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AN APPROACH FROM BANKRUPTCY RULES APPLIED TO THE APPORTIONMENT PROBLEM IN PROPORTIONAL ELECTORAL SYSTEMS

(Discrete) bankruptcy problems associated with apportionment problems have been defined. The authors studied which allocations for apportionment problems have been obtained when (discrete) bankruptcy rules were applied to the associated bankruptcy problems. They have shown that the (discrete) constrained equal losses rule coincides with the greatest remainder method for apportionment problems. Furthermore, new properties related to governability have been proposed for apportionment methods. Finally, several apportionment methods satisfying governability properties have been applied to the case of the Spanish Elections in 2015.

Keywords: *apportionment rules, bankruptcy rules, standard of comparisons, proportional electoral systems, governability*

1. Introduction

Many situations of social and economic interest fall under the umbrella of allocation issues [9], including apportionment problems and bankruptcy problems, on which one can find an extensive literature. The apportionment problem consists of determining how to divide a given (nonnegative) integer number among a set of individuals according to their respective sizes. In electoral systems with proportional representation, the apportionment problem arises in two situations: (i) in the allotment of seats to constitu-

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encies, if any, and (ii) in the allocation of seats to the political parties within each constituency. There are many apportionment methods, but none satisfies all desirable basic properties (see [5, 22]). Nevertheless, electoral systems do not only involve the distribution of seats, but also other questions must be taken into account. Horowitz considered six aims for analyzing or designing an electoral system [20], but not all of them are mutually compatible. Two of them are proportionality of seats to votes and durable governments. These are not compatible and, hence, a balance between them is necessary.

The bankruptcy problem consists of determining how to divide a given estate (perfectly divisible) among a set of creditors according their claims on the estate. There are also many bankruptcy rules satisfying various natural and reasonable properties. Bankruptcy rules are characterized using different subsets of these properties [34, 35]. In the literature, we can also find bankruptcy problems with indivisibilities, so-called discrete bankruptcy problems (see [28, 29, 17–19, 23, 24, 10, 11] among others). Therefore, they could be approached in a similar way to apportionment problems. In this paper, we do the converse, we approach apportionment problems from the point of view of (discrete) bankruptcy problems. As far as we know, this approach is new. With this purpose in mind, for each apportionment problem we define a bankruptcy problem associated with it; we can thereby apply bankruptcy rules and obtain allotments for the corresponding apportionment problem. We show that the discrete constrained equal losses rule (DCEL) provides the same allocation as the greatest remainder method. Similarly, the M-up method for bankruptcy rules, associated with a proper standard of comparison will again correspond to the greatest remainder method. With this aim, we introduce several properties related to governability and the corresponding up-methods are defined. In addition, we also compare the performance of these up-methods with the greatest remainder method and the d'Hondt method for the case study of the Spanish elections in 2015.

The rest of the paper is organized as follows. Section 2 contains basic concepts on apportionment problems and bankruptcy problems. In Section 3, we introduce bankruptcy problems associated with apportionment problems. We show that the discrete constrained equal losses rule coincides with the greatest remainder method. We also introduce properties related to governability and define new apportionment rules based on bankruptcy rules. In Section 4 we apply the apportionment rules defined in Section 3 to the case study of the Spanish elections in 2015. Section 5 concludes.

2. Preliminaries. Apportionment problems and bankruptcy problems

In this section, some notation and basic concepts related to apportionment problems and bankruptcy problems will be introduced.

Let $\Im$ be the set of all nonnegative integer numbers {0, 1, 2, 3, ...}. Given $N = \{1, 2, 3, ..., n\}$, V^N is the set of all nonnegative integer n-vectors $v = (v_1, v_2, ..., v_n)$. Let $\Re_+^N$ be the set of all nonnegative n-vectors $d = (d_1, d_2, ..., d_n)$.

2.1. Apportionment problems

Apportionment problems concern allocating available resources in integral parts to a number of individuals according to a set of nonnegative integers, each representing the size of one individual. An example in politics is the allocation of seats in a legislature among political parties or constituencies according to the number of votes or inhabitants, respectively. The main goal in an apportionment problem is to find an allocation as proportional as possible to the set of nonnegative integers associated to the claimants. The statement of the problem is very simple, but its solution is not easy and there are several different methods for solving it (for details on apportionment problems, see [5] and [22]).

An apportionment problem is defined by a 3-tuple (N, v, h), where N is a finite set of individuals (for example, political parties), $v \in V^N$ is the vector of their sizes (for example, the number of votes) and $h \in \Im$ is the amount to be distributed (for example, the number of seats).

An apportionment method A is a function from $V^N \times \Im$ to V^N such that

$$A((v_1, v_2, ..., v_n); h) \subset V^N$$

and for each $(a_1, a_2, ..., a_n) \in A((v_1, v_2, ..., v_n); h)$, $\sum_{i=1}^{n} a_i = h.$

Therefore, when the apportionment method does not provide a unique allocation vector, another criterion must be considered to select just one allocation, for example, some kind of priority.

