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INDIRECT CONTROL AND POWER

To determine who has the power within a stock corporate company can be a quite complex problem, especially when control is achieved through alliances between shareholders. This problem arises especially in cases of indirect control of corporations, that is, in situations involving shareholders and companies with cross-shareholdings. The first to solve the problem of measuring power in the case of indirect share control were Gianfranco Gambarelli and Guillermo Owen in [10]. In the following years, numerous other models were introduced. In this paper, we critically examine the models of: Gambarelli and Owen, Denti and Prati, Crama and Leruth, Karos and Peters, as well as Mercik and Lobos, taking into account two well-known, illustrative examples, one with an acyclic corporate structure and the other with a cyclic structure.

Keywords: *game theory, indirect control, corporations, power indices*

1. Introduction

To determine who has the power within a stock corporate company can be a quite complex problem, especially when control is achieved through alliances between shareholders. This problem arises especially in the cases of indirect control of corporations, that is, in situations involving shareholders and companies with cross-shareholdings. In these circumstances, there is the need to know what coalitions of firms can control a given company.

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It is a problem as complex as it is crucial. It is indeed easy to find situations in which a holding company controls other companies, resulting in a pyramidal construction that allows the holding company to gradually multiply the capital controlled, starting from a relatively low initial investment. The effects of this phenomenon are also accentuated by the dispersion of actions made by an ocean of small investors. Therefore, in practice, to acquire control of a company, it is often sufficient to possess a much lower percentage of the capital than the theoretical proportion (50% of the shares plus one share).

The work of Gambarelli and Owen [10], published in 1994, is one of the pioneering papers in this field. In the following years, numerous other models were introduced. For us it is difficult, if not impossible, to cite all the articles on indirect control of corporations in one paper, because there is a vast literature on this topic. Anyway, Crama and Leruth [5] and Karos and Peters [13] made a vast literature review of the most relevant and illustrative references in this field. For this reason, we refer interested readers to those sources and here we limit ourselves to a brief review of the relevant literature related to measuring power in corporate structures.

Gambarelli and Owen [10] and Denti and Prati [8, 9] focused on determining the winning coalitions in a control structure. Kołodziej and Stach [14] proposed a computer program based on the approach of Denti and Prati to enable simulations. On the other hand, the works of Hu and Shapley [11], Crama and Leruth [4, 5] and Crama, Leruth and Wang [6], Karos and Peters [13], as well as Mercik and Lobos [15], are dedicated to modelling indirect control relationships in corporate structures and using power indices to evaluate the power of players.

This paper is organized as follows. In Section 2, some preliminary definitions and notation are given. Section 3 provides two illustrative examples. In Section 4, brief overviews of some approaches to measuring indirect control are described. Section 5 gives some conclusions and open problems.

2. Preliminaries

A game is given by a set of rules describing a strategic situation. In cooperative games, players can collaborate to obtain common benefits. Let $N = \{1, 2, ..., n\}$ be the set of all players, indexed by the first n natural numbers. A cooperative n-person game in characteristic function form is an ordered pair (N, v), where $v : 2^N \to R$ is a real--valued function on the family 2^N of all subsets of N such that $v(\varnothing) = 0$. The real-valued function v is called the characteristic function of the game. Any subset S of N is called a coalition and $v(S)$ is the worth of the coalition S in the game. By $|S|$ we denote the cardinality of the set S. In this paper, we denote a cooperative game (N, v) simply by its characteristic function v.

A cooperative game v is monotonic if $v(S) \leq v(T)$ when $S \subset T$. A simple game is a monotonic game v (in N, omitted hereafter), which assumes values in the set {0, 1}: i.e. $v(S) = 0$ or $v(S) = 1$ for all the coalitions $S \subseteq N$. In the first case, a coalition is said to be losing, in the second – winning. Let $W(v)$ denote the set of all winning coalitions in game v. A player i is critical in a winning coalition S if $v(S\backslash\{i\}) = 0$. A simple game is said to be *proper*, if and only if the following is satisfied: for all $T \subset N$, if $v(T) = 1$, then $v(N \setminus T) = 0$. A coalition S is called a minimal winning coalition if $v(S) = 1$, but $v(T) = 0$ for all $T \subset S$, $T \neq S$. $W^m(v)$ denotes the set of all minimal winning coalitions in v. Such a game can be defined either by the family of winning coalitions W or equivalently by the set of minimal winning coalitions W^m. Every player i who belongs to a minimal winning coalition $S \subseteq N$ is critical in S. We will use the following notation: $C_i = \{S \subseteq N : i \text{ is critical in } S\}$. A coalition S is called vulnerable if S contains at least one critical player, i.e. $\exists i \in S$ such that $v(S \setminus \{i\}) = 0$.

Let ($w_1, ..., w_n$) be a vector with non-negative components such that $\sum_{i \in N} w_i = 1$. For any coalition S, $w(S) = \sum_{i \in S} w_i$ is the weight of the coalition. Let $q > 0$ be the majority quota that establishes winning coalitions (usually $q > w(N)/2$). We call the simple game: $v(S) = 1$ if $w(S) > q$ and otherwise $v(S) = 0$ a weighted majority game and denote it by $[q; w_1, ..., w_n]$. Weighted majority games are suitable for describing many voting situations: the weights can be shares owned in a company, seats of political parties, etc.

A power index is a function that maps an n-person simple game v, to an n-dimensional real vector and is a measure of the influence of the players in such games. The literature has proposed many power indices based on diverse axiomatic assumptions and/or models of bargaining. Below, we recall the definitions of only those power indices that are used in the models considered; namely, the definitions of the Shapley –Shubik, Banzhaf–Penrose, and Johnston indices.

The Shapley–Shubik index was introduced by Shapley and Shubik in [19]. For the sake of simplicity, we set $|S| = s$. The Shapley–Shubik index σ, for any v and $i \in N$ is expressed as

$$\sigma_i(v) = \sum_{S \in C_i} \frac{(s-1)!(n-s)!}{n!}$$

For further explanations see, e.g., [20].

The measure called the absolute Banzhaf index β by Banzhaf [1] is sometimes called the Penrose–Banzhaf index, since it actually goes back to Penrose [17, 18]. The absolute Banzhaf index, for any v and $i \in N$, is defined as:

$$\beta_i(v) = \frac{|C_i|}{2^{n-1}}$$

For further explanations, see e.g., [3].

The Johnston index was introduced by Johnston in [12]. Denote by *VC* the set of all vulnerable coalitions. For any $S \in VC$, by $r(S)$ we denote the reciprocal of the number of critical players in *S* and we define $r_i(S)$ in the following way: if *i* is critical in *S* then $r_i(S) = r(S)$, otherwise $r_i(S) = 0$. The raw (absolute) Johnston index is defined as: $\overline{\gamma}_i(S) = \sum_{S \in VC} r_i(S)$ and the Johnston index is obtained after normalization, i.e.

$$\gamma_i(S) = \frac{\sum_{S \in VC} r_i(S)}{\sum_{i=1}^{n} \sum_{S \in VC} r_i(S)}$$

Note that the Johnston index treats all coalitions with critical voters equally, and within each coalition the power is divided equally among critical voters.

