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ON TYPES OF RESPONSIVENESS IN THE THEORY OF VOTING

In mathematics, monotonicity is used to denote the nature of the connection between variables. Hence for example, a variable is said to be a monotonically increasing function of another variable if an increase in the value of the latter is always associated with an increase in the other variable. In the theory of voting and the measurement of a priori voting power one encounters, not one, but several concepts that are closely related to the mathematical notion of monotonicity. We deal with such notions focusing particularly on their role in capturing key aspects of plausible opinion aggregation. Further, we outline approaches to analyzing the relationship of opinion aggregation and voting power and thereby contribute to our understanding of major components that determine the outcome of voting.

Keywords: *plausible opinion aggregation, monotonicity, no-show paradox*

1. Introduction

Rule of the people, by the people and for the people was the expressed goal of institutional design of Abraham Lincoln. But how can we determine that this goal has been achieved? It would seem that the first aspect – rule of the people – has been achieved when an overwhelming majority of the population accepts the decisions of the rulers as binding. The second aspect – the rule by the people – can be regarded as achieved when the views of the people are reflected in the decisions of the rulers. The third aspect, in turn, refers to the degree to which the decisions of the rulers serve the interest of the

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people. Clearly, all three aspects deal with matters of degree and are thus inherently vague. This is particularly true of the second and third aspects. Our focus in this paper is on the second aspect: to what extent are the views of the population reflected in the decisions that are regarded as collectively binding? We shall gloss over several things that play a role in political decision making in modern democracies such as the fact that binding decisions are typically made by elected representatives rather than the population at large. Our primary interest is more theoretical, viz. how to ascertain that the views of the voters are reasonably well reflected in the decisions made by the voters themselves, i.e. without mediation of representatives. What kind of correspondence between collective decisions and voter opinions would be desirable? What is the role of voting power in the selection of the outcome and how do we measure such power?

Obviously, different choice making rules establish somewhat different correspondences between voter opinions and collective decisions. Of the desiderata one could impose on collective choice rules those pertaining to responsiveness seem particularly important: collective decisions should be responsive to changes in voter opinions. To put it in another way: unresponsive rules cannot be viewed as compatible with the government by the people or for the people, for that matter. As examples of obviously unresponsive rules one could mention dictatorial rules as well as "trivial ones". The former always result in outcomes preferred by one individual, regardless of the opinions of others, while trivial ones exclude some outcomes, no matter what the distribution of opinions of the voters might be. Apart from these kinds of completely unresponsive rules, there are others that respond to changes in voter preferences in counterintuitive ways. For example, a rule might specify as the collective choice an alternative that is ranked worst by the largest number of voters. In what follows, we shall discuss some properties that capture various aspects of responsiveness. It turns out that there are, indeed, several such notions. Responsiveness can also be called into question if an increase in vote shares results in a decrease of a priori voting power. We discuss measures of voting power that indicate such results. Our contribution is thus of mainly a conceptual and theoretical nature. We aim to provide conceptual clarification to issues pertaining to how decision making systems respond to the opinions of their members. Related issues have been discussed in depth and with nice results by Gambarelli (see e.g., [20–22].)

This article is organized as follows. The next section deals with the principal notion of the responsiveness of voting procedures, viz. monotonicity, i.e. the requirement that additional support does not render winners into non-winners. Section 3 discusses a related desideratum, viz. that of invulnerability to the no-show paradox. This paradox occurs whenever a group of individuals is better off by not voting at all than by voting according to their preferences, *ceteris paribus*. Thereafter a particularly striking example of the no-show paradox is touched upon: the P-BOT paradox. Section 5 discusses a notion that is often confused with the concept of monotonicity just defined, viz. that of Maskin monotonicity. The latter has a crucial role in the theory of mechanism design, but is largely absent in the theory of voting. We then move on to the monotonicity of measures of voting power, a topic that has been of particular interest to Gianfranco Gambarelli. We also discuss how monotonicity can be achieved and, finally, suggest some future lines of research.