Given a set of individuals N, a vector $v = (v_1, v_2, ..., v_n) \in V^N$ and $h \in \Im$, three allotments can be defined for each individual $i \in N$:

- Exact allotment: $q_i = \dfrac{v_i}{\sum_{j=1}^{n} v_j} h$.
- Lower allotment: $\lfloor q_i \rfloor$ is the integer part of q_i.
- Upper allotment: $\lceil q_i \rceil = \lfloor q_i \rfloor + 1$, if q_i is not an integer number and $\lceil q_i \rceil = \lfloor q_i \rfloor$, if q_i is an integer number.

In general, two different types of apportionment methods can be distinguished: quotient/quota methods and divisor methods, but other methods can be found in the literature, for example, minimax methods [13, 14].

Using quotient methods, a fixed quota C is chosen for distributing h and the size of each electorate is divided by C. Hence, the number of units allocated to each electorate is given by $[v_i/C]$, and when $\sum_{i=1}^{n} \lfloor v_i/C \rfloor$ does not sum to h, then some criterion is used to adapt the allocation to h. This quota C measures the minimum size required to have a right to one unit. Depending on the selected quota and the tie breaking rule, different quotient methods are obtained.

The method of greatest remainders (GR) (also known as Hamilton's method) is the quotient method using the so-called natural quota[3] or Hare quota[4] given by $H = \sum_{j=1}^{n} v_j/h$ and when $\sum_{i=1}^{n} \lfloor q_i \rfloor$ sums to less than h, the priority or deservingness criterion used to assign one extra unit to adapt the allocation to h is the greatest remainder $q_i - \lfloor q_i \rfloor$.

Using divisor methods, the h units are allocated in h steps one by one, in each step the size of each electorate is divided by the divisor $d(x)$, which is a function of the number of units x allocated to that electorate in the previous steps, and the current unit is allocated to the individual with the highest quotient. In the initial step, all individuals have previously received 0 units. Depending on the divisor function used, different divisor methods are obtained. Some well-known examples are the following:

- D'Hondt method (also known as the greatest divisors method or Jefferson's method): $d(x) = x+1,\ x = 0,\ 1, 2, ...$
- Sainte–Laguë method (also known as the major fractions method or Webster's method): $d(x) = 2x+1,\ x = 0,\ 1,\ 2, ...$
- The geometric mean method (also known as the equal proportions method or the Hill–Huntington method): $d(x) = \sqrt{(x+1)(x+2)},\ x = 0,\ 1,\ 2, ...$

There are several properties that may be desirable for an apportionment method. In the paper by Balinski and Ramirez [2] scale-invariance, exactness and anonymity are considered as the three most fundamental properties:

- Scale invariance: for every $\lambda > 0$, $A((\lambda v_1, \lambda v_2, ..., \lambda v_n); h) = A((v_1, v_2, ..., v_n); h)$.

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[3]Note that the exact allotment is given by $q_i \equiv v_i/H$.

[4]Another common quota used in quotient methods is the Droop quota given by $D = \left\lfloor 1 + \sum_{j=1}^{n} v_j/(h+1) \right\rfloor$.

- Exactness: If $\sum_{j=1}^{n} v_j = h,$ then $A((v_1, v_2, ..., v_n); h) = \{(v_1, v_2, ..., v_n)\}$.
- Anonymity: If π is a permutation of N, then $A((v_{\pi(1)}, v_{\pi(2)}, ..., v_{\pi(n)}); h) = \pi(A((v_1, v_2, ..., v_n); h))$.

These three properties are satisfied by all divisor methods and by the method of greatest remainders. However, there are more properties which may be desirable for apportionment problems:

- Balancedness: if $a \in A((v_1, v_2, ..., v_n); h),$ then $v_i = v_j$ implies $|a_i - a_j| \le 1.$
- Monotonicity: If $a \in A((v_1, v_2, ..., v_n); h)$, then there exists $a' \in A((v_1, v_2, ..., v_n); h+1)$ such that $a' \ge a.$
- Responsiveness: If $a \in A((v_1, v_2, ..., v_n); h)$, then $v_i > v_j$ implies $a_i \ge a_j$.
- Lower allotment: $\forall a \in A((v_1, v_2, ..., v_n); h),\ a_i \ge \lfloor q_i \rfloor,\ \forall i \in N.$
- Upper allotment: $\forall a \in A((v_1, v_2, ..., v_n); h),\ a_i \le \lceil q_i \rceil,\ \forall i \in N.$

The greatest remainder method does not satisfy monotonicity (the Alabama paradox) and the divisor methods do not satisfy, in general, the allotment properties. Using these properties and other properties, the different apportionment methods can be characterized. For example, divisor methods are characterized by Balinski and Young [4, 5].

2.2. Bankruptcy problems

Bankruptcy problems concern allocating available resources (estate) to a number of individuals (claimants) according to their demands/claims when all these demands add up to more than the available resources. In a classical bankruptcy situation, the available resources are perfectly divisible, but over the last years the case of indivisible resources has also been studied (see, for example [28, 29, 17–19, 23, 24, 16, 10, 11, 6]). The main goal in a bankruptcy problem is to find an allocation which is as fair as possible, taking into account the demands of the claimants. Many solutions have been proposed depending on the principle(s) of fairness used (see, for a survey on bankruptcy problems, [34, 35]).

