OPERATIONS RESEARCH AND DECISIONS

No. 2 2016

DOI: 10.5277/ord160207

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POWER ON DIGRAPHS

It is assumed that relations between n players are represented by a directed graph or digraph. Such a digraph is called invariant if there is a link (arc) between any two players between whom there is also a directed path. We characterize a class of power indices for invariant digraphs based on four axioms: Null player, Constant sum, Anonymity, and the Transfer property. This class is determined by $2n - 2$ parameters. By considering additional conditions about the effect of adding a directed link between two players, we single out three different, one-parameter families of power indices, reflecting several well-known indices from the literature: the Copeland score, β- and apex type indices.

Keywords: *digraph, power index, transfer property, link addition*

1. Introduction

We consider situations where the relations between n players are reflected by a directed graph or digraph. There are several interpretations possible. A directed link (or arc) from player i to player j may reflect that player i controls player j, for instance i is an investor who has the majority of the shares in firm j. With this interpretation, our paper is related to Gambarelli and Owen [7], where the players are firms or investment companies. It is also a special case of the approach by Hu and Shapley [10, 11] and of the mutual control structures of Karos and Peters [12]. Somewhat related, our model

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can also be seen as a permission structure as introduced by Gilles et al. [9]. A directed link from player i to player j then reflects the fact that player j needs the permission of player i in order to cooperate with other players. Still another interpretation of such a graph is as an information structure: player j can get information from player i only if there is a directed link from player i to player j. These are applications where the links are between decision making agents. Other types of applications may have arcs representing the results of matches played among teams in a sports competition, or the structure of links between internet pages.

A digraph will be called invariant if for any two players who are connected via a directed path, there is also a direct link. For instance, in the control interpretation, if player i controls player j and j controls k, then player j controls k indirectly; in an invariant digraph there will be a direct link between i and k. The term invariance refers to the fact that such indirect control relations are already included – alternatively, such digraphs are often called transitive.

Our aim in this paper is to study power indices, reflecting the power of players as a consequence of their positions in the digraph. We impose four axioms on a power index: Null player, Constant sum, Anonymity, and the Transfer property. The Transfer property (first introduced in a different format by Dubey [3]) is an additivity condition in the spirit of [14]. It is the main tool to ensure that a power index is completely determined by its values on digraphs of an elementary form. In such elementary digraphs, there is a set M of players who each have a link to one and the same player j. Due to the Anonymity condition, only the cardinality of M matters, but it makes a difference whether or not player j is in M: this is why we arrive at a class of power indices with $2(n-1)$ degrees of freedom (parameters)[4]. The Null player axiom ensures that players outside M (other than possibly j) have zero power. Combined with the Constant sum axiom, we obtain that the sum of the players' powers is always 0. This is in contrast with classical power indices (e.g., [15], see also [6, 8, 1]), where power is between 0 and 1, and sums to 1. In our approach, it is natural to allow negative power. A null player is a player who has neither incoming nor outgoing links in the digraph, and it is natural to assign power 0 to such a player. In turn then, it is equally natural to allow negative power for a player who has only incoming links.

An invariant digraph is a special case of an invariant mutual control structure studied by Karos and Peters [12]. Our main result here (Theorem 4.4) is related to the main result in [12], but the axioms imposed in the present paper are weaker, since they apply to power indices on a considerably smaller class of mutual control structures. Therefore, compared to that result, Theorem 4.4 requires a new and different, though less complicated, proof.

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[4]More precisely, by the Null player and Constant sum axioms the case where $M=\{j\}$ is determined, so that by Anonymity we have $n-1$ cases with $j \in M$, and $n-1$ cases with $j \notin M$.

The Constant sum axiom and the possibility of negative power are two features by which the indices in this paper distinguish themselves from digraph power measures such as the outdegree and β-measure, axiomatized by van den Brink and Gilles [18], which both satisfy a normalization that implies that the sum of the powers of the players depends on the digraph. We find three subclasses, related to the Copeland score, β-measure and apex-measure, by imposing three different additional conditions with respect to link addition[5]. First, we impose the condition that adding a link from player i to player j does not change the power of the players who already had a link to player j. This may make sense, for instance, in a situation where player j has to do tasks for players who have a link to j – assuming there is no capacity constraint on what player j can do. Imposing this condition (called Link addition 1) on top of the four basic conditions above singles out a one-parameter family of power indices closely related to the Copeland score [2] from social choice theory: the power of a player is proportional to the number of players to whom he has a link minus the number of players who have a link to him.

The next condition requires that adding a link does not change the power of player j to whom this link is incoming. This implies that the players that already had a link to j now have to share the power they had with the newcomer. Also, this condition (Link addition 2) singles out a one-parameter family of power indices, this time closely related to the concept of a β-measure, as in [17, 18].

The final condition that we consider (Link addition 3) requires that adding a link from player i to player j equally increases or reduces the power of i and j. Again, we obtain a one-parameter family of power indices, with the property that incoming links have no effect on a player's power. These power indices are similar to the apex-type power index in [16].

The organization of the paper is as follows. Section 2 introduces invariant digraphs, Section 3 the main axioms for a power index, Section 4 the main characterization result, and Section 5 the refinements based on Link addition axioms. Section 6 concludes.

2. Preliminaries

For a set A we denote by $P(A)$ the set of all subsets of A, and by $P_0(A)$ the set of all nonempty subsets of A. By $|A|$ we denote the number of elements of A.

[5]The effect of link addition was introduced in communication graph games by Myerson [13], who introduced the axiom of fairness, stating that deleting an undirected communication link between two players has the same effect on their payoffs. Together with so-called component efficiency this characterizes a Shapley type solution, later referred to as the Myerson value.