2. Monotonicity in opinion aggregation

Informally stated, monotonicity says that additional support, *ceteris paribus*, never turns winners into non-winners. The *ceteris paribus* clause is very important here. Slightly more formally, let $R = (R_1, ..., R_n)$ be an n-person profile of complete and transitive preference relations. Denote by P_i the asymmetric part of the relation. We consider a preference profile $R = (R_1, ..., R_n)$, a set of alternatives A and x in A so that $F(A, R) = x$, i.e. rule F when applied to profile R over the set of alternatives A results in x being elected. Consider now any $R' = (R_1', ..., R_n')$ such that for all y, z in A: $(x \neq y, x \neq z)$ and for all i in N: yR_iz if and only if $yR'z$, while for at least one i in N and w in A: wP_ix but $xP_i'w$. Now, F is monotonic if under the preceding conditions $F(A, R') = x$.

Many collective choice rules are, indeed, monotonic, for example, the plurality rule which gives each voter one vote. The winner is the alternative that has been voted for by more voters than any other alternative. Suppose that this system is implemented so that each voter submits his/her full ranking of alternatives and the system then singles out the first ranked ones, whereupon the alternative ranked first by more voters than any other is declared the winner. It is easy to see that improving the winner's position in at least one voter's ranking without making any other changes in the preferences cannot make it a non-winner in the changed profile, provided that the same rule is applied. Improving a winning alternative's position *vis-a-vis* some other alternative cannot bring about a new winner, since after such a change the former winner has at least the same number of first positions as before, while all other alternatives have at most the same number of first positions as before the change.

Even though monotonicity is an intuitively quite plausible property, there are relatively commonly used rules that do not satisfy it. To wit, plurality runoff and Hare's system are well-known and widely used methods that are non-monotonic (see e.g., [42]). Typically, non-monotonic methods involve several stages of computing the result. For example, according to plurality runoff one first examines whether there is a candidate that is ranked first by more than half of the voters. If there is, then this candidate is the winner. Otherwise, a real or fictitious second round is conducted. In this round, only the two front-runners from the first round are presented, whereupon the one that gets more votes is elected. According to Hare's system there are potentially several rounds of restricting the set of candidates, but the criterion of winning is basically the same as in plurality runoff: the winner has the support of at least half of the electorate. Another concept that pertains to responsiveness of invulnerability to the no-show paradox.

3. Invulnerability to the no-show paradox

A no-show paradox occurs whenever an alternative, say *x*, is elected in a given profile, but *y* is elected if the original profile is enlarged by a group of voters that prefer *x* to y, *ceteris paribus*. So, in essence this paradox occurs when a group of identically-minded voters is better off by abstaining than by voting according to their preferences. The change in outcomes may be large or small, it may also pertain to outcomes that are ranked lowly in the abstainers' preferences. The essential point is that abstaining, *ceteris paribus*, brings about a better outcome for the abstainers than voting according to their preferences.

This is the now established view of the no-show paradox. In fact, the original notion of the no-show paradox, as expressed by Fishburn and Brams, says something quite different: *The addition of identical ballots with candidate x ranked last may change the winner from some other candidate to x* [17]. In this original version, the no-show paradox pertains to situations where, in order to avoid the worst outcome, a voter is better off by not voting at all. This original notion of the no-show paradox is similar in spirit to the currently established view, but different in focusing specifically on avoiding the worst possible outcome rather than improving the outcome in more general terms. It seems that the now established definition appeared for the first time in [39] and from there spread throughout the social choice community. We shall here adopt the established view.

Table 1. The no-show paradox and plurality runoff

26 voters	47 voters	2 voters	25voters
A	B	B	C
B	C	C	A
C	A	A	B

Table 1 illustrates this paradox in the context of the plurality runoff system. We have three alternatives (A, B, C) and 100 voters. The latter are divided into four groups of voters, each with identical preferences over the alternatives. The preferences of each group are indicated by listing the alternatives in the order of preference from top to bottom. Thus, the group consisting of 26 voters has A as their first, B as their second and C as their lowest-ranked alternative. Assuming that all voters reveal their preferences in voting, there will be a runoff between A and B, whereupon A wins in the second round. Suppose now that the group of 47 voters indicated in the second column decide not to vote at all, *ceteris paribus*. It follows that there will be a runoff between A and C in which the latter defeats the former, i.e. C wins. The outcome is thus better for the abstainers than the one resulting from their voting according to their preferences[4].