A bankruptcy problem is defined by a 3-tuple (N, d, E), where N is the finite set of claimants, $d \in \Re_+^N$ is the vector of their demands/claims and $E \in \Re_+$ is the amount of available resources (perfectly divisible), such that $\sum_{i=1}^{n} d_i \ge E,$ i.e., there are not enough resources to fully satisfy the demands of the claimants.

A bankruptcy rule B is a function from $\Re_+^N \times \Re_+$ to $\Re_+^N$ such that

$$B((d_1, d_2, ..., d_n); E) \in \Re_+^N$$

$$\sum_{i=1}^{n} d_i \geq E \text{ and } \sum_{i=1}^{n} B_i((d_1, d_2, ..., d_n); E) = E$$

There are many different bankruptcy rules, but in this paper we only consider three of them: the proportional rule, the constrained equal awards (CEA) rule and the constrained equal losses (CEL) rule.

- The proportional rule (P): $P((d_1, d_2, ..., d_n); E) = \frac{E}{\sum_{i=1}^{n} d_i} (d_1, d_2, ..., d_n)$.
- The constrained equal awards rule (CEA): CEA$((d_1, d_2, ..., d_n); E) = (\min\{d_1, \lambda\}, \min\{d_2, \lambda\}, ..., \min\{d_n, \lambda\})$, where $\lambda > 0$ is chosen so that $\sum_{i=1}^{n} \min\{d_i, \lambda\} = E$.
- The constrained equal losses rule (CEL): CEL$((d_1, d_2, ..., d_n); E) = (\max\{0, d_1 - \lambda\}, \max\{0, d_2 - \lambda\}, ..., \max\{0, d_n - \lambda\})$, where $\lambda > 0$ is chosen so that $\sum_{i=1}^{n} \max\{d_i - \lambda\} = E$.

There are several properties that are very natural for a bankruptcy rule and some of them are completely analogous to those for apportionment rules.

- Homogeneity: for every $\lambda > 0$,
 $B((\lambda d_1, \lambda d_2, ..., \lambda d_n); \lambda E) = \lambda B((d_1, d_2, ..., d_n); E)$.
- Efficiency: $\sum_{j=1}^{n} B_j((d_1, d_2, ..., d_n); E) = E$.
- Anonymity: If π is a permutation of N, then $B((d_{\pi(1)}, d_{\pi(2)}, ..., d_{\pi(n)}); E) = \pi(B((d_1, d_2, ..., d_n); E))$.
- Equal treatment of equals: If $d_i = d_j$, then $B_i(d; E) = B_j(d; E)$.
- Monotonicity: If $\sum_{i=1}^{n} d_j \geq E' > E$, then $B((d_1, d_2, ..., d_n); E') \geq B((d_1, d_2, ..., d_n); E')$.
- Order-preservation: If $d_i > d_j$, then $B_i(d; E) \geq B_j(d; E)$ and $d_i - B_i(d; E) \geq d_j - B_j(d; E)$.
- Respect of minimal rights: $B_i(d; E) \geq \max\{E - \sum_{j:j\neq i} d_j, 0\}$, for all i.
- Boundedness of claims: $B_i(d; E) \leq d_i$, for all i.

An excellent survey on axiomatic analysis of bankruptcy rules is given by Thomson [34, 35].

3. The apportionment problem as a bankruptcy problem

As was pointed out in Section 2.2, over the last years bankruptcy problems with indivisibilities have also been studied. Thus, if we consider bankruptcy situations in which the demands are nonnegative integer numbers and the estate consists of a nonnegative integer number of indivisible units, then the definitions of the discrete bankruptcy problem and discrete bankruptcy rule are similar to the classical ones:

A discrete bankruptcy problem is defined by a 3-tuple (N, d, E), where N is the finite set of claimants, $d \in V^N$ is the vector of their demands/claims and $E \in \Im$ is the amount of available resources, such that $\sum_{i=1}^{n} d_i \geq E$.

A discrete bankruptcy rule B is a function from $V^N \times \Im$ to V^N given by

$$B((d_1, d_2, ..., d_n); E) \subset V^N$$

such that for each $(b_1, b_2, ..., b_n) \in B((d_1, d_2, ..., d_n); E)$, $\sum_{i=1}^{n} b_i = E$.

We should highlight that the only difference between the definition of the bankruptcy problem and the definition of the discrete bankruptcy problem is that in the former the demands and the estate are non-negative real numbers and in the latter they are non-negative integers. Considering the rules, in addition to the difference between using real or integer numbers, bankruptcy rules are single valued, while discrete bankruptcy rules are set valued. Therefore, when a discrete bankruptcy rule does not provide a unique allocation vector, another criterion must be considered to select a single allocation, for example, some kind of priority or deservingness.