We can now analyse the problem of measuring the power of indirect control in corporations by means of these power indices. Naturally, the problem of determining the percentage of shares in a certain company indirectly owned by investors affects the measurement of power. For example, suppose that companies A, B, C, and D have a share capital composed of 100 shares each. Now assume that A holds 40 shares of company B and the remaining shares are equally divided between two other investors. Suppose also that B owns 51 shares of C, which has in turn 25 shares of D, and that the remaining shares of D are divided equally among three other investors. In the above situation, we could say that A has (1/3)×(1)×(1/4) = 1/12 of the power in company D. In general, it would seem logical to assign to each shareholder a measure of "power via indirect control" given by the product of the appropriate power indices. Unfortunately, this method can lead to situations where the total quota of shares in the controlled company exceeds 100%. It is therefore necessary to consider other approaches that will be covered in the next sections.

But firstly it is necessary to analyse typical situations that may arise in the case of indirect control. The simplest case is characterized by the absence of cycles (loops). In this situation, each investor holds shares in a number of companies; investee companies may themselves hold voting rights in other companies, but there can be no cross-shareholdings. The presence of a loop occurs when there are two or more companies with cross-shareholdings.

Although in each country the legislator regulates the indirect control of corporations more or less decisively, this does not mean that the phenomenon of indirect control does not exist. The models recalled here do not consider legislative constraints. Hereafter, by an investor we mean a firm which is not controlled either directly or indirectly by any other firm in a corporate network and by a company we mean a stock corporation i.e. a corporation which has shareholders.

3. Illustrative examples

In our treatment, we will refer to two illustrative examples of corporate control structures. In the first example, there are no cycles, whereas in the second one, looped relationships among shareholders exist. In both examples, we use networks (weighted directed graphs) to show corporate control structures: firms (corporation and investors) are represented by vertices, while share ownership is represented by weighted edges. The number corresponding to an edge connecting, e.g., firm i to firm j represents, in percentage terms, how many voting rights firm i holds in firm j.

As in this paper we adopt a game-theoretical approach, where corporate structures are often described by simple voting games (in particular weighted majority games), in defining the examples we call the firms players and we talk about the majority quota q. Our illustrative examples refer to real corporate groups but reflect their past shareholding structures. Although over time the real corporate structures presented in both examples have changed, we have chosen not to update the situation to maintain the effectiveness of the examples.

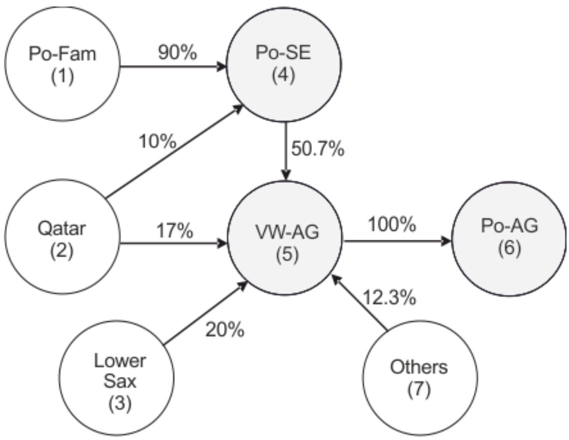

Fig. 1. The shareholding structure of the Porsche–Volkswagen case

Example 1. Let us consider the corporate structure presented in Fig. 1 with majority quota q = 80%. The players are: Po-Fam (1), Qatar (2), Lower Sax (3), Po-SE (4), VW--AG (5), Po-AG (6), and Others (7). In this case, we have seven players, where four of which (1, 2, 3, 7) are investors and three, (4, 5, 6), are companies. This example refers to the Porsche–Volkswagen case, which was considered by Karos and Peters [13]. Here, we modify a little the corporate structure, in order to be able to compare all the models considered. This means that we aggregate the 12.3% of voting rights of all the undefined shareholders to player 7 (Others), in accordance with the Volkswagen shareholder structure [21]. For a full description of this real case, see [13] and [21].

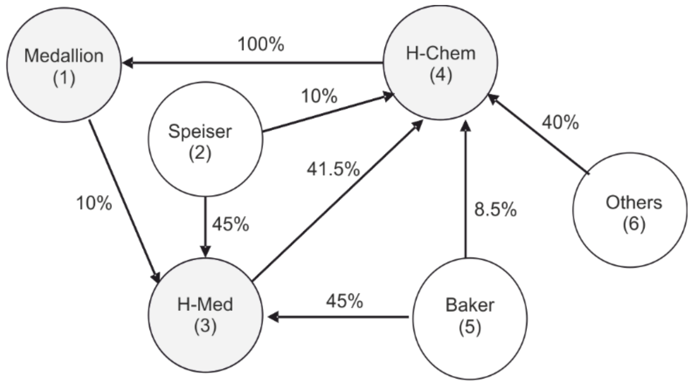

Fig. 2. The shareholding structure of the Speiser-Baker case;
H-Med – HealthMed, H-Chem – HealthChem

Example 2. This example deals with the Speiser and Baker case [15]. In this case, we also slightly simplify the corporate structure (Fig. 2). In this network, there are six players, where four of which are investors. Namely, there are the following players: Medallion (1), Speiser (2), HealthMed (3), HealthChem (4), Baker (5), Others (6). For this case, we consider a simple majority quota $q = 50\%$.

4. Approaches for measuring indirect control

In the literature on this topic, some general game-theoretic models to describe control relationships in corporate structures have been developed. In the following subsections, we limit ourselves to only consider those proposed by the following: Gambarelli and Owen, Denti and Prati, Crama and Leruth, Karos and Peters, as well as Mercik and Lobos. For each of these approaches, we provide only a brief overview, necessary to perform the calculation of the power indices considered by these models. Then, we try to compare these models taking into account the two illustrative examples. In describing these models, we try to keep the original notation, but sometimes, for clarity, we make some minor changes.

4.1. The Gambarelli and Owen model

Gambarelli and Owen [10] were first to define a power measure in the case of indirect control of corporations. They developed a mathematical model to determine the control (direct or indirect) that coalitions of investors have in the firms within a closed shareholding system. In particular, these two authors devised a very refined process capable of transforming a set of various linked majority games into a single game. This method is based on the concept of a multilinear extension, introduced by Owen [16]. One of the advantages of this methodology is that it can be used with any power index. In fact, this method constructs a resultant game, in which the index considered most suitable to describe the situation under consideration can be applied. The Gambarelli and Owen model enables us to solve all concatenated games without cycles and a class of cyclic games. The model recognizes other games as being unstable [2, 10]. In cases of instability, the Gambarelli and Owen model can only be interpreted via the intervention of exogenous factors to provide a statistical value; for example: the mean power index of the players involved in the cycle.

Let us introduce some necessary concepts. Let N be the set of all companies and M be the set of all investors in a shareholding system. Direct control in such a stock system can be described by a formal game system (f.g.s.), i.e. a n-tuple $[W_1, ..., W_n]$, of simple monotonic games over the set $N \cup M$. Let W_j denote the weighted majority game played in company j and in accordance with the notation in Section 2, it can be described by the set of all winning coalitions of shareholders in firm j, i.e. coalitions which hold the required majority of voting rights in j. In order to consider indirect control, Gambarelli and Owen proposed a reduction operation. A reduction for firms N and investors M is an n-tuple $(V_1, ..., V_n)$ of voting games over a set of investors M. In cases without loops, a recursive procedure to find the so-called effective reduction $(V_1, ..., V_n)$ starting from the original f.g.s. is provided [10]. In the cyclic case, such a procedure would repeat itself forever. Therefore, Gambarelli and Owen introduced a more general concept, namely a so-called consistent reduction which, however, is not necessarily unique. In a shareholding structure without loops, a consistent reduction coincides with the effective reduction. For this reason, hereafter we will only use the name "consistent reduction". Gambarelli and Owen provided a technique to obtain a consistent reduction. This technique relies on the idea of multilinear extensions. For details see [10].

