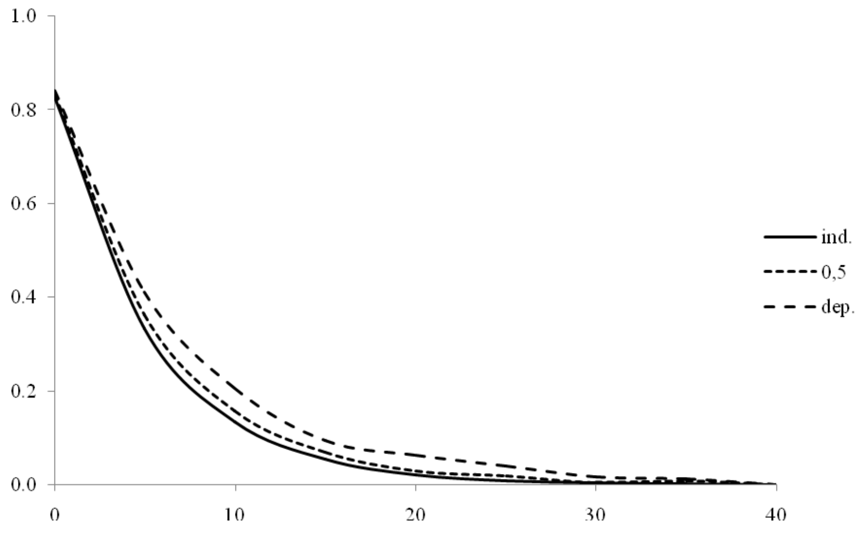

Fig. 2. Graph of the function ψ_{and}

4. The sum of the processes

Now we will investigate the model which represents the aggregation of two risk processes:

$$U(t) = u + ct - S(t), \tag{3}$$

where $S(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i$ and $N_j(t) = M_j(t) + M_0(t)$, $j = 1, 2$.

The two-dimensional counting process $\mathbf{N}(t) = [N_1(t), N_2(t)]^{\mathrm{T}}$ can be described as

$$\mathbf{N}(t) = \mathbf{AM}(t),$$

where $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{M}(t) = [M_1(t), M_2(t), M_0(t)]$. If we want to find the distribution of the sum $S(t)$, then we can use the result of Ambagaspitya in [1]. Let $\mathbf{N} = [N_1, ..., N_p]^{\mathrm{T}}$ be a p-dimensional random vector, $\mathbf{M} = [M_1, ..., M_k]^{\mathrm{T}}$ be a k-dimensional random vector, where M_j, $j = 1, ..., k$, are independent, Poisson distributed random variables with intensities λ_j and $\mathbf{A} = (a_{ij})$ be a $p \times k$-dimensional matrix with natural elements. Suppose the vector $\mathbf{N}$ is of the form

$$\mathbf{N} = \mathbf{AM}.$$

Consider the sum

$$S = \sum_{i=1}^{p} \sum_{j=1}^{N_i} X_{ij} ,$$

where the X_{ij} are independent random variables, independent of M_j and they are identically distributed for fixed i with cdf F_{X_i}. S has a compound Poisson distribution CP(λ, H), where $\lambda = \sum_{j=1}^{k} \lambda_j$, $H(x) = \frac{1}{\lambda} \sum_{j=1}^{k} \lambda_j (F_{X_1}^{*(a_{1j})} * F_{X_2}^{*(a_{2j})} * ... * F_{X_p}^{*(a_{pj})})(x)$, and $F_{X_i}^{*(a_{ij})}$ is the a_{ij} – time convolution of the cdf F_{X_i}. We assume that $F_{X_i}^{*(0)} \equiv 1$.

Using the above results, it follows that the sum $S(t)$ is a compound Poisson process $CP(\lambda, H)$, where $\lambda = \lambda_1 + \lambda_2 + \lambda_0$ and

$$H(x) = \frac{1}{\lambda}(\lambda_1 F_X(x) + \lambda_2 F_Y(x) + \lambda_0 F_X * F_Y(x)) .$$

Hence, we can derive the probability of ruin for the risk process (3) using classical methods based on the compound Poisson process with the aggregated claims Z_i having cdf $H(x)$. The expected value of such a claim is equal to

$$E(Z_i) = \frac{\lambda_1 + \lambda_0}{\lambda} m_X + \frac{\lambda_2 + \lambda_0}{\lambda} m_Y = m\,.$$