Note that both the apportionment problem and the discrete bankruptcy problem are very similar, the only difference is the condition $\sum_{i=1}^{n} d_i \geq E$. Bankruptcy problems with indivisibilities are usually sorted out using methods of assigning priority. The simplest method of assigning priority is to consider a priority order defined over the set of claimants to break possible ties for the very last units to be allocated. Another alternative is to use the idea of a standard of comparison [36]. A standard of comparison is a strict binary relation ρ defined over all agent–claim pairs such that $(i, x+1)\rho(i, x)$ for all $x \in \Im$, where $(i, x)\rho(j, y)$ means that agent i with claim x units has priority over agent j with claim y units. A standard of comparison is called monotonic if $(i, x+1)\rho(j, x)$ for all claimants i, j, and for all $x \in \Im$ [19]. Likewise, Herrero and Martínez [19] illustrate how to define a monotonic standard of comparison which coincides with the d'Hondt method used in apportionment problems. In general, any method of apportionment could be used as a discrete bankruptcy rule with a suitable standard of comparison.

Next we do the converse, that is, we will use discrete bankruptcy rules for solving apportionment problems. To do this, first we must define the discrete bankruptcy problem associated with a given apportionment problem.

Definition 1. Given an apportionment problem (N, v, h), we define an associated discrete bankruptcy problem (N, d, E) as follows:

- N is the set of claimants,
- $d_i = \lfloor q_i \rfloor + 1$, for all $i \in N$,
- $E = h$.

It is obvious that the problem (N, d, E) is a bankruptcy problem because $\sum_{r \in N} d_i \geq E$. Now we can apply any discrete bankruptcy rule to this problem and obtain an allocation for the apportionment problem (N, v, h). One of these rules is the discrete constrained equal losses rule (DCEL), which is applied in two steps. Firstly, the CEL rule is applied to the problem as if it was a bankruptcy problem without indivisibilities and each claimant is assigned the corresponding integer part. In the second step, an extra unit is allocated to each claimant whose allocation according to CEL is not integer, following a priority order. For this discrete rule, Giménez-Gómez and Vilella [16] prove that the DCEL rule is the recursive discrete P-rights rule[5] with P = {WOP}, where WOP is the property of weak order preservation, which is given by:

- If $d_i > d_j$, then $B_i(d; E) \geq B_j(d; E)$ and $d_i - B_i(d; E) \geq d_j - B_j(d; E)$.
- If $d_i = d_j$, then $\left| B_i(d; E) - B_j(d; E) \right| \leq 1$.

Note that condition (i) is related to the property of responsiveness and (ii) is related to the property of being balanced for apportionment problems.

Theorem 2. Let (N, v, h) be an apportionment problem and (N, d, E) the associated discrete bankruptcy problem. Let σ be the priority rule defined over the set N as follows:

$$i \sigma j \Leftrightarrow q_i - \lfloor q_i \rfloor > q_j - \lfloor q_j \rfloor$$

Thus DCEL(d; E) = GR(v; h), provided that the tie breaking rule used is the same for both methods.

Proof. We will first prove that $\lfloor CEL_i(d; E) \rfloor = \lfloor q_i \rfloor$ for all $i \in N$. Indeed, we have the following:

[5]See [15] for details on recursive P-rights processes.

$$\sum_{i\in N} d_i - E = \sum_{i\in N} \lfloor q_i \rfloor + n - h \le \sum_{i\in N} q_i + n - h = n \tag{1}$$

where $N = \{1, 2, ..., n\}$. This implies that

$$e = \frac{\sum_{i\in N} d_i - E}{n} \le 1$$

Hence, taking into account that $d_i \ge 1$ for all $i \in N$, we can subtract e from each d_i for all $i \in N$, and all these differences are nonnegative. Thus, we have

$$\sum_{i\in N} \max\{0, d_i - e\} = \sum_{i\in N} (d_i - e) = E$$

Therefore, by definition of the CEL rule and the structure of the claims, we obtain that $\mathrm{CEL}_i(d; E) = \lfloor q_i \rfloor + 1 - e$, and this implies that $\lfloor \mathrm{CEL}_i(d; E) \rfloor = \lfloor q_i \rfloor$. This completes the first step of the DCEL rule.

For the second step, we know that $\sum_{i\in N} d_i - E \le n$, therefore each claimant will receive at most one extra unit. Applying the priority rule σ, the extra units will go to the claimants with the greatest remainders $q_i - \lfloor q_i \rfloor$. Thus $\mathrm{DCEL}_i(d; E) = \lfloor q_i \rfloor + 1$ for the claimants with highest priorities (greatest remainders) and $\mathrm{DCEL}_i(d; E) = \lfloor q_i \rfloor$ for the claimants with lowest priorities.