Now, let us apply the Gambarelli and Owen approach to Example 1. In this example, {4, 5, 6} is the set of firms, and {1, 2, 3, 7} – the set of individual investors. Considering direct control, we have the formal game system: W_4 = {{1} and supersets}, W_5 = {{2, 3, 4}, {3, 4, 7} and supersets}, W_6 = {{5} and supersets}. Note that player 1 has control over player 4. Thus, concerning indirect control, we can see that investors 1, 2 and 3 jointly have indirect control over all the firms, 4, 5, and 6. Also, coalition {1, 3, 7} has control over all the companies. Considering multilinear extensions, we obtain the same result and the so-called, in the Gambarelli–Owen terminology, consistent reduction. Namely, the multilinear extensions for firms 4, 5, and 6 are:

$$MLE_4 = x_1(1-x_2) + x_1x_2 = x_1$$

$$MLE_5 = x_2x_3x_4(1-x_7) + x_3x_4x_7(1-x_2) + x_2x_3x_4x_7 = x_4(x_2x_3 + x_3x_7 - x_2x_3x_7)$$

$$MLE_6 = x_5$$

Solving this system of 3 equations with 7 variables by substituting $MLE_j = x_j$ (j = 4, 5, 6), we obtain: $x_4 = x_1$, $x_5 = x_6 = x_3x_4(x_2 + x_7 - x_2x_7)$ and the reduced extension: $RE_4 = x_1$, $RE_5 = RE_6 = = x_1x_3(x_2 + x_7 - x_2x_7)$. Since $x_1x_3(x_2 + x_7 - x_2x_7) = 1$ when $x_1 = x_2 = x_3 = 1$ or $x_1 = x_3 = x_7 = 1$ or $x_1 = x_2 = x_3 = x_7 = 1$, then we obtain the following consistent reduction: V_4 = {{1}}, $V_5 = V_6$ = {{1, 2, 3}, {1, 3, 7}, {{1, 2, 3, 7}}.

Now, for any firm j (j = 4, 5, 6), we can calculate the power of the investors using the three power indices considered in Section 2. Therefore, regarding company 4, we see that player 1 has total control over this player and the other investors have power measures equal to zero. The power of investors is presented in Table 1.

Table 1. Power of investors in companies 4, 5, and 6 in Example 1

Power index	Company 4	Companies 5 and 6			
	Player 1	Player 1	Player 2	Player 3	Player 7
Shapley–Shubik index	1	0.4166	0.0833	0.4166	0.0833
Absolute Banzhaf index		0.3750	0.1250	0.3750	0.1250
Johnston index		0.3889	0.1111	0.3889	0.1111

Player 3 (Lower Saxony) with only 20% of the voting rights in company 5 has the same power as player 1 (Porsche Families) in both companies 5 and 6. This result does not depend on the power index applied. It is sufficient that the index satisfies the symmetry condition (Table 1).

Let us see how the Gambarelli and Owen approach works in a stock system with loops. Namely, let us consider Example 2 (the Baker-Speiser case). Firstly, we calculate the multilinear extensions for players 1, 3, and 4:

$$MLE_1 = x_4, \quad MLE_3 = x_1x_2(1-x_5) + x_1x_5(1-x_2) + x_2x_5(1-x_1) + x_1x_2x_5$$

$$MLE_4 = x_2x_3(1-x_5)(1-x_6) + x_2x_{35}(1-x_6) + x_2x_3x_6(1-x_5) + x_2x_5x_6(1-x_3) + x_3x_6(1-x_2)(1-x_5) + x_3x_5x_6(1-x_2) + x_2x_3x_5x_6$$

After some algebraic steps, we obtain the following functions:

$$MLE_1 = x_4$$

$$MLE_3 = x_1x_2 + x_1x_5 + x_2x_5 - 2x_1x_2x_5$$

$$MLE_4 = x_2x_3 + x_3x_6 + x_2x_5x_6 - x_2x_3x_6 - x_2x_3x_5x_6$$

Now we solve this system of 3 equations with 6 variables by substituting $MLE_j = x_j$ (j = 1, 3, 4). This structure has loops. To solve the problem of cycles, Gambarelli and Owen proposed calculating a reduced multilinear extension (*RMLE*) for each company, thus we set $x_j = RMLE_j$ and solve the resulting system of equations for these variables. Hence, we obtain the following reduced extensions:

$$RE_1 = RE_4 = \frac{x_2x_5(x_2 + 2x_6 - x_2x_6 - x_2x_5x_6)}{X_1 + X_2}$$

$$RE_3 = \frac{x_2x_5(1 + x_2x_6 + x_5x_6 - 2x_2x_5x_6)}{X_1 + X_2}$$

where

$$X_1 = 1 - x_2^2 - x_2x_5 + 2x_2^2x_5 - x_2x_6 - x_5x_6 + 3x_2x_5x_6$$

$$X_2 = x_2^2x_6 - 2x_2^2x_5x_6 + x_2^2x_5x_6 + x_2x_5^2x_6 - 2x_2^2x_5^2x_6$$

As $x_j^2 = x_j$ for $x_j \in \{0, 1\}$, these functions can be reduced by lowering the exponents of the variables x_j (here, j = 2, 5, 6) to the first degree, i.e.:

$$RE_1 = RE_4 = \frac{x_2x_5(x_2 + 2x_6 - x_2x_6 - x_2x_5x_6)}{1 - x_2 + x_2x_5 - x_5x_6 + x_2x_5x_6}$$

$$RE_3 = \frac{x_2x_5(1 + x_2x_6 + x_5x_6 - 2x_2x_5x_6)}{1 - x_2 + x_2x_5 - x_5x_6 + x_2x_5x_6}$$

It is now necessary to evaluate the function RE_j. Because we are trying the find the set of winning coalitions, it is interesting to know when this function takes the value 1.

Considering the values of the numerators and denominators of RE_j, we have the following cases (Table 2).

Table 2. Value of RE_j (j = 1, 3, 4) for $x_i \in \{0, 1\}$, i = 2, 5, 6,
i.e. at the vertices of the unit hypercube

Case	x_2	x_5	x_6	Numerator of RE_j	Denominator of RE_j	RE_j
1	0	0	0	0	1	0
2	0	0	1	0	1	0
3	0	1	0	0	1	0
4	0	1	1	0	0	0/0
5	1	0	0	0	0	0/0
6	1	1	0	1	1	1
7	1	0	1	0	0	0/0
8	1	1	1	1	1	1

Source: Authors' calculations.

1. If the numerator of RE_j equals 1, then the denominator also equals 1, and in consequence the quotient equals 1 and the set of all winning coalitions required to control the three companies is composed of {2, 5} and {2, 5, 6}. Thus, investors 2 and 5 together have full control over all three companies.

2. If the numerator of RE_j equals 0 then

2.1. The denominator equals 0 for coalitions {2}, {2, 6}, and {5, 6} and the quotient becomes the indeterminate form 0/0, which means that hidden solutions are possible. Investors 5 and 6 alone do not have control. Investor 2 may or may not have control. The situation is unclear. On the one hand, if player 2 could convince the management of any of the companies 1, 3 or 4 to cooperate with him, this would give him full control over all the companies. On the other hand, if investor 5 could get firm 3's management to oppose investor 2, then 5 could keep player 2 indefinitely from getting full control over all the companies. From the theoretical point of view, a coalition between investors 5 and 6 would get company 3's management to cooperate with them. This would give coalition {5, 6} full control over all the companies. Another possibility is that the coalition of investors {2, 6} could act jointly and convince company 3's management to cooperate with them and thus they would obtain full control over all the companies.