Let $N=\{1,...,n\}$ with $n \geq 2$ denote the set of players. Elements of $P(N)$ are called coalitions. A directed graph or digraph on N is a map $C: P(N) \to P(N)$ satisfying $C(\varnothing)=\varnothing$ and $C(S)=\cup_{i\in S} C(i)$ for every $S \in P_0(N)$. Hence, a digraph C is completely determined by its values $C(i)$ for $i \in N$. The graphical interpretation of a digraph C is, indeed, that there is a link from $i \in N$ to $j \in N$ if and only if $j \in C(i)$[6]. As mentioned in the Introduction, various interpretations with respect to applications are possible. The set $C(i)$ can be interpreted as the set of players controlled by player i (cf. [10–12]), or the set of players who need permission from i [9]; or the set of players to whom player i can communicate, etc. The reason for defining a digraph not just for singletons lies in the axiomatic approach to power indices presented later on. Observe that a digraph C is trivially monotonic: if $S, T \in P(N)$ and $S \subseteq T$ then $C(S) \subseteq C(T)$.

For a digraph C, a directed path from $i \in N$ to $j \in N$ is a sequence $(i_1, i_2), (i_2, i_3), ..., (i_k, i_{k+1}) \in N \times N$ for some $k \in \mathbb{N}$ such that $i_{\ell+1} \in C(i_\ell)$ for each $\ell = 1, ..., k$, $i = i_1$, and $j = i_{k+1}$. A digraph C is invariant if for all $i, j \in N$, if there is a directed path from i to j, then $j \in C(i)$[7]. The expression “invariant” refers to the fact that in an invariant digraph adding a link between players having a directed path connecting them, does not change the digraph.

Note that an arbitrary digraph can be made invariant by simply adding links for every pair of players between whom there exists a directed path. The following observation, the easy proof of which is omitted, will be useful in the sequel.

Lemma 2.1. Let C be a digraph. Then C is invariant if and only if $C(i \cup C(i)) = C(i)$ for all $i \in N$.

We denote the set of all digraphs (on N) by $\mathcal{D}$ and the set of all invariant digraphs by $\mathcal{D}^*$.

In the sequel, the following observations will be relevant. For $C, D \in \mathcal{D}$ define $C \cup D$ and $C \cap D$ by $(C \cup D)(S) = C(S) \cup D(S)$ and $(C \cap D)(S) = C(S) \cap D(S)$ for all $S \in P(N)$. Then $C \cup D \in \mathcal{D}$, but even if $C, D \in \mathcal{D}^*$ then it does not necessarily follow that $C \cup D \in \mathcal{D}^*$. See the following example.

Example 2.2. Let $N = \{1, 2, 3\}$ and let $C, D \in \mathcal{D}^*$ be defined by $C(1) = 2$, $D(2) = 3$, and $C(i) = \varnothing$ and $D(i) = \varnothing$ in the remaining cases[8]. Then $(C \cup D)(1) = 2$

[6]Usually, what we call players are called nodes or vertices in a digraph. Because of the applications we have in mind, we refer to them as players.

[7]In the literature, these digraphs are usually referred to as transitive digraphs.

[8]Here and elsewhere we often omit set braces if confusion is unlikely.

and $(C \cup D)(2) = 3$ but $3 \notin (C \cup D)(1)$, so that $C \cup D \notin \mathcal{D}^*$. (See also Remark 2.9 in [12].)

Further, if $C, D \in \mathcal{D}^*$, then it does not necessarily follow that $C \cap D \in \mathcal{D}$, but if $C \cap D \in \mathcal{D}$, then also $C \cap D \in \mathcal{D}^*$. The latter is straightforward to prove by using Lemma 2.1; for the former see the following example.

Example 2.3. Let $n \geq 4$ and let $C(1) = 3$, $C(2) = 4$, $D(1) = 4$, $D(2) = 3$, and $C(i) = D(i) = \emptyset$ for all $i \neq 1, 2$. Then $C, D \in \mathcal{D}^*$, but $C \cap D \notin \mathcal{D}$, since $(C \cap D)(i) = \emptyset$ for all *i*, but $(C \cap D)(12) = 34$.

Example 2.3 in fact shows that our definition of the intersection $C \cap D$ of two digraphs cannot be restricted to taking intersections for singleton coalitions only and then extending to arbitrary coalitions by taking unions.

3. Axioms for a power index

We consider power indices for invariant digraphs. A power index is a map $\phi : \mathcal{D}^* \to R^N$. We propose four basic axioms, in the spirit of the standard axioms of the Shapley [14] value for cooperative games. We will then prove that these four axioms characterize a large, 2(*n* – 1)-parameter family of power indices.

Player $i \in N$ is *a null player* in $C \in \mathcal{D}^*$ if $C(i) = \emptyset$ and $i \notin C(N)$: i.e. player i is an isolated node in the digraph. The digraph in which every player is a null player, is denoted by *O*, i.e., $O(S) = \emptyset$ for all $S \subseteq N$. This corresponds to a graph without any links.

The first axiom requires that null players have zero power.

Null Player (NP). $\phi_i(C) = 0$ for every null player *i* in *C*, for every $C \in \mathcal{D}^*$.

The second axiom states that the sum of the powers of all the players is the same for any digraph.

Constant sum (CS). $\sum_{i \in N} \phi_i(C) = \sum_{i \in N} \phi_i(D)$ for all $C, D \in \mathcal{D}^*$.

The combination of these two axioms has as a simple consequence that the powers of all the players in a digraph always add up to zero.