Note that if $\lfloor q_i \rfloor$ is an integer, then agent i will not receive an extra unit. Indeed, let us consider the set $I = \{i \in N : q_i \in \Im\} \subset N$ and $|I|$ its cardinality, then we have

$$E - \sum_{i\in N} \lfloor q_i \rfloor = \sum_{i\in N} q_i - \sum_{i\in N} \lfloor q_i \rfloor = \sum_{i\notin I} (q_i - \lfloor q_i \rfloor) \le n - |I|$$

Thus, the number of extra units to be allocated in the second step is less than or equal to $n - |I|$, therefore the agents in I will not receive any extra unit, because they are the agents with the lowest priorities according to σ.

If we apply the GR method to the apportionment problem, then we obtain the same results, therefore DCEL(d; E) = GR(v; h).

Finally, in the case of a tie we could use the same tiebreak rule in both the DCEL rule and the GR method and the result holds. If no tiebreak rule is applied, then we can conclude that the allocation sets defined by the DCEL rule and the GR method coincide. This completes the proof.

In the following example, we show that the DCEA rule does not coincide, in general, with the GR method for apportionment problems.

Example 3. Let us consider the apportionment problem (N, v, h), where N = {1, 2, 3, 4, 5}, v = (1000, 500, 300, 150, 50) and h = 8. In this case, the exact allotment vector is q = (4, 2, 1.2, 0.6, 0.2) and GR(v; h) = (4, 2, 1, 1, 0). The associated bankruptcy problem is given by N = {1, 2, 3, 4, 5}, d = (5, 3, 2, 1, 1) and E = 8. CEA(d; E) = (2, 2, 2, 1, 1) = DCEA(d; E). However, CEL(d; E) = (4.2, 2.2, 1.2, 0.2, 0.2) and DCEL(d; E) = (4 + 0, 2 + 0, 1 + 0, 0 + 1, 0 + 0) = (4, 2, 1, 1, 0) = GR(v; h), as expected.

Remark 4. In Example 3, if we take $d_i = \lceil q_i \rceil$, then d = (4, 2, 2, 1, 1). This implies CEL(d; E) = (3.6, 1.6, 1.6, 0.6, 0.6) and DCEL(d; E) = (3 + 0, 1 + 0, 1 + 1, 0 + 1, 0 + 1) = (3, 1, 2, 1, 1) $\neq$ GR(v, h). Therefore, we have to consider a new priority rule, for example, $i \sigma j \Leftrightarrow q_i - \lfloor \mathrm{CEL}_i(d;\, E) \rfloor > q_j - \lfloor \mathrm{CEL}_j(d;\, E) \rfloor$, and then DCEL($d$; E) = (3 + 1, 1 + 1, 1 + 0, 0 + 1, 0 + 0) = (4, 2, 1, 1, 0) = GR(v, h).

M-up methods and M-down methods are other discrete bankruptcy rules [19]. These methods are applied unit by unit according to a monotonic standard of comparison. In each step, one unit is allocated (M-up methods) or subtracted (M-down methods) to the agent corresponding to the agent-claim pair with the highest priority and this agent-claim pair is removed for the next step. Herrero and Martínez [19] show that any M-down method can be interpreted as a discrete version of the constrained equal awards rule. This rule makes it advantageous for parties to split and this is not a desirable property for apportionment problems in electoral systems. In fact, rules which make it advantageous for parties to merge are preferred, because there is an incentive to form coalitions before and not after the election. For this reason, in this paper we will focus on M-up methods (see [8] for details on manipulations involving coalitions in bankruptcy problems).

Given an apportionment problem (N, v, h) and the associated discrete bankruptcy problem (N, d, E), we consider the following monotonic standard of comparison ρ^* defined over all agent-claim pairs:

- $(i,\, x)\rho^*(j,\, y) \Leftrightarrow x > y$, for all $i, j \in N$ and $x, y \in \Im$.
- $(i,\, x)\rho^*(j,\, x) \Leftrightarrow q_i - \lfloor q_i \rfloor > q_j - \lfloor q_j \rfloor$, for all $i, j \in N$ such that $i \neq j$, and $x \in \Im$.

Theorem 5. Let (N, v, h) be an apportionment problem and (N, d, E) the associated discrete bankruptcy problem. Let U^{ρ^*} denote the solution of the bankruptcy problem according to the M-up method with the monotonic standard of comparison ρ^*. Then, $U^{\rho^*}(d;\, E)$ = GR(v; h), provided that the tie breaking rule is the same for both methods.

Proof. For each $i \in N = \{1, 2, ..., n\}$, there are d_i pairs $(i, 1), (i, 2), ..., (i, d_i)$. By Eq. (1), we know that after $\sum_{i \in N} \lfloor q_i \rfloor$ steps the U^{ρ^*} method has allocated exactly $\lfloor q_i \rfloor$ units to agent i, for all $i \in N$. At this point, the remaining pairs are $(1, 1), (2, 1), ..., (n, 1)$ and the remaining units are allocated to the agents with the highest priority according to criterion (ii) in the definition of ρ^* above. These agents are those with the greatest remainders, so $U^{\rho^*}(d; E) = \mathrm{GR}(v; h)$.