2.2. The denominator equals 1 for other coalitions, so the ratio is 0. These coalitions are therefore losing.

Thus we see that in the presence of loops the calculation becomes more complicated and consistent reduction may not be unique. We find that the three simple games in the consistent reduction must all be equivalent and contain the following winning coalitions: {2, 5}, {2, 5, 6}. In addition to this, they might contain one, two, all, or none of the three following coalitions: {2}, {2, 6}, {5, 6}. Hence, there are eight possible consistent reductions. But some of these consistent reductions are improper, e.g. those with the two coalitions {2}, {5, 6} as in any consistent reduction with these coalitions, the coalition {2, 5, 6} is also present. However, a much more serious problem is the occurrence of conflicting consistent reductions, as in this example. More precisely, a consistent reduction containing {2} but not {5, 6}, and another one with {5, 6} but not {2} are conflicting. If investor 2 can put his creatures in control of the three companies, he will be able to keep control indefinitely. If {5, 6} can, acting jointly, put their creatures in control, they can effectively shut investor 2 out. Hence, the result seems to hinge on which investor(s) manage to move first.

4.2. The Denti and Prati model

To the authors' knowledge, the first publication about the Denti and Prati model dates back to 1996 [7]. This model was developed and improved in [8, 9]. Denti and Prati, compared to the Gambarelli and Owen approach, extend the set of winning coalitions to all alliances able to achieve control of the "target" firm. Namely, such a model is not limited to coalitions of investors alone, but also considers coalitions formed by companies, and by companies and investors together. In [8], they proposed an algorithm to check whether a preset coalition of firms, in a corporate shareholding structure either with or without loops, is winning or not. Then, they extended their approach and assumed that shareholders can abstain or oppose others [9]. Therefore, not all the winning coalitions have the same relevance, i.e. controlling power. Hence, Denti and Prati [9] proposed three algorithms to determine the winning coalitions of various levels of relevance. More precisely, they proposed algorithms to calculate so-called: potentially winning coalitions, potentially stably winning coalitions and stably winning coalitions. For details, see [8, 9]. These algorithms have exponential computational complexity. Thus computational problems can arise if the number of firms is large.

Based on Denti and Prati's algorithm [8], a computer program was implemented, which enables to perform simulations [14]. More precisely, it is possible to find all the minimal winning coalitions which control a preset coalition of firms, or check whether a certain coalition is able to control a preset coalition of firms. So regarding Example 1, we find that there are two minimal winning coalitions ({1, 2, 3} and {1, 3, 7}) that control companies 4, 5, and 6. On the other hand, player 6 can be controlled directly or indirectly by five minimal winning coalitions: {5}, {1, 2, 3}, {1, 3, 7}, {2, 3, 4}, {3, 4, 7}. Of course, by considering only coalitions of investors, the results obtained by the Denti and Prati algorithm coincide with the results obtained by the Gambarelli and Owen approach.

Considering Example 2, player 2 and 5 jointly control companies 1, 3, and 4. This is a unique such minimal winning coalition. In addition, there are five minimal winning coalitions that control company 3: {1, 2}, {1, 5}, {2, 4}, {2, 5}, {4, 5}, and also five that have control over player 4: {1, 2}, {2, 3}, {2, 5}, {3, 6}, {1, 5, 6}. Thanks to the computer program, we can provide, of course, more simulations.

Denti and Prati, as mentioned before, focus only on the determination of winning coalitions in a corporate network, without considering power indices. But once the winning coalitions have been obtained, it is possible to measure the control power by power indices.

4.3. The Crama and Leruth approach

Crama and Leruth [4, 5] focus on the use of power indices to model control relationships in corporate structures. More precisely, they focus on an algorithmic approach to estimating the Banzhaf index (Section 2) in corporate networks, both with and without the presence of loops. It should be added that they also took into consideration in their algorithmic approach some difficult issues that exist in complex corporate structures, like modelling the set of small shareholders, etc. In large corporate networks, where there are a lot of firms involved, they proposed a Monte-Carlo approach to compute control power. They model a corporate structure by a network, i.e. direct graph. A precise graph-theoretic model is given in [4] and an intuitive description is provided in [5]. Therefore, here we only provide the necessary details.

Let V be the set of all firms involved in a corporate structure. Each $j \in V$ is associated with a 0–1 variable x_j. If $x_j = 1$, then firm j votes "yes" and $x_j = 0$ means that firm i votes "no". Let N be a set of n investors. Thus, in order to measure the control power of investor j in target firm t by power indices, they define indirect games v_j. However, it should be noted that in order to define an indirect game, they used an equivalent definition of a simple game. Namely, for the set of players N, they modelled a simple game v as a Boolean function $f_v : \{0, 1\}^n \to \{0, 1\}$, where the value of the function reflects the outcome of the vote for each vector of individual votes. More precisely, for all $X = (x_1, x_2, ..., x_n) \in \{0, 1\}^n$, if $v(\{i: x_i = 1\}) = 1$, then $f_v(X) = 1$, otherwise $f_v(X) = 0$. The direct game g_j for firm $j \in V \setminus N$ is the weighted majority game with the player-set composed of all direct shareholders of firm j. Now, for any acyclic network and any firm $j \in V$, the indirect game v_j is defined as the composition of the direct weighted majority games (g) associated with the direct shareholders of firm j. In particular, for all $X = (x_1, x_2, ..., x_n) \in \{0, 1\}^n$ and $j \in V$, the corresponding indirect game is defined recursively as follows: if $j \in N$, then $v_j(X) = x_j$, otherwise $v_j(X) = g_j(v_{i_1}(X), v_{i_2}(X), ..., v_{i_k}(X))$, where $i_1, i_2, ..., i_k$ denote the direct shareholders of firm $j \in V \setminus N$. For details see [4].

The Z index, proposed by Crama and Leruth [4] to measure the amount of a priori voting power held by a firm $j \in N$ in target company t, is given as follows:

$$Z_t(j) = \frac{1}{2^{n-1}} \left(\sum_{X \in \{0,1\}^n : x_j = 1} v_t(X) - \sum_{X \in \{0,1\}^n : x_j = 0} v_t(X) \right) \tag{1}$$

$Z_t(j)$ is simply the Banzhaf index of firm j in the indirect game v_t associated with firm t.

Let us apply the Crama and Leruth approach to Example 1 (the Porsche –Volkswagen case). In this example, there are 4 investors or sources in the terminology of the Crama and Leruth approach: players 1, 2, 3 and 7. Thus $N = \{1, 2, 3, 7\}$ and $n = 4$. The majority quota q equals 80%. There are $2^4 = 16$ different patterns (x_1, x_2, x_3, x_7) of possible votes of investors (Table 3). Calculating index Z, in this example, is rather simple, as in the games corresponding to companies 4 and 6, there are dictator players. For firm 4, player 1 is a dictator. Thus, the result of the vote in game v_4 depends only on player 1, so $v_4 = x_1$. For firm 5, we have three winning and vulnerable coalitions: {2, 3, 4}, {3, 4, 7}, and {2, 3, 4, 7}. Considering only direct control in company 5, we see that players 3 and 4 are indispensable to passing any decision. In game v_6, we also have a dictator, player 5, so $v_6 = v_5$. Thus, players 2 and 7 have the same control power, as these players are symmetric in game v_5 and so also in v_6.