The cdf of the record variable *V,* is of the form

$$F_V(y) = \frac{1}{m}\int_0^y (1 - H(x))dx$$

$$= \frac{1}{(\lambda_1 + \lambda_0)m_X + (\lambda_1 + \lambda_0)m_Y}\int_0^y (\lambda - \lambda_1 F_X(x) - \lambda_2 F_Y(x) - \lambda_0 F_X * F_Y(x))dx$$

and the number of records *K* is geometrically distributed with parameter

$$p = 1 - \psi(0) = \frac{(\lambda_1 + \lambda_0)m_X + (\lambda_1 + \lambda_0)m_Y}{c}\,.$$

Now we will study the impact of the degree of dependence, described by the intensity λ_0 common to both classes, on the probability of ruin. As in the previous model, we assume, that the claims X_i, X_i' and Y_i, Y_i' have the same distribution for each *i* and the conditions $\lambda_i + \lambda_0 = \lambda_i' + \lambda_0' = v_i$ are satisfied. Then $\lambda = v_1 + v_2 - \lambda_0$,

$$H(x;\lambda_0) = \frac{1}{\lambda}((v_1 - \lambda_0)F_X(x) + (v_2 - \lambda_0)F_Y(x) + \lambda_0 F_X * F_Y(x))\,,$$

$$m = \frac{v_1 m_X + v_2 m_Y}{v_1 + v_2 - \lambda_0}$$

and

$$p = \frac{v_1 m_X + v_2 m_Y}{c}\,.$$

We see that the distribution of the random variable *K* does not depend on λ_0. But the distribution of the record variable *V* depends on this parameter. The cdf of the random variable *V* is equal to

$$F_V(y;\lambda_0)$$

$$= \frac{1}{v_1 m_X + v_2 m_Y}\int_0^y (v_1 + v_2 - \lambda_0 - (v_1 - \lambda_0)F_X(x) - (v_2 - \lambda_0)F_Y(x) - \lambda_0 F_X * F_Y(x))dx.$$

When $\lambda_0 = 0$, the processes $U_1(t)$ and $U_2(t)$ are independent. The sum $S(t)$ is a compound Poisson process, where $\lambda = v_1 + v_2$ and $H(x) = \frac{1}{\lambda}(\lambda_1 F_X(x) + \lambda_2 F_Y(x))$. When λ_0

takes its greatest potential value, i.e. $\lambda_0 = \min\{v_1, v_2\}$, we obtain the greatest degree of dependence. We assume that $v_1 \le v_2$ and it follows that the sum $S(t)$ is a compound Poisson process with $\lambda = v_2$ and $H(x) = \frac{1}{v_2}((v_2 - v_1)F_Y(x) + v_1 F_X * F_Y(x))$. Theorem 2 describes the relation between the value of the parameter λ_0 and the probability of ruin.

Theorem 2. If $\lambda_0 \le \lambda_0'$, then $\psi_s(u) \le \psi_s'(u)$.

Proof: Let $\lambda_0' = \lambda_0 + d$, where $d > 0$. Let us derive the distribution of the difference between records. Straightforward computation gives

$$F_V(y) - F_{V'}(y) = \frac{d}{v_1 m_X + v_2 m_Y} \int_0^y (1 + F_X * F_Y(x) - F_X(x) - F_Y(x))dx .$$

Now, let us show that the above integral is nonnegative for any $y \ge 0$, i.e.

$$\int_0^y (1 + F_X * F_Y(x) - F_X(x) - F_Y(x))dx \ge 0 . \tag{4}$$

The inequality $(X + Y - y)_+ \ge (X - y)_+ + (Y - y)_+$ is satisfied for any $y \ge 0$ almost everywhere (see [4] (10.9)). This implies $E[(X + Y - y)_+] \ge E[(X - y)_+] + E[(Y - y)_+]$, i.e.

$$\int_y^\infty (\overline{F_X * F_Y})(x)dx \ge \int_y^\infty (\overline{F_X})(x)dx + \int_y^\infty (\overline{F_Y})(x)dx ,$$

since $E[(X - y)_+] = \int_y^\infty (1 - F_X(x))dx$ [4]. Using $\int_0^\infty \overline{F}_X(x)dx = E(X)$, we obtain

$$\int_0^y (\overline{F_X * F_Y})(x)dx \le \int_0^y \overline{F_X}(x)dx + \int_0^y \overline{F_Y}(x)dx .$$

This inequality implies

$$0 \le \int_0^y (\overline{F_X}(x) + \overline{F_Y}(x) - \overline{F_X * F_Y})(x))dx = \int_0^y (1 + F_X * F_Y(x) - F_X(x) - F_Y(x))dx$$

and gives (4). Hence, we obtain $F_V(y) \ge F_{V'}(y)$ for any $y \ge 0$. This fact implies the following stochastic dominance: $V \le_{st} V'$ and hence $L \le_{st} L'$ [4], because the variables K and K' counting the number of records are identically distributed, since this distribution does not depend on λ_0. The last dependence implies $\psi_s(u) \le \psi'_s(u)$, because $\psi(u) = \overline{F_L}(u)$.