Finally, in the case of a tie we could use the same tiebreak rule for both the U^{ρ^*} method and the GR method and the result holds. If no tiebreak rule is applied, then the sets of allocations defined by the U^{ρ^*} method and the GR method coincide. This completes the proof.

Given an apportionment problem (N, v, h) and the associated discrete bankruptcy problem (N, d, E), we can consider the following monotonic standard of comparison ρ^g defined over all agent-claim pairs:

- $(i, x)\rho^g(j, y) \Leftrightarrow x > y$, for all $i, j \in N$ and $x, y \in \Im$.
- $(i, x)\rho^g(j, x) \Leftrightarrow v_i > v_j$, for all $i, j \in N$ such that $i \neq j$, and $x \in \Im$.

Condition (i) of the monotonic standard of comparison ρ^g is the same as for ρ^*, but condition (ii) favors agents with the largest sizes (number of votes) instead of agents with the greatest remainders. Therefore, the apportionment obtained by applying U^{ρ^g} does not satisfy the upper allotment property, in general, but it exceeds the upper allotment by at most one unit. Nevertheless, in electoral systems, this shortcoming could be seen as an interesting quality in terms of governability because it could make the formation of durable government coalitions easier. For example, in [20] six goals for an electoral system are established, one of them is durable governments. Related to this goal, we introduce the following property for apportionment methods:

- Governability. If $a \in A((v_1, v_2, ..., v_n); h)$, then $v_i > v_j$ implies $a_i - \lfloor q_i \rfloor \geq a_j - \lfloor q_j \rfloor$.

Example 6. Let us consider again the apportionment problem (N, v, h), where $N = \{1, 2, 3, 4, 5\}$, $v = (1000, 500, 300, 150, 50)$ and $h = 8$. The associated discrete bankruptcy problem is given by $N = \{1, 2, 3, 4, 5\}$, $d = (5, 3, 2, 1, 1)$ and $E = 8$. To calculate $\rho^g(d; E)$, we consider the chain of the first 8 pairs with highest priorities:

$$(1, 5)\rho^g(1, 4)\rho^g(1, 3)\rho^g(2, 3)\rho^g(1, 2)\rho^g(2, 2)\rho^g(3, 2)\rho^g(1, 1).$$

Therefore, $U^{\rho^g}(d; E) = (5, 2, 1, 0, 0)$.

It is easy to check that the apportionment obtained by applying U^{ρ^g} satisfies the property of governability, but the GR method does not. A stronger version of governability can be stated in the following terms

• Strong governability. If $a \in A((v_1, v_2, ..., v_n); h)$ and $v_i > v_j$, then

$$\begin{cases} a_i - \lfloor q_i \rfloor > a_j - \lfloor q_j \rfloor & \text{if } a_j - \lfloor q_j \rfloor > 0 \\ a_i - \lfloor q_i \rfloor \geq a_j - \lfloor q_j \rfloor & \text{if } a_j - \lfloor q_j \rfloor = 0 \end{cases}$$

This property means that no agent with fewer votes can obtain a better or equal extra allocation than another with more votes, with respect to their lower bounds.

The extreme case which favors the most popular party while respecting the lower bounds is the following:

• All for the winner If $v_i = \max\{v_1, v_2, ..., v_n\}$, there exists $a \in A(v; h)$ such that

$$a_i = \lfloor q_i \rfloor + \left(h - \sum_{j \in N} \lfloor q_j \rfloor \right)$$

In all cases, the lower bound is respected, if we consider a bankruptcy situation in which the vector of lower bounds plays the role of a reference point. Additionally, each agent has an associated upper bound, which represents the maximum number of units that can be allocated to her, so that the upper bounds add up to more than the available resource units. This situation resembles bankruptcy problems with references [31, 32]. Therefore, we could also define a bankruptcy problem with references associated with an apportionment problem, including a priority rule or standard of comparison for allocating all the units exceeding the sum of the reference point.

To finish this section, we consider the definition of the bankruptcy problem associated with the apportionment problem given in Remark 4, i.e., $d_i = \lceil q_i \rceil$. Given an apportionment problem (N, v, h) and the associated discrete bankruptcy problem (N, d, E) with $d_i = \lceil q_i \rceil$ for all $i \in N$, we consider the following (non-monotonic) standard of comparison ρ^{BY} defined over all agent-claim pairs:

$$(i, x)\rho^{BY}(j, y) \Leftrightarrow \frac{v_i}{x} > \frac{v_j}{y} \text{ for all } i, j \in N \text{ and } x, y \in \Im$$

The $U^{\rho^{BY}}$ method is very similar (or equivalent) to the quota method for apportionment problems [3]. It is a variant of the d'Hondt method, but respecting the upper bound.

In fact, if we set $d_i = h$ for all $i \in N$ in the associated discrete bankruptcy problem, then we obtain the d'Hondt method.