[4]Of course, the 49 voters in the two middle columns may force the outcome C by voting strategically for C in the first round or – better yet – by convincing the A-supporters to vote for B in the first round

A more dramatic and at the same time more important version of the no-show paradox occurs when the outcome resulting from abstaining – again *ceteris paribus* – is ranked first in the preferences of the abstainers. This version is known as the strong no-show paradox [46]. Table 2 demonstrates that Black's procedure may lead to the strong version of the no-show paradox. Black's procedure is a hybrid of Condorcet's winner criterion and the Borda count: if a Condorcet winner exists, it will be the winner, otherwise the Borda winner is elected. In Table 2, alternative D is the Condorcet winner and is, thus, elected by Black. However, if the right-most voter abstains, there is no Condorcet winner any longer. Thus, the Borda winner E becomes the Black winner. E is the first-ranked alternative of the abstainer [44][5]. Hence, by abstaining this voter brings about his first-ranked alternative, while by voting according to his preferences (i.e. sincere voting) he brings about a worse outcome for himself[6].

Under specific circumstances, vulnerability to the strong no-show paradox confronts a voter with a rather bizarre choice: either to reveal his preferences and settle for an outcome that is not his best or to abstain in order to secure his best outcome. One could, with some justification, argue that procedures vulnerable to the strong paradox fly in the face of the basic rationale of elections: to disclose popular opinion about the alternatives.

Table 2. Black's procedure
and the strong no-show paradox

1 voter	1 voter	1 voter	1 voter	1 voter
D	E	C	D	E
E	A	D	E	B
A	C	E	B	A
B	B	A	C	D
C	D	B	A	C

The best-known results on no-show paradoxes are due to Moulin [39] and Pérez [45, 46]. Moulin proved a theorem stating the incompatibility of the Condorcet principle and invulnerability to the no-show paradox. The Condorcet principle states that if there is a Condorcet winner alternative in a profile, then this alternative, and only this one,

whereupon B would win. It is also conceivable that the A-supporters may convince the C-supporters (all of them) to vote for A in the first round to avoid B being elected, and so on. To succeed all these stratagems require complete information about the profile and perfect coordination of balloting. It should be emphasized, though, that the no-show paradox is not about strategic voting, but about bizarre voting outcomes under the assumption that the voters do what the voting systems expect them to do: reveal their opinions about the alternatives at hand.

[5]A much earlier report, [47], provides a similar example showing that Black's procedure violates a criterion called voter adaptability. This criterion is almost the same as invulnerability to no-show paradox.

[6]For a comprehensive overview of similar paradoxes, see [13].

should be elected. Rules that always satisfy the Condorcet principle are known as Condorcet extensions. Pérez subsequently strengthened Moulin's result by showing that, apart from a couple of exceptions, all Condorcet extensions are vulnerable to the strong no-show paradox. The exceptions are the maximin rule and Young's procedure. So, examples such as Table 2 can be found for nearly all rules that satisfy the Condorcet principle.

The twin paradox is Moulin's finding [39]. It is similar in spirit to the no-show paradox and turns out to characterize largely the same class of rules. Denote by N a set of n voters. Suppose that in a given profile, the rule F results in alternative x being chosen. Let us now add an $(n + 1)$th voter j with an identical ranking over the alternatives as that of some voter, say i in N. Denote the augmented profile of $n + 1$ voters by N^+. The twin paradox occurs if the choice resulting from applying F to N^+ is y and, moreover, x is strictly preferred to y by voters i and j. In other words, the twin paradox occurs when a voter (or a group with identical preference rankings) is better off with a smaller number of identically-minded partners than with a larger number of them.

Table 3. Nanson's rule and the twin paradox

5 voters	5 voters	6 voters	1 voter	2 voters
A	B	C	C	C
B	C	A	B	B
D	D	D	A	D
C	A	B	D	A

In Table 3, Nanson's rule results in B [44]. Here we have an instance of the twin paradox: if the right-most group of 2 voters lost one of its members, C would win. If this voter returns, B wins. Clearly the twin is not welcome by the remaining member of the right-most group. In fact, there would be an incentive for this group to coordinate so that only one of the members vote in order to secure the best possible outcome for the group.