Lemma 3.1. Let ϕ be a power index satisfying NP and CS. Then $\sum_{i \in N} \phi_i(C) = 0$ for every $C \in \mathcal{D}^*$.

Proof. By NP, $\phi_i(O) = 0$ for every $i \in N$. Hence, by CS, $\sum_{i\in N}\phi_i(C) = \sum_{i\in N}\phi_i(O) = 0$ for every $C \in \mathcal{D}^*$. □

Usually in the literature, power indices take values between 0 and 1 and sum up to 1 (e.g., [15]). In our case, we allow the possibility that power can be negative. Typically, for instance, player j with $C(j) = \varnothing$, but $j \in C(i)$ for some other player i, seems to have less power than a null player, so that we may wish to assign negative power to such a player. A null player in a simple game does not add any power to any coalition. However, in a simple game players cannot be subordinate to other players, so that a null player is then a player with minimal power. In our model, this is different, and there can be players who have less power than a "null player". The expression null player suits its definition above well, whether in its quantitative meaning of "zero", or in its qualitative meaning of "without value or consequence, amounting to nothing".

Let $\pi : N \to N$ be a permutation. Then for $C \in \mathcal{D}^*$ we define $\pi C \in \mathcal{D}^*$ by

$$(\pi C)(S) = \pi(C(\pi^{-1}(S)))$$

The next axiom is standard[9].

Anonymity (AN). $\phi_{\pi(i)}(\pi C) = \phi_i(C)$ for every player $i \in N$, every permutation π of N, and every $C \in \mathcal{D}^*$.

The final axiom replaces the usual additivity condition known from the Shapley value. For cooperative games, it was first introduced by Dubey [3] in the format (1) presented in the next section. See, further, [4, 5].

Transfer property (TP). $\phi(C) - \phi(C') = \phi(D) - \phi(D')$ for all $C, C', D, D' \in \mathcal{D}^*$ such that $C' \subseteq C$, $D' \subseteq D$, and $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ for every $S \subseteq N$.

An alternative and stronger version of the Transfer property would be obtained by imposing the condition $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ only for singletons, i.e., replacing it by $C(i) \setminus C'(i) = D(i) \setminus D'(i)$ for every $i \in N$. This, however, is too strong for our purposes: it would have the same effect as adding the Link addition 1 condition, see Section 5, and it would exclude other power indices, for instance those characterized in that section.

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[9]Nevertheless, it may be interesting to investigate the consequences of dropping this condition and assuming that there may be asymmetries between the players apart from those resulting from the digraphs.

4. Characterization of power indices satisfying NP, CS, AN, and TP

The main result of this section is Theorem 4.4, which is a characterization of all power indices satisfying Null player, Constant sum, Anonymity, and the Transfer property. We start out by proving a consequence of the Transfer property.

Lemma 4.1. Let ϕ be a power index satisfying TP. Then

$$\phi(C \cap D) + \phi(C \cup D) = \phi(C) + \phi(D) \tag{1}$$

for all $C, D \in \mathcal{D}^*$ with $C \cup D, C \cap D \in \mathcal{D}^*$.

Proof. Let $C, D \in \mathcal{D}^*$ with $C \cup D, C \cap D \in \mathcal{D}^*$. Clearly,

$$(C(S) \cup D(S)) \setminus C(S) = D(S) \setminus (C(S) \cap D(S))$$

for all $S \subseteq N$. Hence by TP, $\phi(C \cup D) - \phi(C) = \phi(D) - \phi(C \cap D)$, implying (1). □

For $C \in \mathcal{D}^*$ and $j \in N$ we define the digraph C_j by

$$C_j(i) = \begin{cases} \{j\} & \text{if } j \in C(i) \\ \varnothing & \text{otherwise} \end{cases}$$

It is easy to see that $C_j \in \mathcal{D}^*$. Observe that in a graphical representation of C_j, the graph has only links pointing to *j*, namely from those players *i* with $j \in C(i)$.

Lemma 4.2. Let ϕ be a power index satisfying TP and NP, and let $C \in \mathcal{D}^*$. Then $\phi(C) = \sum_{j \in N} \phi(C_j)$.

Proof. We first show that $\cup_{k \in T} C_k \in \mathcal{D}^*$ for every $T \in P_0(N)$. Let $S, T \in P_0(N)$, then

$$\bigcup_{k \in T} C_k(S) = \bigcup_{k \in T} \bigcup_{j \in S} C_k(j) = \bigcup_{j \in S} \bigcup_{k \in T} C_k(j)$$

which implies that $\cup_{k \in T} C_k \in \mathcal{D}$. Let $i \in N$ and $j \in \cup_{k \in T} C_k(i \cup (\cup_{h \in T} C_h(i)))$, then there exists an $\ell \in T$ such that $j \in C_\ell(i \cup (\cup_{h \in T} C_h(i)))$. In turn, since $C_\ell \in \mathcal{D}$, this implies that $j \in C_\ell(i)$ or there exists an $m \in T$ such that $j \in C_m(i)$. Hence, $j \in \cup_{k\in T} C_k(i)$, so $\cup_{k\in T} C_k(i \cup (\cup_{h\in T} C_h(i))) \subseteq \cup_{k\in T} C_k(i)$. The converse inclusion follows by monotonicity. Hence $\cup_{k\in T} C_k \in D^*$ by Lemma 2.1.

Next, for every $T \in P_0(N)$ and $k \notin T$, we have $C_k \cap (\cup_{\ell\in T} C_\ell) = O \in D^*$ (the empty digraph). By NP, $\phi(O) = 0 \in R^n$. Also, $C = \cup_{i\in N} C_i$.