4. Case study. Spanish elections in 2015

The discussion on the need to reform an electoral system recurrently arises in democratic countries after elections, because there are different groups which consider the final result does not correspond with the reality of the society. In the case of Spain, several proposals for reforming the Spanish electoral system can be found in the literature (see, for example, [26]). Two common complaints have traditionally been made (i) the non-proportional distribution of seats in a parliament regarding the number of votes obtained by each party and (ii) the number of voters represented by a seat. However, perfect proportionality is not possible and there are several methods for measuring the proportionality or disproportionality of an electoral system (see, for example, [21, 12]). In particular, in electoral systems aimed at proportional representation, there are two main variants, national lists and constituency lists, as well as combinations of both (see, for an example of bi-apportionment by Maier et al. [25]). It is usually claimed that the method of apportionment used to allocate first the seats among the constituencies and then, within each constituency, the seats among the parties is responsible for the resulting disproportionality (for example, in papers by Sánchez-Soriano [23] and Curiel [7], the Spanish Electoral System and the Electoral System of Surinam, respectively, have been analyzed from a mathematical point of view). However, this is not the only controversial aspect of an electoral system. For example, in [20], Mercik analyzes the veto power of the President of Poland. In fact, the goals of an electoral system are more than just proportionality. In the paper by Horowitz [20], six goals for an electoral system have been established: proportionality of seats to votes, accountability to constituents, durable governments, victory of the Condorcet winner, interethnic and interreligious conciliation, and office holding by minorities.

In this section, we analyze the distribution of seats in the Spanish Congress after the Spanish elections in 2015. The Congress consists of 350 members. To elect the members of the Congress, the country is divided into 52 constituencies, 50 provinces and 2 autonomous cities. The seats are allocated to the constituencies in proportion to their populations using the greatest remainder method. Because the populations of the constituencies are quite different, the constituencies are quite asymmetric in terms of the number of seats. For each constituency, the political parties present ordered lists of their candidates for election and the seats are allocated to political parties using the d'Hondt method. We will use the results of the Spanish elections in 2015 to check the effect of using several of the methods of apportionment introduced in Section 3 with regard to the property of governability. The goal is to analyze how to avoid directly giving seats to the winning party in elections, which favors governability, as is done in the Italian and Greek electoral systems, among others.

When we use quotient methods, we must make a decision on how to address the minimum thresholds required to obtain a seat. In particular, what should be done when some parties do not obtain enough votes to reach the threshold, because, depending on the decision, the quotas of the other political parties can vary. In our analysis, we assume that there is no minimum threshold and we will use the following methods of apportionment: d'Hondt, the greatest remainders (GR), the Up method with a monotonic standard of comparison satisfying governability (U^{ρ^g}), the Up method with a standard of comparison satisfying strong governability $(U^{\rho^{sg}})$ and the Up method with a standard of comparison satisfying "all for the winner" (U^{ρ^w}).

Table 1. Distribution of seats in the Spanish Congress for five apportionment rules
52 constituencies (50 provinces and 2 autonomous cities)

Political party	Exact allotment	Present (d'Hondt)	GR	U^{ρ^g}	$U^{\rho^{sg}}$	U^{ρ^w}
PP	101.22	123	103	125	162	181
PSOE	77.56	90	86	95	78	66
Podemos	44.74	42	49	44	32	30
Ciudadanos	49.16	40	52	28	21	21
En Comú Podem	13.01	12	11	13	14	16
Compromís–Podemos	9.42	9	8	10	9	7
ERC	8.42	9	8	10	9	6
DIL	7.93	8	7	8	11	13
En Marea	5.74	6	6	7	5	2
PNV	4.23	6	4	6	6	5
IU	12.96	2	9	1	1	1
Bildu	3.06	2	3	3	2	2
CC	1.14	1	2	–		
UDC	0.91	–	1	–	–	–
MÉS	0.47	–	1	–	–	–
Total seats	–	350	350	350	350	350

In Table 1, we observe that the effect of the all-for-the-winner method (W) is similar to that of elections in uninominal constituencies. When the constituencies have few seats (less than 5) to be elected, all (or almost all) of the seats go to the winner and there are several parties with more than 15% of the votes. Consequently, this method of apportionment reaches the desired effect of facilitating the formation of a government. The winner obtains an absolute majority of the seats, but proportionality is drastically reduced. Furthermore, we observe that the method of apportionment satisfying governability performs similarly to the d'Hondt method, only Ciudadanos is clearly harmed in favor of the other parties. Therefore, in this scenario, the method of apportionment satisfying governability does not get the desired effect of facilitating governance, but the method of apportionment satisfying strong governability does. Of course, three relatively popular parties, PSOE, Podemos and Ciudadanos, are clearly harmed in favor of the winning party, PP, but a coalition between PP and Ciudadanos would be sufficient to form a government. Therefore, this method of apportionment would improve the possibilities of forming a stable government. As it is well-known, the greatest remainder method produces a greater dispersion in the distribution of seats among political parties. Finally, as for the number of political parties represented in the Congress, all of these methods except for the GR method give almost the same range of ideologies.