We see that we have the same relation as in Theorem 1a. A greater degree of dependence leads to a greater probability of ruin. Independence is the "best" case. The probability of ruin is the smallest in this situation.

Let us investigate the case of exponential claims. We can analytically compute the probability of ruin. Let X_i, Y_i have exponential distributions: Exp(m_X) and Exp(m_Y), where the parameters m_X, m_Y are the expected values. The aggregated value of claims $S(t)$ is a compound Poisson process $CP(\lambda, H)$. The random variable Z with cdf $H(x)$ has a phase-type distribution, because the exponential distribution is phase-type. The convolution and convex combination of phase-type distributions are also phase-type distributions [6]. The sum $X_i + Y_i$ has a phase-type distribution $PH(\boldsymbol{\alpha}_1, \mathbf{B}_1)$, where $\boldsymbol{\alpha}_1$ = (1, 0) and $\mathbf{B}_1 = \begin{pmatrix} -1/m_X & 1/m_X \\ 0 & -1/m_Y \end{pmatrix}$. Hence, the random variable Z has distribution PH(α, $\mathbf{B}$), where $\alpha = (\lambda_1/\lambda, \lambda_2/\lambda, \lambda_0/\lambda, 0)$ and

$$\mathbf{B} = \begin{pmatrix} -1/m_1 & 0 & 0 & 0 \\ 0 & -1/m_2 & 0 & 0 \\ 0 & 0 & -1/m_1 & 1/m_1 \\ 0 & 0 & 0 & -1/m_2 \end{pmatrix}.$$

Then the probability of ruin $\psi(u)$ for the process $U(t)$ is described by the formula [6]

$$\psi(u) = p\alpha^s \exp(u(\mathbf{B} + p\mathbf{b}^{\mathrm{T}}\alpha^s))\mathbf{e}^{\mathrm{T}}, \tag{5}$$

where $p = \frac{1}{1+\theta}$, $\alpha^s = -\frac{1}{m_Z}\alpha\mathbf{B}^{-1}$, $\mathbf{b}^{\mathrm{T}} = -\mathbf{B}\mathbf{e}^{\mathrm{T}}$, and $\exp(\mathbf{A}) = \mathbf{I} + \frac{\mathbf{A}}{1!} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \ldots$ for matrix $\mathbf{A}$ and identity matrix $\mathbf{I}$.

Example 2. Let $v_1 = v_2 = v = 1$, $c = 2.2$, $u_1 = u_2$ and let the claims X_i, Y_i have the same exponential distribution Exp(1) with cdf $F(x) = 1 - e^{-x}$. We will study the relation of the probability of ruin with the degree of dependence between the two processes, i.e. with the value of the parameter λ_0. To this end, we will investigate five values for this parameter: 0 (independence), 0.25, 0.5, 0.75 and 1 (strict dependence). The process $S(t)$ is a compound Poisson process $CP(\lambda, H)$, where $\lambda = 2 - \lambda_0$, and $H(x) = \frac{2-2\lambda_0}{2-\lambda_0}F(x) + \frac{\lambda_0}{2-\lambda_0}F * F(x)$. The sum of random variables $X_i + Y_i$ with cdf $F{*}F(x) = 1 - xe^{-x}$ has an Erlang distribution, Erl(2, 1), and the random variable Z with distribution $H(x)$ has a phase-type distribution $PH(\alpha, \mathbf{B})$, where α =

$\left(\frac{2-2\lambda_0}{2-\lambda_0}, \frac{\lambda_0}{2-\lambda_0}, 0\right)$, $\mathbf{B} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and expected value $m_Z = \frac{2}{2-\lambda_0}$. The probability of ruin is described by formula (5), where $p = \frac{1}{1.1}$, $\alpha^s = \left(\frac{2-2\lambda_0}{2}, \frac{\lambda_0}{2}, \frac{\lambda_0}{2}\right)$ and $\mathbf{b} = (1, 0, 1)$. This probability takes the following form in each particular case:

a) $\lambda_0 = 0$ (independence):

$$S(t) \sim CP(2; F),$$

$$\psi(u) = 0.9091 e^{-u/11},$$

b) $\lambda_0 = 0.25$:

$$S(t) \sim CP\left(1.17; \frac{6}{7}F + \frac{1}{7}F * F\right),$$

$$\psi(u) = 0.9103e^{-0.0809u} - 0.0012e^{-1.1236u},$$

c) $\lambda_0 = 0.5$:

$$S(t) \sim CP\left(1.5; \frac{2}{3}F + \frac{1}{3}F * F\right),$$

$$\psi(u) = 0.9128e^{-0.073u} - 0.0037e^{-1.2452u},$$

d) $\lambda_0 = 0.75$:

$$S(t) \sim CP(1.25; \frac{2}{5}F + \frac{3}{5}F * F)$$

$$\psi(u) = 0.9159e^{-0.0666u} - 0.0068e^{-1.3652u},$$

e) $\lambda_0 = 1$ (strict dependence):

$$S(t) \sim CP(1; F*F)$$

$$\psi(u) = 0.9192e^{-0.0613u} - 0.01009e^{-1.4842u}$$

Figure 3 gives graphs of the probability of ruin $\psi(u)$ for three values of the parameter λ_0: 0, 0.5 and 1.

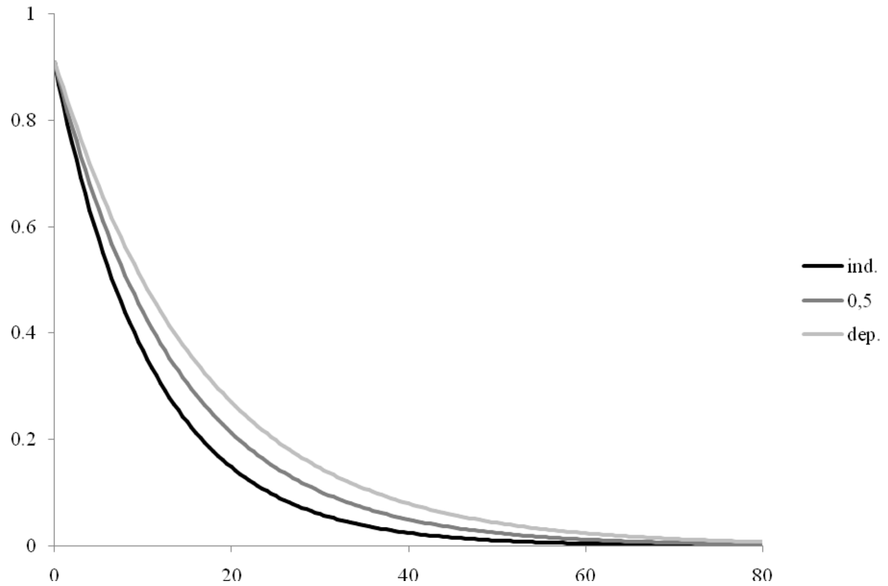

Fig. 3. Graphs of the function $\psi(u)$ for different values of λ_0

5. Conclusion

Three models of ruin for a two-dimensional, dependent Poisson process are investigated. A dependence between the number of claims of different classes is induced by common factors. A different ruin time is studied in each model. The time to ruin in both classes, the time to the first ruin and the time to ruin for the sum of the processes are investigated. An analysis of the relation of the probability of ruin with the degree of dependence between the components of the process is presented. The cases are illustrated by examples, where claims have an exponential distribution. The probabilities of ruin are estimated using simulation in the first two cases, but an analytical formula based on phase-type distributions is used in the last case.

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Prawdopodobieństwo ruiny dla zależnego, dwuwymiarowego procesu Poissona

W pracy rozpatrywany jest dwuwymiarowy, zależny proces ryzyka Poissona. Wielkości wypłat podzielono na dwie klasy. W każdej klasie wypłaty mają ten sam rozkład, natomiast wypłaty należące do różnych klas mogą mieć różne rozkłady, a procesy liczące wypłaty mogą być zależne. Zależność ta jest generowana przez wspólny czynnik. Rozpatrywane są trzy modele ruiny, oparte na różnych sposobach wyznaczania czasu ruiny: czas wystąpienia pierwszej ruiny, pierwszy moment wystąpienia ruiny w obydwu klasach oraz ruina dla sumy procesów. Badane jest prawdopodobieństwo wystąpienia ruiny oraz wpływ stopnia zależności klas na to prawdopodobieństwo. Rozpatrzono przykłady, w których wypłaty mają rozkłady wykładnicze. W dwóch pierwszych modelach prawdopodobieństwa ruiny zostały wyznaczone metodami symulacyjnymi. W trzecim modelu wykorzystano metodę opartą na rozkładach fazowych.

Słowa kluczowe: *proces Poissona, model ruiny, wypłata, symulacja, rozkład fazowy*