Table 3. Calculation of v_i (i = 4, 5, 6)
based on the voting pattern

Voting pattern				Game		
x_1	x_2	x_3	x_7	v_4	v_5	v_6
0	0	0	0	0	0	0
1	0	0	0	1	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
1	1	0	0	1	0	0
1	0	1	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	0	0	0
0	1	0	1	0	0	0
0	0	1	1	0	0	0
1	1	1	0	1	1	1
1	1	0	1	1	0	0
1	0	1	1	1	1	1
0	1	1	1	0	0	0
1	1	1	1	1	1	1

Source: Authors' calculations.

Hence, based on the results in Table 3, we obtain: $Z_4(1)=1$, $Z_4(j)=0$, for j = 2, 3, 7, and $Z_i(1)=Z_i(3)=3/8$, $Z_i(2)=Z_i(7)=1/8$, for i = 5, 6. The same power indices were obtained for these investors based on calculating β according to the Gambarelli and Owen approach (Table 1).

For a shareholding structure with loops, the index (1) is not well defined and, for this reason, Crama and Leruth [4] proposed a heuristic approach to calculating the influence of a firm-investor in a company. More precisely, in the presence of cycles, they proposed an iterative procedure called MIX, in order to attempt to find a stable voting pattern and estimate the value of the game when the outcome of the game g_i $(j \in V \setminus N)$ is not perfectly defined. A voting pattern $X=(x_1, x_2, ..., x_n) \in \{0,1\}^n$ is stable if $x_j = g_j(X)$ for each $j \in V$. The concept of a stable pattern is closely related to the concept of a "consistent reduction" as introduced by Gambarelli and Owen [10].

Let us see how this method works in Example 2 (the Speiser and Baker case), where a loop of shareholding companies exists. Namely, let us calculate $Z_i(j)$, the influence of investor j (j = 2, 5, 6) on firm i (i = 1, 3, 4) in game $g_i(x_2, x_5, x_6)$, using the Banzhaf index. In this example, there are eight possible voting patterns for the combination of players 2, 5, and 6, see Table 6. We have to find $g_i(x_2, x_5, x_6)$ for each i = 1, 3, 4. For some voting patterns (x_2, x_5, x_6), the values of $g_i(x_2, x_5, x_6)$ for each i = 1, 3, 4 are perfectly defined, as in cases 1, 3, 4, 5, and 8 in Table 6. Let us consider case 5 when (x_2, x_5, x_6) = (1, 1, 0), then necessarily g_3 = 1 (2 and 5 form a winning coalition in g_3, since the combined voting rights of 2 and 5 in company 3 are greater than 50%). Also, g_4 = 1 and g_1 = 1. Hence, in this case, $g_i(x_2, x_5, x_6) = g_i(1, 1, 0)$ is perfectly determined for each i = 1, 3, 4 and this reasoning is valid independently of the initial votes of firms 1, 3, 4 and thus does not require the consideration of stable patterns. But in the other cases (2, 6, or 7 in Table 6), we cannot immediately deduce the vote of firms (1, 3, 4), and hence we must resort to the MIX procedure. The problem is that g_i is not uniquely defined when there is a cycle. So the underlying idea is to replace $g_i(x_2, x_5, x_6)$ by its expected value, assuming that all the firms whose votes are not entirely determined by (x_2, x_5, x_6) initially vote randomly and that they keep updating their votes until a stable pattern emerges. Since convergence to a stable pattern is not guaranteed, we estimate (by simulation) the expected value of $g_i(x_2, x_5, x_6)$ over all the values that it can take. More precisely, if, for example, (x_2, x_5, x_6) = (0, 1, 1), and the initial pattern of firms 1, 3, 4 is (x_1, x_3, x_4) = (1, 1, 0), then, using the MIX procedure, firms 1, 3, 4 will change their votes to (0, 1, 1), then to (1, 0, 1), then back to the initial state (1, 1, 0). As a stable pattern does not exist in this case, the value of $g_3(0, 1, 1)$ is taken to be 2/3, meaning that x_3 = 1 in 2/3 of the states of the cycle. However, we still have to take into account all the other possible initial votes for (x_1, x_3, x_4), and average over all these initial votes, which in this case gives 4/8 (Tables 4 and 5).

In Table 4, we present the results of the MIX procedure for cases when the votes of firms 2, 5, 6 are $(x_2, x_5, x_6) = (1, 0, 0)$, $(1, 0, 1)$, or $(0, 1, 1)$. The outcomes $g_i(x_2, x_5, x_6)$ obtained for any initial pattern (x_1, x_3, x_4) and $i = 1, 3, 4$ are shown in Table 5, and subsequently the average of $g_i(x_2, x_5, x_6)$ is calculated over all initial patterns for each i, $i = 1, 3, 4$. This average is defined to be the outcome of $g_i(x_2, x_5, x_6)$, $i = 1, 3, 4$, for a fixed voting pattern (x_2, x_5, x_6) (Table 6).

Table 4. The MIX procedure for $g_i(x_2, x_5, x_6)$, $i = 1, 2, 3$, $(x_2, x_5, x_6) = (1, 0, 0), (1, 0, 1), (0, 1, 1)$, and initial voting patterns: (0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1) for firms 1, 3, 4

<table>
<tr><th rowspan="2">Step k</th><th colspan="3">Voting pattern</th><th colspan="3">Voting pattern</th><th colspan="3">Voting pattern</th><th colspan="3">Voting pattern</th></tr>
<tr><th>x_1^k</th><th>x_3^k</th><th>x_4^k</th><th>x_1^k</th><th>x_3^k</th><th>x_4^k</th><th>x_1^k</th><th>x_3^k</th><th>x_4^k</th><th>x_1^k</th><th>x_3^k</th><th>x_4^k</th></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>1</td><td>1</td><td>0</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>2</td><td></td><td></td><td></td><td>0</td><td>0</td><td>1</td><td>1</td><td>0</td><td>1</td><td></td><td></td><td></td></tr>
<tr><td>3</td><td></td><td></td><td></td><td>1</td><td>0</td><td>0</td><td>1</td><td>1</td><td>0</td><td></td><td></td><td></td></tr>
<tr><td>$g_i(x_2, x_5, x_6)$</td><td>0</td><td>0</td><td>0</td><td>1/3</td><td>1/3</td><td>1/3</td><td>2/3</td><td>2/3</td><td>2/3</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td rowspan="2">Stable pattern (ST)</td><td rowspan="2" colspan="3">ST exists</td><td colspan="6">ST does not exist.
The same result occurs for initial patterns:</td><td rowspan="2" colspan="3">ST exists</td></tr>
<tr><td colspan="3">(0, 1, 0) and (0, 0, 1)</td><td colspan="3">(1, 0, 1) and (0, 1, 1)</td></tr>
</table>

Source: Authors' calculations.

Table 5. Calculation of the expected values of $g_i(x_2, x_5, x_6)$ for $i = 1, 3, 4$ and $(x_2, x_5, x_6) = (1, 0, 0), (1, 0, 1), (0, 1, 1)$

<table>
<tr><th colspan="3">Initial pattern</th><th colspan="3">Game</th><th rowspan="2">Summary of MIX procedure</th></tr>
<tr><th>x_1</th><th>x_3</th><th>x_4</th><th>$g_1(x_2, x_5, x_6)$</th><th>$g_3(x_2, x_5, x_6)$</th><th>$g_4(x_2, x_5, x_6)$</th></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>Stable pattern exists.</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>1/3</td><td>1/3</td><td>1/3</td><td rowspan="6">Stable pattern does not exist.</td></tr>
<tr><td>0</td><td>1</td><td>0</td><td>1/3</td><td>1/3</td><td>1/3</td></tr>
<tr><td>0</td><td>0</td><td>1</td><td>1/3</td><td>1/3</td><td>1/3</td></tr>
<tr><td>1</td><td>1</td><td>0</td><td>2/3</td><td>2/3</td><td>2/3</td></tr>
<tr><td>1</td><td>0</td><td>1</td><td>2/3</td><td>2/3</td><td>2/3</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>2/3</td><td>2/3</td><td>2/3</td></tr>
<tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>Stable pattern exists.</td></tr>
<tr><td colspan="3">Total</td><td>4</td><td>4</td><td>4</td><td></td></tr>
<tr><td colspan="3">Average</td><td>4/8</td><td>4/8</td><td>4/8</td><td></td></tr>
</table>

Source: Authors' calculations.