Table 2. Distribution of seats in the Spanish Congress for five apportionment rules
19 constituencies (17 regions and 2 autonomous cities)

Political party	Exact allotment	Present (d'Hondt)	GR	U^{ρ^g}	$U^{\rho^{sg}}$	U^{ρ^w}
PP	101.22	114	104	113	127	136
PSOE	77.56	82	79	87	81	77
Podemos	44.74	49	48	47	44	41
Ciudadanos	49.16	46	49	44	40	40
En Comú Podem	13.01	13	12	12	14	15
Compromís–Podemos	9.42	8	8	9	8	8
ERC	8.42	8	8	8	8	7
DIL	7.93	7	7	8	7	7
En Marea	5.74	6	6	6	6	5
PNV	4.23	5	4	5	5	4
IU	12.96	7	15	6	6	6
Bildu	3.06	3	4	3	2	2
CC	1.14	1	1	1	1	1
Pacma	3.08	–	2	–	–	–
Nós–Candidatura Galega	0.99	1	1	1	1	1
UDC	0.91	–	1	–	–	–
MÉS	0.47	–	1	–	–	–
Total seats	–	350	350	350	350	350

In Table 2, the number of seats in each constituency (region) has been calculated as the sum of the number of seats in the provinces belonging to it. For d'Hondt, the apportionment satisfying governability and the greatest remainder methods, we can derive the distribution of seats are similar to those in Table 1. In this scenario, with fewer constituencies, we observe that the effect of the properties of strong governability and all for the winner declines significantly with respect to the scenario with more constituencies. Nonetheless, the winning party still receives a more than proportional share of the seats, without affecting too much the proportionality of the distribution of seats in the Congress. However, this bonus (13 or 23 seats, respectively) would have a positive effect on the possibilities of forming a government, because a coalition between PP and Ciudadanos would have a total of 167 seats (close to the 176 seats required for an absolute majority) using the Up-method with the strong governability property and 176 seats (an absolute majority) using the all for the winner property. Regarding the number of political parties represented in the Congress, all of the methods except for the GR method give the same range of ideologies.

In Table 3, we observe that all of the analyzed methods of apportionment work similarly, the only difference concerns the number of political parties represented in the Congress.

Table 3. Distribution of seats in the Spanish Congress for five apportionment rules based on the national vote with no constituencies

Political party	Exact allotment	d'Hondt	GR	U^{ρ^g}	$U^{\rho^{sg}}$	U^{ρ^w}
PP	101.22	104	101	102	106	113
PSOE	77.56	80	78	78	81	77
Podemos	44.74	46	45	45	45	44
Ciudadanos	49.16	50	49	50	51	49
En Comú Podem	13.01	13	13	14	13	13
Compromís-Podemos	9.42	9	9	10	9	9
ERC	8.42	8	8	9	8	8
DIL	7.93	8	8	8	7	7
En Marea	5.74	5	6	6	5	5
PNV	4.23	4	4	5	4	4
IU	12.96	13	13	13	12	12
Bildu	3.06	3	3	4	3	3
CC	1.14	1	1	1	1	1
Pacma	3.08	3	3	3	3	3
UPyD	2.17	2	2	2	2	2
Nós-Candidatura Galega	0.99	1	1	–	–	–
UDC	0.91	–	1	–	–	–
Vox	0.81	–	1	–	–	–
Recortes 0–Grupo Verde	0.68	–	1	–	–	–
MÉS	0.47	–	1	–	–	–
PCPE	0.44	–	1	–	–	–
Geroa Bai	0.43	–	1	–	–	–
Total seats	–	350	350	350	350	350

In view of these results, we can conclude that the effect of the governability properties decreases when the size of constituencies increases (or, equivalently, the number of constituencies decreases).

5. Conclusions

We have analyzed apportionment problems from the perspective of (discrete) bankruptcy problems. As far as we know, this approach is new. For this purpose, we have defined bankruptcy problems associated with apportionment problems. One important decision in the definition of the bankruptcy problem is how to determine the appropriate claims of the agents. We have defined the claim of each agent as her lower allotment plus one, but other criteria can be followed, for example based on upper allotments. Another alternative is to directly use the exact allotments of the agents involved in the apportionment problem as claims in the associated bankruptcy problem. In this case, bankruptcy problems with an integer estate and non-integer claims are obtained [10, 11]. But even bankruptcy problems with references could be associated with apportionment problems where the references are the lower allotments and the utopian claims can range from the upper allotments to the total number of units to be distributed [31, 32].

We have also shown that the discrete constrained equal losses rule applied to the associated discrete bankruptcy problem coincides with the greatest remainder method applied to the corresponding apportionment problem. But we have also used various standards of comparisons and the associated Up-methods to define new apportionment rules with new properties related to governability. These properties try to catch the idea that political parties with more votes should benefit in the allotment of seats, in order to facilitate, somehow, the formation of governments. Therefore, the link between apportionment problems and bankruptcy problems could be used to analyze both and further research can be done in this direction.

Finally, we have applied Up-methods with different properties of governability to the case of the Spanish election in 2015 and compared these methods with the greatest remainder and d'Hondt methods. We have observed that the effect of the governability properties decreases when the size of constituencies increases (or, equivalently, the number of constituencies decreases).

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