By the preceding arguments and by repeatedly applying (1), we obtain

$$\phi(C) = \phi(C_1 \cup (\cup_{i=2,\dots,n} C_i)) = \phi(C_1) + \phi(\cup_{i=2,\dots,n} C_i) - \phi(C_1 \cap (\cup_{i=2,\dots,n} C_i))$$
$$= \phi(C_1) + \phi(\cup_{i=2,\dots,n} C_i) = \sum_{j\in N} \phi(C_j),$$

which concludes the proof of the lemma. □

By Lemma 4.2 we may concentrate on digraphs of the form C_j. More generally, for $M \in P_0(N)$ and $j \in N$ the digraph $U_{M,j}$ is defined by

$$U_{M,j}(i) = \begin{cases} \{j\} & \text{if } i \in M \\ \varnothing & \text{otherwise} \end{cases} \tag{2}$$

Clearly, $U_{M,j}$ is invariant.

Lemma 4.3. Let the power index φ on D^* satisfy NP, CS, and AN. Then there exist $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_n \in R$, with $\alpha_0 = 0, \beta_1 = 0$ such that for every $M \in P(N)$ and $j \in N$, with $m = |M|$:

(a) if $j \notin M$, then for every $i \in N$

$$\phi_i(U_{M,j}) = \begin{cases} 0 & \text{if } i \notin M \cup j \\ \alpha_m / m & \text{if } i \in M \\ -\alpha_m & \text{if } i = j \end{cases}$$

(b) if $j \in M$, then for every $i \in N$

$$\phi_i(U_{M,j}) = \begin{cases} 0 & \text{if } \notin M \\ \beta_m / m & \text{if } i \in M \backslash j \\ \beta_m / m - \beta_m & \text{if } i = j \end{cases}$$

Proof. Straightforward from the axioms. □

Observe that for $C \in \mathcal{D}^*$ and $j \in N$, we have $C_j = U_{M_j^C, j}$ where $M_j^C = \{i \in N : j \in C(i)\}$

From Lemmas 4.2 and 4.3 we obtain our main result.

Theorem 4.4. A power index ϕ on $\mathcal{D}^*$ satisfies NP, CS, AN, and TP if and only if there exist $\alpha_0, \alpha_1, ..., \alpha_{n-1}, \beta_1, ..., \beta_n \in R$, with $\alpha_0 = 0$ and $\beta_1 = 0$, such that for each $C \in \mathcal{D}^*$ we have $\phi(C) = \sum_{j \in N} \phi(C_j)$, with each $\phi(C_j) = \phi(U_{M_j^C, j})$ defined as in (a) and (b) of Lemma 4.3.

Proof. The only-if direction follows from Lemmas 4.2 and 4.3. For the if-direction, let $\alpha_0, \alpha_1, ..., \alpha_{n-1}, \beta_1, ..., \beta_n \in R$, with $\alpha_0 = 0$ and $\beta_1 = 0$, and with $\phi(C) = \sum_{j \in N} \phi(C_j)$, where each $\phi(C_j) = \phi(U_{M_j^C, j})$ is defined as in (a) and (b) of Lemma 4.3. It is obvious that φ satisfies AN and CS. For NP, note that *i* is a null player in $C \in \mathcal{D}^*$ if and only if $C_i = O$ and $i \notin M_j^C$ for all $j \in N$. This implies that $\varphi_i(C_j) = 0$ for all $j \in N$ and therefore $\phi_i(C) = 0$.

For TP, let $C', C, D', D \in$ (HTML translation failed) D^* such that $C' \subseteq C$, $D' \subseteq D$, and $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ for every $S \subseteq N$. Then it is straightforward to show that $C'_j \subseteq C_j$, $D'_j \subseteq D_j$, and $C_j \setminus C'_j = D_j \setminus D'_j$ for all $j \in N$. Hence, to show TP, it is sufficient to show that

$$\varphi(C_j) - \varphi(C'_j) = \varphi(D_j) - \varphi(D'_j) \quad (*)$$

for all $j \in N$. Now fix *j* and observe that $C'_j \subseteq C_j$ and $D'_j \subseteq D_j$ imply $M_j^{C'} \subseteq M_j^C$ and $M_j^{D'} \subseteq M_j^D$. If $M_j^{C'} = M_j^C$, then $C'_j = C_j$ and therefore $D'_j = D_j$, so that (*) follows. Otherwise, both $M_j^{C'} \subset M_j^C$ and $M_j^{D'} \subset M_j^D$ (where $\subset$ denotes strict inclusion). Then, for $S = N \setminus M_j^{C'}$ we have $C_j(S) \setminus C'_j(S) = \{j\} = D_j(S) \setminus D'_j(S)$. This implies $S \cap M_j^{D'} = \varnothing$, hence $M_j^{D'} \subseteq M_j^{C'}$. Similarly, $M_j^{C'} \subseteq M_j^{D'}$, hence $M_j^{D'} = M_j^{C'}$. This implies that also $M_j^D = M_j^C$, so $C_j = D_j$ and $C'_j = D'_j$, and (*) follows again. □

In the concluding Section 6 we provide examples to show that the four axioms in Theorem 4.4 are logically independent. In the next section we consider specific subclasses of the family of power indices satisfying the conditions of Theorem 4.4, following from the possible effects of adding links in a digraph. Here, we suggest a scaling condition on a power index ϕ by which we can single out a unique member of this family[10].

Scaling (SC). For all $C \in D^*$

• If $j \in N,\ C(j) = \varnothing$, and $j \in C(k)$ for some $k \in N,$ then $\phi_j(C) = -1$.