The values of the game $g_i(x_2, x_5, x_6)$, $i = 1, 3, 4$ for all possible voting patterns of players 2, 5, and 6 are summarized in Table 6. We can calculate the Banzhaf indices

($Z_i(j)$) of players $j = 2, 5, 6$ in game $i = 1, 3, 4$. Thus: $Z_i(2) = 5/8$, $Z_i(5) = 3/8$, $Z_i(6) = 1/8$, $i = 1, 3, 4$.

Table 6. Values of g_i ($i = 1, 3, 4$) for all possible voting patterns of players 2, 5, and 6

Case	Player vote			Game $g_i(x_2, x_5, x_6)$			Summary
	x_2	x_5	x_6	g_1	g_3	g_4	
1	0	0	0	0	0	0	$g_i(0, 0, 0) = 0$ is perfectly determined for all i.
2	1	0	0	1/2	1/2	1/2	We cannot immediately deduce how firms (1, 3, 4) vote, and must resort to the MIX procedure.
3	0	1	0	0	0	0	$g_i(0, 1, 0) = 0$ is perfectly determined for all i.
4	0	0	1	0	0	0	$g_i(0, 0, 1) = 0$ is perfectly determined for all i.
5	1	1	0	1	1	1	$g_i(1, 1, 0) = 1$ is perfectly determined for all i.
6	1	0	1	1/2	1/2	1/2	We cannot immediately deduce how firms (1, 3, 4) vote, and must resort to the MIX procedure.
7	0	1	1	1/2	1/2	1/2	We cannot immediately deduce how firms (1, 3, 4) vote, and must resort to the MIX procedure.
8	1	1	1	1	1	1	$g_i(1, 1, 1) = 1$ is perfectly determined for all i.

Source: Authors' calculations.

Firms 2 and 3 both have 45% of the stocks in company 3 but the influence of investor 2 (Speiser) in company 3, as measured by the Z index, is greater than the power of investor 5 (Baker). Player 6 has many more stock rights in company 4 than players 2 and 5 but the power of player 6 in this company is much lower than the power of the others.

In Example 2, player 6 (others) is an undefined set of small shareholders. As we mentioned at the beginning of this Section, Crama and Leruth [4] proposed an algorithmic method to take such a set into account. Thus the power of such a set could be calculated in a similar way (cf. also [5]).

4.4. The Karos and Peters approach

Karos and Peters [13] model relations of indirect control in a shareholding structure in two equivalent ways: by the so called invariant mutual control structure (a map which assigns the set of controlled players to each coalition), and by a simple game structure where each simple game indicates who controls the corresponding player. Hence, they propose a large class of indices, based on the concept of dividends that satisfy four axioms and can measure the power of players in a shareholding network. By adding one more axiom, called the controlled player condition, they obtain a uniquely defined power index Φ. Everything is rigorously defined in [13]. Here, we only provide the details necessary to estimate the power of players, using the index Φ, in the two examples.

A mutual control structure represents direct control in a shareholding structure. Formally, a mutual control structure C is a function assigning to each nonempty coalition $T \subseteq N$ $(T \neq \emptyset)$ another coalition $C(T) = S$ such that each player of S is controlled by the coalition T and the following monotonicity condition holds: if T controls S, then any coalition containing T also controls S. In order to capture the idea of indirect control in a shareholding structure, a mutual control structure should be invariant, i.e. satisfy the condition of indirect control, which states that for all coalitions R, S, T, if T controls S, and S and T jointly control R, then T indirectly controls R. An invariant mutual control structure is denoted by C^*.

Equivalently, a mutual control structure can be characterized by a simple game structure, i.e. a vector of simple games. More precisely, given a mutual control structure C for any player i, the simple game v_i^C is defined as follows: $v_i^C(S) = 1$ if $i \in C(S)$ and $v_i^C(S) = 0$ otherwise, i.e. the winning coalitions of v_i^C are exactly those that control player i.

Let $N = \{1, 2, ..., n\}$ be a set of players, and C^* be the set of all invariant mutual control structures based on N. The power index $\Phi : C \to R^n$, which satisfies the following five axioms: null player, constant sum, anonymity, transfer, and controlled player, is given by the following formula:

$$\Phi_i(C) = \sum_{k \in N} \sigma_i(v_k^C) - v_i^C(N) \qquad \text{for each } i \in N$$

where σ is nothing else than the Shapley–Shubik index. Note that the range of the index is $\Phi \geq -1$. The minimum value (–1) is obtained by the least powerful players, i.e. players who do not control any firm but are controlled by at least one coalition. Moreover, for all investors the value of Φ is non-negative. Finally, the sum of this index over all the players is equal to 0.

Let us calculate the power of the players in Example 1 (i.e. the Porsche–Volkswagen case) applying the index Φ. In this case, the mutual control structure C is defined as follows: for any coalition $S \subseteq N = \{1, ..., 7\}$, we have:

$$4 \in C(S) \Leftrightarrow 1 \in S, \quad 5 \in C(S) \Leftrightarrow \{2, 3, 4\} \subseteq S \;\text{ or }\; \{3, 4, 7\} \subseteq S, \text{ and } 6 \in C(S) \Leftrightarrow 5 \in S$$

Now, let us consider indirect control. Applying the updating procedure to C [13], we obtain the invariant mutual control structure C^* as follows: $4 \in C^*(S) \Leftrightarrow 1 \in S$,

$$5 \in C^*(S) \Leftrightarrow \{2, 3, 4\} \subseteq S \;\text{ or }\; \{1, 2, 3\} \subseteq S \;\text{ or }\; \{3, 4, 7\} \subseteq S \;\text{ or }\; \{1, 3, 7\} \subseteq S$$

$$6 \in C^*(S) \Leftrightarrow 5 \in S \text{ or } \{2, 3, 4\} \subseteq S \text{ or } \{1, 2, 3\} \subseteq S \text{ or } \{3, 4, 7\} \subseteq S \text{ or } \{1, 3, 7\} \subseteq S$$

Since players 1, 2, 3, and 7 are not controlled by any coalition, for i = 1, 2, 3, 7 and any coalition $S \subseteq N$ 7/60 we have $v_i^{C^*}(S) = 0$. For players i = 4, 5, 6, the simple games $v_i^{C^*}$ are defined by the sets of minimal winning coalitions W_i^m, where W_4^m = {{1}}, W_5^m = {{1, 2, 3}, {1, 3, 7}, {2, 3, 4}, {3, 4, 7}}, W_6^m = {{5}, {1, 2, 3}, {1, 3, 7}, {2, 3, 4}, {3, 4, 7}}. In Table 7, we present the values of the Shapley–Shubik power index (cf. Section 2) calculated for each player in the simple game $v_i^{C^*}$, i = 1, 2, ..., 7.