4. Invulnerability to P-BOT paradoxes

To clarify the distinctions between various types of voting paradoxes, Felsenthal and Tideman introduce the concepts of P-TOP and P-BOT paradoxes [16]. The former class consists of those that have been called strong no-show paradoxes above. To quote the authors: *According to this paradox ... if a candidate, say candidate x, has been elected initially, then it is possible that another candidate, y, will be elected if, ceteris paribus, additional voters whose top-ranked candidate is x join the electorate*.

The latter class exhibits equally, if not more, bizarre behavior. To wit, an instance of the P-BOT paradox occurs *if one of the candidates, say candidate c, who has not been elected originally, may be elected if, ceteris paribus, the electorate is increased as a result of additional voters whose bottom-ranked candidate is c join the electorate* … [16]. Systems vulnerable to P-BOT paradoxes present a voter with an obvious dilemma: should one express one's preferences by voting and risk helping one's lowest-ranked candidate to be elected, or should one abstain thereby contributing to the election of some more "tolerable" candidate. It should be added, though, that vulnerability to a paradox does not mean that this kind of dilemma would be faced by a voter in all or even in a majority of elections. Nonetheless the occurrence of a P-BOT paradox is certainly an unpleasant surprise for a group that joins the original electorate only to find that its lowest-ranked candidate got elected as a result of the group's activity.

Vulnerability to the P-BOT paradox is in fact quite common among voting procedures [15]. Table 4 illustrates P-BOT paradox in terms of Kemeny's method. This method – it will be recalled – decomposes any preference profile into a set of pairwise rankings of alternatives. These are the compared with the pairwise rankings of each logically possible strict preference ranking. Thus, e.g., if the alternatives are A, B and C, one of the possible rankings, viz. $A \succ B \succ$ C, is decomposed into the following ordered pairs: (A≻B), (A ≻ C), (B ≻ C). One then counts the support in the profile for each of these pairwise rankings, support here meaning the number of voters having each particular ordered pair in their preference relations. Finally, the Kemeny winner is the ranking that has the largest support in the profile.

Table 4. Kemeny's method
is vulnerable to the P-BOT paradox

5 voters	3 voters	3 voters
D	A	A
B	D	D
C	C	B
A	B	C

In the profile illustrated by Table 4, A is the strong Condorcet winner and – since Kemeny's method is a Condorcet extension – is the Kemeny winner as well. Suppose now that a group of 4 voters with the unanimous ranking B ≻ C ≻ A ≻ D joins the electorate. Computing the support for all 24 rankings yields D ≻ B ≻ C ≻A as the Kemeny ranking in the expanded profile. Thus, A was the winner in the original 11 voter profile, but D – the lowest-ranked alternative of the 4 new entrants – becomes the winner in the expanded one.

5. Maskin monotonicity

In the preceding definitions, the *ceteris paribus* clause is to be taken seriously. The importance of this statement becomes obvious when one turns to another concept of monotonicity that plays a prominent role in mechanism design, viz. Maskin monotonicity [38]. It is similar to the concept of monotonicity outlined above, but dispenses altogether with the *ceteris paribus* clause. More specifically Maskin monotonicity is the following property of a procedure. Assume that alternative x wins in profile Q over the set of alternatives A. Also suppose that profile Q' is formed over A so that for any alternative y in A, x is preferred to y by at least as many voters in Q' as in Q. Maskin monotonicity requires that x is still chosen in Q'. The important point here is that no restrictions are imposed on those pairs of alternatives in Q' that do not include x. It is easy to show that Maskin monotonicity is not satisfied by typical voting procedures [43]. Consider the plurality procedure and Table 5 [43].

Table 5. The plurality rule
is not Maskin monotonic

2 voters	1 voter	1 voter	1 voter
A	B	C	D
B	C	B	C
C	A	A	B
D	D	D	A

Call the profile illustrated by Table 5 profile Q. Clearly alternative A wins in Q. Suppose now that the profile is transformed into Q' so that A is ranked at least as high by all voters and strictly higher by at least one voter. To wit, in Q' we keep the rankings of all alternatives the same for the two left-most voters who rank A first, we lift A ahead of C in the next ranking of P', we interchange the ranks of B and C for the next voter in Q' and lift B ahead of both C and D in the right-most ranking in Q′. These changes keep the position of A the same or higher in Q' than in Q for all voters. However, as a result of these changes, A is no longer the plurality winner in Q'; it is B. Hence the plurality rule does not satisfy Maskin monotonicity.