• If $i,\ j \in N,\ C(i) = C(j),\ j \in C(k)$ for some $k \in N,\ i \notin C(k)$ for all $k \in N$, then $\phi_j(C) = \phi_i(C) - 1$.

The first bullet point fixes the power of a player without any outgoing link but with at least one incoming link to -1. The second point fixes the difference in power between two players with the same outgoing links to 1, if one player has no incoming links but the other player has at least one incoming link. We leave the proof of the following corollary to the reader. Denote a power index ϕ as in Theorem 4.4 by $\phi^{\alpha,\beta}$, where $\alpha = (\alpha_1, ..., \alpha_{n-1}) \in R^{n-1}$ and $\beta = (\beta_2, ..., \beta_n) \in R^{n-1}$.

Corollary 4.5. A power index ϕ on D^* satisfies NP, CS, AN, TP, and SC if and only if $\phi = \phi^{\alpha,\beta}$ with $\alpha = \beta = (1, ..., 1)$.

Clearly, in the digraph O, which has no arcs at all, every power index $\phi^{\alpha,\beta}$ assigns 0 to every player. We conclude with a consideration of the following question: for which digraphs do all players have zero power?

Remark 4.6. Suppose that every component of a digraph $C \in D^*$ is a complete graph on the nodes in that component, i.e., for all $i,\ j$ in the same component there is a link from i to *j*, as well as a link from j to *i*; in particular, there is a link from i to *i*. Then it can be deduced from the description in Lemma 4.3 that $\phi^{\alpha,\beta}(C) = (0, ..., 0)$ for every power index satisfying the conditions in Theorem 4.4. Examples of such digraphs are the complete digraph on all nodes (players), which is the invariant digraph associated with, for instance, a circular digraph; and the digraph without any links, where all the players are isolated and therefore null players.

5. Adding links: Copeland, β- and apex type indices

In this section we consider the class of power indices characterized in Theorem 4.4 and study refinements following from the effect of adding a directed link (arc) between

[10]This axiom is similar to the Controlled player axiom in [12].

player i and player j. Let Φ denote the class of all power indices satisfying NP, CS, AN, and TP. As above, a generic element of Φ is denoted as $\phi^{\alpha,\beta}$, where $\alpha = (\alpha_1, ..., \alpha_{n-1}) \in R^{n-1}$ and $\beta = (\beta_2, ..., \beta_n) \in R^{n-1}$.

The first axiom says that if we add an additional link to player j from player i, then this should not change the power of the players who already have a link to j. In the parlance of control or permission: if player j becomes additionally controlled by some player i, then this should not change the power of the players who were already controlling j.

Recall that for a digraph $C \in D^*$ and player j, M_j^C is the set of players with a link to j, i.e., $M_j^C = \{i \in N : j \in C(i)\}$.

Link addition 1 (LA1). $\phi_h(C) = \phi_h(C')$ for all $C, C' \in D^*$, $j \in N$, and $h \in M_j^C \setminus \{j\}$, such that there is $i \in N$ with $i \notin M_j^C$, $C'(i) = C(i) \cup \{j\}$ and $C'(\ell) = C(\ell)$ for all $\ell \in N \setminus \{i\}$.

Theorem 5.1. Let $\phi = \phi^{\alpha,\beta} \in \Phi$. Then ϕ satisfies LA1 if and only if there exists a $c \in R$ such that $\alpha_k = kc$ for all $k = 1, ..., n-1$ and $\beta_k = kc$ for all $k = 2, ..., n$.

Thus, under LA1 we obtain a one-parameter family of power indices of the form

$$\phi_i^c(C) = \sum_{j \in C(i)} c - |M_i^C| c = c\left(|C(i)| - |M_i^C|\right) \tag{3}$$

where $c \in R$. For instance, for $c = 1$ and using the terminology of control, the power of player i is equal to the number of players controlled by player i minus the number of players controlling player i, which yields the Copeland score in social choice theory [2][11, 12]

Proof. For the if-direction let $c \in R$ and let ϕ^c be as in (3). We show that ϕ^c satisfies LA1. Let C, C', h, j, i be as in the statement of the axiom. Then $h \neq i$ and therefore $\phi_h^c(C) = c(|C(h)| - |M_h^C|) = c(|C'(h)| - |M_h^{C'}|) = \phi_h^c(C')$.

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[11] In the social choice theory, for a given preference profile the Copeland score of an alternative x is the number of alternatives beaten by x minus the number of alternatives beating x in pairwise comparison.

[12] An alternative way of characterizing the class $\{\phi^C : c \in R\}$ is by strengthening the transfer property in the way indicated at the end of Section 3.

For the only-if direction, let $\phi = \phi^{\alpha,\beta} \in \Phi$ satisfy LA1. First suppose $M \in P_0(N)$ and $i, j \in N \setminus M$ with $i \neq j$ and $|M| = k$ for some $k \in \{1, ..., n-2\}$. Then for $h \in M$, by LA1 with $U_{M,j}$ in the role of C and $U_{M\cup\{i\},j}$ in the role of C', $\phi_h(U_{M,j}) = \phi_h(U_{M\cup\{i\},j})$, hence $\alpha_k / k = \alpha_{k+1}/(k+1)$. Let $c = \alpha_1$, then this implies that $\alpha_k = kc$ for all $k = 1, ..., n-1$. Now suppose $M' \in P_0(N)$ with $j \notin M'$ and $|M'| = k \in \{1, ..., n-1\}$. For $h \in M'$, again by LA1, $\phi_h(U_{M',j}) = \phi_h(U_{M'\cup\{j\},j})$, hence $\alpha_k / k = \beta_{k+1}/(k+1)$. This implies $\beta_k = kc$ for all $k = 2, ..., n-1$. □

The following axiom requires that it is player j whose power does not change if an additional link is added from some player i to j. In terms of control: if a player j becomes additionally controlled by some player i then this should not change the power of player j.