Table 7. The σ index calculated for each player and a simple game defined for Example 1

Simple game $v_i^{C^*}$	Player						
	1	2	3	4	5	6	7
i = 4	1	0	0	0	0	0	0
i = 5	7/60	7/60	32/60		0	0	7/60
i = 6	3/60	3/60	10/60	3/60	38/60	0	
i = 1, 2, 3, 7	0	0	0	0	0	0	0

Source: Authors' calculations.

Now, we calculate the index Φ taking into account the results from Table 7 and thus obtaining $\Phi_1(C^*) =$ (60 + 7 + 3)/60 = 70/60, $\Phi_2(C^*) = \Phi_7(C^*) = 1060$, $\Phi_3(C^*) = 42/60$, $\Phi_4(C^*) = 10/60 - 1 = -50/60$, $\Phi_5(C^*) = 38/60 - 1 = -22/60$, and $\Phi_6(C^*) = -1$.

Note that Karos and Peters excluded player 7 (others) in the Porsche–Volkswagen case. Others mean investors who hold less than 3% of the shares and are therefore not mentioned in any reports. Thus they assigned to others a power equal to zero. Here, it is different. Keeping player 7 results in a slight increase in the power of player 1 and, what is interesting, a significant difference in the powers of players 2 and 3. In [13], players 2 and 3 had equal power and here $\Phi_2(C^*) < \Phi_3(C^*)$ and the difference is significant.

Consider Example 2 (the Speiser–Baker case). Taking into consideration the Karos and Peters model, we can describe direct control by a mutual control structure C. For any coalition $S \subseteq N = \{1, ..., 6\}$, we have:

$$1 \in C(S) \Leftrightarrow 4 \in S, \quad 3 \in C(S) \Leftrightarrow \{1, 2\} \subseteq S \text{ or } \{1, 5\} \subseteq S \text{ or } \{2, 5\} \subseteq S$$

$$4 \in C(S) \Leftrightarrow \{2, 3\} \subseteq S \text{ or } \{3, 6\} \subseteq S \text{ or } \{2, 5, 6\} \subseteq S$$

Now, taking into account indirect relationships, we obtain the invariant mutual structure C*:

$$1 \in C^*(S) \Leftrightarrow 4 \in S \text{ or } \{2, 3\} \subseteq S \text{ or } \{3, 6\} \subseteq S$$
$$\text{or } \{2, 5, 6\} \subseteq S \text{ or } \{1, 2\} \subseteq S \text{ or } \{1, 5, 6\} \subseteq S$$

$$3 \in C^*(S) \Leftrightarrow \{1, 2\} \subseteq S \text{ or } \{1, 5\} \subseteq S \text{ or } \{2, 5\} \subseteq S \text{ or } \{2, 4\} \subseteq S$$
$$\text{or } \{4, 5\} \subseteq S \text{ or } \{2, 3\} \subseteq S \text{ or } \{3, 5\} \subseteq S,$$

$$4 \in C^*(S) \Leftrightarrow \{2, 3\} \subseteq S \text{ or } \{3, 6\} \subseteq S \text{ or } \{2, 5, 6\} \subseteq S$$
$$\text{or } \{1, 2\} \subseteq S \text{ or } \{1, 5, 6\} \subseteq S \text{ or } \{2, 4\} \subseteq S \text{ or } \{4, 5, 6\} \subseteq S$$

Given C*, for any $i \in N$ we define the following simple game: $v_i^{C*}(S) = 1$ if $i \in C^*(S)$ and $v_i^{C*}(S) = 0$ otherwise. Since players 2, 5, and 6 are not controlled by any coalition, for i = 2, 5, 6 and any coalition $S \subseteq N$ we have $v_i^{C^*}(S) = 0$. For players i = 1, 3, and 4, the simple games $v_i^{C^*}$ are defined by the following sets of minimal winning coalitions W_i^m : W_1^m = {{4}, {1, 2}, {2, 3}, {3, 6}, {2, 5, 6}, {1, 5, 6}}, W_3^m = {{1, 2}, {1, 5}, {2, 5}, {2, 4}, {4, 5}, {2, 3}, {3, 5}}, W_4^m = {{1, 2}, {2, 3}, {2, 4}, {3, 6}, {2, 5, 6}, {1, 5, 6}, {4, 5, 6}}. In Table 8, we present the values of the Shapley–Shubik power index calculated for each player in the simple games $v_i^{C^*}$, i = 1, ..., 6.

Table 8. The σ index calculated for each player and simple game defined for Example 2

Simple game $v_i^{C^*}$	Player					
	1	2	3	4	5	6
i = 1	5/60	10/60	8/60	28/60	2/60	7/60
i = 3	6/60	21/60	6/60	6/60	21/60	0
i = 4	5/60	22/60	10/60	5/60	14/60	4/60
i = 2, 5, 6	0	0	0	0	0	0

Source: Authors' calculations.

Now, we apply the index Φ to Example 2 with invariant mutual structure C*. Taking into account the results from Table 8, we obtain $\Phi_1(C^*) = -44/60$, $\Phi_2(C^*) = 53/60$, $\Phi_3(C^*) = -36/60$, $\Phi_4(C^*) = -21/60$, $\Phi_5(C^*) = 37/60$, $\Phi_6(C^*) = 11/60$.

Players 2 and 5 both have 45% of the stocks of firm 3. Player 5 has only a 1.5% lower share of stocks in company 4 than player 2, but in the whole shareholder structure, the difference in power, calculated according to the Karos and Peters index, seems to be greater.

4.5. The Mercik and Lobos approach

Mercik and Lobos in [15] proposed a measure of reciprocal ownership, called the index of implicit power, as a modification of the Johnston power index [12]. Also, they focused on application of this index to measure indirect power in cyclic shareholder structures. The implicit power index takes into account not only the power of the individual entities constituting the companies (investors), but also the impact of the companies themselves on implicit relationships.

Mercik and Lobos suggested a three-step algorithm to calculate the implicit power index. They assumed that there should be at least two companies, i.e. stock corporations, in the corporate structure. Here, we only give a brief sketch of this algorithm, for a full description see [15]. Namely, in step 1, the absolute value of the Johnston index is calculated for each company, taking into account only direct ownership. In step 2, for each shareholder–company, each value of the power index calculated in step 1 must be divided equally among all its shareholders. They call this first degree regression. In step 3, for each company, the absolute value of the implicit power index is calculated by summing up the appropriate values in the whole corporate network. For each investor, the absolute value of the implicit power index is calculated by summing up the appropriate values across the entire system of companies. Then, these absolute values are appropriately standardized to obtain the implicit power index of each shareholder.

Table 9. Fractional critical defections and the value of the raw Johnston index

<table>
<tr><td rowspan="3">Vulnerable coalition</td><td colspan="7">Investors in company</td></tr>
<tr><td colspan="2">4</td><td colspan="4">5</td><td>6</td></tr>
<tr><td>1</td><td>2</td><td>2</td><td>3</td><td>4</td><td>7</td><td>5</td></tr>
<tr><td>{1}</td><td>1</td><td>0</td><td></td><td></td><td></td><td></td><td rowspan="7">1</td></tr>
<tr><td>{1, 2}</td><td>1</td><td>0</td><td></td><td></td><td></td><td></td></tr>
<tr><td>{5}</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>{2, 3, 4}</td><td></td><td></td><td>1/3</td><td>1/3</td><td>1/3</td><td>0</td></tr>
<tr><td>{3, 4, 7}</td><td></td><td></td><td>0</td><td>1/3</td><td>1/3</td><td>1/3</td></tr>
<tr><td>{2, 3, 4, 7}</td><td></td><td></td><td>0</td><td>1/2</td><td>1/2</td><td>0</td></tr>
<tr><td>Raw Johnston index</td><td>2</td><td>0</td><td>1/3</td><td>7/6</td><td>7/6</td><td>1/3</td></tr>
</table>

Source: Authors' calculations.