Link addition 2 (LA2). $\phi_j(C) = \phi_j(C')$ for all $C, C' \in D^*$, $j \in N$ with $M_j^C \setminus \{j\} \neq \emptyset$, and $i \in N$ with $i \notin M_j^C$, $C'(i) = C(i) \cup \{j\}$ and $C'(\ell) = C(\ell)$ for all $\ell \in N \setminus \{i\}$.

Theorem 5.2. Let $\phi = \phi^{\alpha,\beta} \in \Phi$. Then ϕ satisfies LA2 if and only if there exists a $c \in R$ such that $\alpha_k = c$ for all $k = 1, ..., n-1$ and $\beta_k = \dfrac{k}{k-1}c$ for all $k = 2, ..., n$.

The power indices characterized in Theorem 5.2 take the form

$$\bar{\phi}_i^c(C) = \sum_{j \in C(i)\setminus\{i\}} \frac{c}{|M_j^C \setminus \{j\}|} - c 1_{\{M_i^C \neq \emptyset\}} \tag{4}$$

where $1_{\{P\}} = 1$ if statement P is true and $1_{\{P\}} = 0$ otherwise. A power index $\bar{\phi}^c$ is similar to the idea of the β-measure as in van den Brink and Gilles [18] or its reflexive variant in van den Brink and Borm [17]: if player i has a link to player j, then he equally shares the amount of power c with the other players linked to j, except possibly j[13]. The difference is that player j loses c in power.

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[13]The β measure is introduced for irreflexive digraphs (i.e. $i \notin C(i)$ for all $i \in N$) and assigns to player i in irreflexive digraph C the score $\beta_i(C) = \sum_{j \in C(i)} 1/\left|M_j^C\right|$. Its reflexive variant assigns the score $\beta_i^{\text{refl}}(C) = \sum_{j \in C(i)\cup\{i\}} 1/|M_j^C|(+1)$. The difference between the two lies in whether the set of loops is taken into account or not. So, the reflexive variant equals the β measure of the reflexive digraph obtained by adding all loops $(i \in C(i)$ for all $i \in N)$.

Proof. For the if-direction let $c \in R$ and $\bar{\phi}^c$ be defined as in (4). We show that $\bar{\phi}^c$ satisfies LA2. Let C, C', j, i be as in the statement of the axiom, then

$$\bar{\phi}_j^c(C) = \sum_{\ell \in C(j) \setminus \{j\}} \frac{c}{|M_\ell^C \setminus \{\ell\}|} - c = \sum_{\ell \in C'(j) \setminus \{j\}} \frac{c}{|M_\ell^{C'} \setminus \{\ell\}|} - c = \bar{\phi}_j^c(C')$$

as is straightforward to verify, both for the case $i \neq j$ and for the case $i = j$.

For the only-if direction, let $\phi = \phi^{\alpha,\beta} \in \Phi$ satisfy LA2. First suppose $M \in P_0(N)$ and $i, j \in N \setminus M$ with $i \neq j$ and $|M| = k$ for some $k \in \{1, ..., n-2\}$. Then by LA2, with $U_{M,j}$ in the role of C and $U_{M \cup \{i\}, j}$ in the role of C', $\phi_j(U_{M,j}) = \phi_j(U_{M \cup \{i\}, j})$, hence $\alpha_k = \alpha_{k+1}$. Let $c = \alpha_1$, then this implies that $\alpha_k = c$ for all $k = 1, ..., n-1$. Now suppose $M' \in P_0(N)$ with $j \notin M'$ and $|M'| = k \in \{1, ..., n-1\}$. Again by LA2, $\phi_j(U_{M',j}) = \phi_j(U_{M' \cup \{j\}, j})$, hence by the previous argument, $c = \frac{\beta_{k+1}}{k+1} - \beta_{k+1} = \frac{-k}{k+1}\beta_{k+1}$, and thus $\beta_{k+1} = \frac{k+1}{k} c$, which implies that $\beta_k = \frac{k}{k-1} c$ for all $k = 2, \ldots, n-1$. □

The final axiom we consider says that if we add a link from player *i* to player *j* then both have the same gain or loss in power.

Link addition 3 (LA3). $\phi_i(C') - \phi_i(C) = \phi_j(C') - \phi_j(C)$ for all $C, C' \in D^*$, $j \in N$ with $M_j^C \neq \emptyset$, and $i \in N$ with $i \notin M_j^C$, $C'(i) = C(i) \cup \{j\}$ and $C'(\ell) = C(\ell)$ for all $\ell \in N \setminus \{i\}$.

Theorem 5.3. Let $\phi = \phi^{\alpha,\beta} \in \Phi$. Then ϕ satisfies LA3 if and only if there exists a $c \in R$ such that $\alpha_k = \frac{2}{k+1} c$ for all $k = 1, ..., n-1$, and $\beta_k = 0$ for all $k = 2, ..., n$.

The power indices characterized in Theorem 5.3 take the form

$$\tilde{\phi}_i^c(C) = \sum_{j \in C(i) \setminus C(j)} \frac{\alpha_{|M_j^C|}}{|M_j^C|} - \alpha_{|M_i^C|} 1_{\{M_i^C \neq \emptyset,\ i \notin C(i)\}} \tag{5}$$

with α_k as in Theorem 5.3. In terms of control, according to a power index $\tilde{\phi}^c$, if player *j* controls himself, then no player, including player *j*, derives (positive or negative) power from controlling *j*. Further, the (negative, if $c > 0$) power from being controlled decreases and converges to zero as the number of controlling players increases. We note that $\tilde{\phi}^c$ is related to the apex power index in [16][14].

Proof. For the if-direction let $c \in R$ and $\tilde{\phi}^c$ be defined as in (5). We show that $\tilde{\phi}^c$ satisfies LA3. Let C, C', j, i be as in the statement of the axiom, and $k = |M_j^C| \in \{1, ..., n-1\}$. First, if $j \notin M_j^C$, $i \neq j$, then

$$\tilde{\phi}_i^c(C') - \tilde{\phi}_i^c(C) = \frac{\alpha_{k+1}}{k+1} = \frac{2c}{(k+1)(k+2)}$$

and

$$\tilde{\phi}_j^c(C') - \tilde{\phi}_j^c(C) = -\alpha_{k+1} + \alpha_k = \frac{2c}{(k+1)(k+2)}$$

so that LA3 holds. Second, if $j \in M_j^C$, $i \neq j$, then

$$\tilde{\phi}_i^c(C') - \tilde{\phi}_i^c(C) = \frac{\beta_{k+1}}{k+1} = 0$$

and

$$\tilde{\phi}_j^c(C') - \tilde{\phi}_j^c(C) = \frac{\beta_{k+1}}{k+1} - \beta_{k+1} - (\frac{\beta_k}{k} - \beta_k) = 0$$

Thus LA3 also holds here.

For the only-if direction, let $\phi = \phi^{\alpha,\beta} \in \Phi$ satisfy LA3. First suppose $M \in P_0(N)$ and $i, j \in N \setminus M$ with $i \neq j$ and $|M| = k$ for some $k \in \{1, ..., n-2\}$. Then $\phi_j(U_{M \cup \{i\}, j}) - \phi_j(U_{M,j}) = -\alpha_{k+1} + \alpha_k$, whereas $\phi_i(U_{M \cup \{i\}, j}) - \phi_i(U_{M,j}) = \alpha_{k+1}/(k+1)$. By LA3 with $U_{M,j}$ in the role of C and $U_{M \cup \{i\}, j}$ in the role of C', this implies $\alpha_k = \frac{2}{k+1}c$ for all $k = 1, ..., n-1$ with $\alpha_1 = c$.

[14]The apex-measure is introduced for irreflexive digraphs and assigns to player i in an irreflexive digraph C the score $a_i(C) = \sum_{j \in C(i)} 2/\left(|M_j^C|(|M_j^C|+1)\right) + \left(\left(|M_i^C| - 1/\left(M_i^C|+1\right)\right)\right) 1_{\{M_i^C \neq \emptyset\}}$. It can be defined for reflexive digraphs in an obvious way.

Now suppose $M' \in P_0(N)$ with $j \in M'$, $i \notin M'$, and $|M'| = k \in \{1, \dots, n-1\}$. Then $\phi_j(U_{M' \cup \{i\}, j}) - \phi_j(U_{M', j}) = -k\beta_{k+1}/(k+1) + (k-1)\beta_k/k$, and $\phi_i(U_{M' \cup \{i\}, j}) - \phi_i(U_{M', j})$ $= \beta_{k+1}/(k+1)$. By LA3, this implies $\beta_{k+1} = ((k-1)/k)\beta_k$ for all $k \in \{1, ..., n-1\}$. Since $\beta_1 = 0$, it follows that $\beta_k = 0$ for all $k = 2, ..., n$. □

Remark 5.4. If we weaken LA3 to condition LA3′ by strengthening the premiss that $M_j^C \neq \emptyset$ to $M_j^C \setminus \{j\} \neq \emptyset$ as in LA1 and LA2, then we obtain a two-parameter family: there exists a $c \in R$ such that $\alpha_k = \frac{2}{k+1}c$ for all k = 1, ..., n – 1, and $d \in R$ such that $\beta_k = \frac{1}{k-1}d$ for all $k = 2, ..., n$. We omit the proof of this claim.

6. Concluding remarks

We provide a summary and further relations with the literature, and finally show that the axioms in Theorem 4.4 are independent.

6.1. Summary and further relations with the literature

We axiomatized a class of power indices for invariant digraphs using the axioms of Constant sum, Anonymity, Null player and the Transfer property, inspired by Karos and Peters [12], see Theorem 4.4. By adding one of the three considered link addition axioms, in each case we obtained a subclass of indices related to the Copeland score, β-measure and apex-measure, respectively. One main difference related to the last two measures is that these satisfy a normalization condition where the total sum of the powers of all the players depends on the digraph, whereas for the indices defined in this paper, this sum is always zero. As a consequence, in the present paper players can have negative power, which is not possible according to the β-measure and apex-measure.

The β- and apex-measures also satisfy the Anonymity and Null player axioms, and the β-measure even satisfies a stronger version of NP, in which every player who has no outgoing arcs has power zero. As we saw above, the Transfer and Null player properties imply that we can find the power values for a digraph by adding up the power values over special elementary digraphs, one associated with each player, where such a digraph consists of all the links going into this player (see Lemma 4.2). For digraphs this is also implied by the property of Additivity over Independent partitions used by van den Brink and Gilles [18] to axiomatize the outdegree and β-measure, which requires that the power index for a digraph is the sum of the powers in a partition of the digraph such that every player has a positive indegree in at most one subgraph in the partition.