Let us calculate the implicit power index for Example 1 (the Porsche–Volkswagen case) following the three-step algorithm mentioned above. Table 9 illustrates the necessary calculations to realize step 1. More precisely, for any shareholder of a company, the raw Johnston index is calculated as the sum of the fractional critical defections over all the vulnerable coalitions in which the given shareholder is critical.

The last row of Table 9 gives the distributions of absolute power in companies 4, 5, and 6, taking into account only direct control by shareholders. But indirect control is also considered in the calculation of the implicit power index. Thus, in step 2, the power of investors is augmented by a fraction according to indirect control. For example, company 4 is a direct shareholder of company 5 with absolute power 7/6. Since investors 1 and 2 directly control company 4 and, consequently, indirectly control company 5, according to this approach, 7/6 units of power are shared equally between investors 1 and 2. Hence, in consequence, the indirect absolute power of investor 1 in company 5 is equal to 7/12, and the power of investor 2 in the same company is 11/12 . The results of all the necessary calculations to complete steps 2 and 3 are provided in Table 10. The Mercik and Lobos approach allows us to measure not only the influence of investors on companies but also the absolute and standardized power of all the firms involved in corporate network.

Table 10. Absolute and standardized values of the implicit power index in Example 1

Company	Investors (members of companies)				Implicit index of a company	
	1	2	3	7	Absolute	Standardized
4	2	0	0		2	$\frac{24}{69} \approx 0.348$
5	$\frac{7}{12}$	$\frac{11}{12}$	$\frac{7}{6}$	$\frac{1}{3}$	3	$\frac{36}{69} \approx 0.522$
6	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4} = 0.75$	$\frac{9}{69} \approx 0.130$
Absolute implicit index of an investor	$\frac{31}{12} \approx 2.583$	$\frac{14}{12} \approx 1.667$	$\frac{17}{12} \approx 1.417$	$\frac{71}{12} \approx 0.583$	$\frac{69}{12}$	
Standardized implicit index of an investor	$\frac{31}{69} \approx 0.449$	$\frac{14}{69} \approx 0.203$	$\frac{17}{69} \approx 0.246$	$\frac{7}{69} \approx 0.101$		1

Source: Authors' calculations.

Mercik and Lobos assessed the implicit power of investors and companies in the Speiser and Baker case (Example 2). Here, we only recall, in Table 11, the values of this index. For detailed calculations see [15].

Table 11. Values of the implicit power index in Example 2

Power index	Player (company)			Player (investor)		
	1	3	4	2	5	6
Standardized implicit index	0.088	0.324	0.588	0.451	0.333	0.216
Absolute implicit index	0.75	2.75	5	3.833	2.833	1.833

Source: [15].

5. Conclusions

In the context of game theory, various studies on indirect control have been made in corporate shareholding systems. In this paper, we have critically examined the models of Gambarelli and Owen [10], Denti and Prati [8, 9], Crama and Leruth [4, 5], Karos and Peters [13], and Mercik and Lobos [15], taking into account two examples of shareholding structure, one with cycles and one without. The reason that we chose these and not other models is that we wanted to compare the Gambarelli and Owen approach (one of the oldest pioneering methods) with those most recently presented in the literature. It should be noted that Crama and Leruth [4–6] presented the broadest approach to measuring indirect power/control in corporate networks. In particular, they took into account not only the presence of cyclic shareholding relationships but also the collection of small, unidentified shareholders called the float. Then, they also considered an aspect of computing indirect power in real-world financial networks. Also, their algorithmic approach allows us to efficiently deal with the complexity of computing power indices in corporate networks, regardless of their size.

Reviewing the literature on the topic, it can be said that most methods use two popular indices (Shapley–Shubik and Banzhaf) or their modifications to measure indirect control. An exception is the Mercik and Lobos [15] approach, which uses the Johnston index. A different approach was proposed by Karos and Peters [13], who do not start with a particular proposition of an index, but from axioms that determine a large class of indices. The Denti and Prati approach only considers the determination of winning coalitions.

There are some similarities between the considered approaches. The concept of a stable pattern in the method of Crama and Leruth is closely related to the concept of a consistent reduction in the method of Gambarelli and Owen. Also, the procedure of making a mutual control structure invariant, as defined by Karos and Peters, shows some resemblance to a reduction operation. As we saw in Example 2, a consistent reduction based on the Gambarelli–Owen approach is not necessarily unique (in contrast to the minimal invariant extensions of Karos and Peters), but a consistent reduction (a vector of voting games over a set of investors) based on the Gambarelli–Owen approach, as well as the simple games in a simple game structure according to the Karos and Peters approach, can be improper. According to the Crama–Leruth and Mercik–Lobos approaches, simple games are considered proper. By the way, simple games in a simple game structure based on the Karos–Peters model are called command games [11].

There can be many differences between these approaches, and they might even result from using different power indices; but above all, however, from the proposed game-theoretical structures describing indirect control. Tables 12 and 13 summarize the measures of power in Examples 1 and 2, respectively.

Table 12. Power of players in Example 1

Power index	Player (company)			Player (investor)			
	4	5	6	1	2	3	7
Shapley–Shubik	–	–	–	0.417	0.083	0.417	0.083
Absolute Banzhaf and Crama–Leruth				0.375	0.125	0.375	0.125
Johnston				0.389	0.111	0.389	0.111
Karos–Peters (Φ)	–0.833	–0.367	–1.000	1.167	0.167	0.700	0.167
Standardized implicit	0.348	0.522	0.130	0.449	0.203	0.246	0.101

Table 13. Power of players in Example 2

Power index	Player (company)			Player (investor)		
	1	3	4	2	5	6
Karos–Peters (Φ)	–0.733	–0.600	–0.350	0.883	0.617	0.183
Standardized implicit	0.088	0.324	0.588	0.451	0.333	0.216
Crama–Leruth (Z)	–	–	–	0.625	0.375	0.125

However, it is difficult to compare the results obtained in these two examples, and there may well be many reasons (even if the indices proposed have different ranges, for example). The Karos–Peters and Mercik–Lobos approaches take into account all of the firms involved in a corporate system in the calculation of a power index, and other methods (Gambarelli–Owen and Crama–Leruth) only consider investors. Subsequently, the Gambarelli–Owen and Crama–Leruth methods calculate the power of investors in a target company, while the Karos–Peters and Mercik–Lobos approaches consider the entire system. We hope that considering the calculation of indices of indirect control using different models in one paper is of value in itself. Of course, in our examples, we noticed that all of these models rank the investors (in terms of control power over companies) or companies (in terms of power in the whole system) in the same way. We suspect that these indices always rank the players in the same way but this is an open problem. Moreover, based on the methods of Gambarelli–Owen, Denti–Prati, and Crama–Leruth, players 1 and 3 are symmetric, as players 2 and 7 are. However, taking into account the whole system and the approach of Karos and Peters, as well as the Mercik–Lobos approach, we see that player 1 is more powerful than player 3, which intuitively seems to be correct. Regarding players 2 and 7, the index Φ confirms the previous statement regarding the symmetry of these players, while the implicit index gives more power to player 2 (which also seems to be the most-expected result).

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