The Copeland score for digraphs assigns to each player the difference between its outdegree and indegree. Therefore, the power indices satisfying LA1 characterized in Theorem 5.1 can be seen as multiples of the Copeland score.

The power indices characterized by adding LA2 (see Theorem 5.2) are similar to multiples of the β-measure, with the exception that players who are dominated but do not dominate have negative power, while they have zero power according to the β-measure (and even positive power according to the reflexive β-measure). This corresponds well with the interpretation of mutual control.

The power indices characterized by adding LA3 (see Theorem 5.3) are similar to multiples of the apex-measure, with the exception that players who are dominated but do not dominate have negative power, while they have positive power according to the apex-measure. Similar to the apex-measure, according to these indices, a player who is dominated obtains a greater power if he becomes dominated by more predecessors. This reflects the concept that control over a player decreases when more players formally have control over this player, and therefore its level of self-control increases but will never become positive.

We illustrate the different classes of power indices we considered here by an example.

Example 6.1. For $N = \{1, ..., n\}$ and $k \in N \backslash \{n\}$, consider the structures $C_n^k \in D^*$ given by $C_n^k(i) = \{n\}$ for all $i \in \{1, ..., k\}$, and $C_n^k(i) = \varnothing$ otherwise. In Table 1, we give the powers of the successor n and predecessors $i \in M_n^C$ for various numbers of predecessors.

Table 1. Power indices for Example 6.1

	1	2	3	4	...	k
$\phi_i^c,\ i \in M_n^C$	c	c	c	c	...	c
ϕ_n^c	$-c$	$-2c$	$-3c$	$-4c$	...	$-kc$
$\overline{\phi}_i^c,\ i \in M_n^C$	c	$\frac{c}{2}$	$\frac{c}{3}$	$\frac{c}{4}$	...	$\frac{c}{k}$
$\overline{\phi}_n^c$	$-c$	$-c$	$-c$	$-c$	...	$-c$
$\tilde{\phi}_i^c,\ i \in M_n^C$	c	$\frac{c}{3}$	$\frac{c}{6}$	$\frac{c}{10}$	...	$\frac{2c}{k(k+1)}$
$\tilde{\phi}_n^c$	$-c$	$-\frac{2c}{3}$	$-\frac{c}{2}$	$-\frac{2c}{5}$	...	$-\frac{2c}{k+1}$

Finally, we remark that for an arbitrary digraph $C \in \mathcal{D}$ one could define $\phi(C) = \phi(C^*)$, where C^* is the invariant extension of C obtained as explained in Section 2, namely by adding links for each pair of players between whom there exists a directed path.

6.2. Independence of the axioms in Theorem 4.4

By providing four examples, we show that the axioms in the main characterization, Theorem 4.4, are logically independent.

Not NP, but CS, AN, TP. For $U_{M,j} \in \mathcal{D}^*$ with $M = \{j\}$ define $\phi_i(U_{M,j}) = 1/(n-1)$ for all $i \in N \backslash \{j\}$ and $\phi_j(U_{M,j}) = -1$. In all other cases, define $\phi(U_{M,j})$ as in Lemma 4.3. Extend to the whole of $\mathcal{D}^*$ by taking sums as in Theorem 4.4.

Not CS, but NP, AN, TP. For $U_{M,j} \in \mathcal{D}^*$ with $M = \{j\}$ define $\phi_i(U_{M,j}) = 0$ for all $i \in N \backslash \{j\}$ and $\phi_j(U_{M,j}) = 1$. In all other cases, define $\phi(U_{M,j})$ as in Lemma 4.3. Extend to the whole of $\mathcal{D}^*$ by taking sums as in Theorem 4.4.

Not AN, but CS, NP, TP. For $U_{M,j} \in \mathcal{D}^*$ with $|M| = 2$ and $j \notin M$ (hence $n \geq 3$) define $\phi_i(U_{M,j}) = 0$ for all $i \notin M \cup \{j\}$, also, for $i \in M$ when $i > k$ and $M = \{i, k\}$, $\phi_k(U_{M,j}) = 1$, and $\phi_j(U_{M,j}) = -1$. In all other cases, define $\phi(U_{M,j})$ as in Lemma 4.3. Extend to the whole of $\mathcal{D}^*$ by taking sums as in Theorem 4.4.

Not TP, but AN, CS, NP. For $C \in \mathcal{D}^*$ define

$$n_1 = |\{i \in N : C(i) \neq \emptyset,\ C_i = O\}|,\ n_2 = |\{i \in N : C(i) = \emptyset,\ C_i \neq O\}|$$

If $n_1, n_2 > 0$, then define

$$\phi_i(C) = \begin{cases} 1/n_1 & \text{if } C(i) \neq \emptyset,\ C_i = O \\ -1/n_2 & \text{if } C(i) = \emptyset,\ C_i \neq O \\ 0 & \text{otherwise} \end{cases}$$

and define $\phi(C) = (0, ..., 0)$ in all other cases. To see that ϕ does not satisfy TP, let $n = 4$ and consider $C, D \in D^*$ with $C(1) = 2$, $D(3) = 4$, and $C(S), D(S) = \varnothing$ in all other cases. Then $\phi(C \cup D) + \phi(C \cap D) = (1/2, -1/2, 1/2, -1/2)$, whereas $\phi(C) = (1, -1, 0, 0)$ and $\phi(D) = (0, 0, 1, -1)$, thus violating (1) and therefore TP.

Acknowledgements

We thank two reviewers for their useful suggestions.

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Received 24 April 2016
Accepted 11 July 2016