Obviously Maskin monotonicity is a very demanding property. The explanation is simple: by excluding *ceteris paribus* clauses it allows all kinds of transformations that include the preference rankings of other alternatives than the original winner. In a way the fact that voting systems do not satisfy Maskin monotonicity is another way of saying that they do not satisfy the independence of irrelevant alternatives from Arrow's theorem: whether x is collectively preferred to y is typically dependent, not just on their mutual rankings, but also on how they are related to other alternatives. The relationship of Maskin monotonicity with independence of irrelevant alternatives and other properties is exhaustively covered by [1] and [2].

6. Is voting power monotonic?

The paradoxical results above are the result of combining voting rules with specific distributions of voting weights and particular preferences of the voters. It is not always obvious which of the three ingredients produces results which are considered paradoxical because of some violation of monotonicity. Intuition says that all three are responsible for such paradoxes. However, paradoxes of non-monotonicity also appear when we analyze the game form of voting games, i.e., if we abstract from preferences. Power indices are standard instruments used to illustrate these issues and discuss their implications and properties.

There are two, quite different, models of monotonicity relevant in applying power indices and discriminating between various measures[7]. One discusses the relationship between a given vote distribution $w = (w_1, ..., w_n)$ and a power distribution $\pi = (\pi_1, ..., \pi_n)$. If $w_i < w_j$ implies $\pi_i \leq \pi_j$, for all w and π, then the power measure is monotonic, i.e., satisfies local monotonicity (LM). The other concept of monotonicity compares two vote distributions w and w' and asks whether $\pi_i \leq \pi_i'$ if $w_i < w_i'$ holds for all voters. The distribution of w' can be seen to represent *changes* compared to w. There is a series of paradoxes, including the paradox of redistribution which demonstrates that all the standard power indices, i.e., Shapley–Shubik index, Banzhaf index, etc., fail to satisfy this *comparative monotonicity* requirement. However, the Shapley–Shubik index and the Banzhaf index satisfy LM. On the other hand, power indices based on minimum winning coalitions, the Deegan–Packel index and the public good index (PGI), violate both LM and comparative monotonicity. Do we have to conclude that voting power is not monotonically responsive to changes in the vote distribution? Of course, the answer depends on the power measure we choose to apply.

The Shapley–Shubik index satisfies global monotonicity. The (normalized) Banzhaf index does not, as demonstrated by [48]. Global monotonicity compares two voting games $v = (d; w)$ and $v' = (d; w')$, where d represents the decision rule, and requires for each pair of voters i and j in N that $\pi_i \leq \pi_i'$ if $w_i < w_i'$ and $w_j > w_j'$, given that all the other voting weights in w and w' are identical.

[7]It has been suggested to speak of an "index" if the values of a power measure add up to one. We will not follow this convention. However, the values of the measures discussed in this paper add up to one – either through standardization or by their very nature. The latter holds for the Shapley–Shubik index.

A notorious example to illustrate that PGI and the Deegan–Packel index violate LM is the voting game $v^\circ = (51; 35, 20, 15, 15, 15)$. The corresponding PGI is $h(v^\circ) = (4/15, 2/15, 3/15,3/15,3/15)$, while the corresponding Deegan–Packel index is $\rho(v^\circ) = (18/60, 9/60, 11/60, 11/60, 11/60)$. In both cases, the second group of voters has less a priori voting power but more votes than the groups with a smaller number of votes.

In what follows, we restrict ourselves to weighted voting games, like v° above, fully described by the decision rule ("quota") d and the vote distribution w. This corresponds to the presentation of the voting paradoxes above. Note, alternatively, that voting games can also be described by their sets of minimum winning coalitions (MWC). A winning coalition S is a MWC if $S\backslash\{i\}$ is a losing one, for all $i \in S$. Obviously, all the players of a MWC coalition have a swing position, i.e., they are decisive (also called crucial) to this coalition winning. The PGI of player i, h_i, is the number of MWCs that have i as a member divided by the number of all the swing positions the players have in all MWCs of the game. If m_i is the number of MWCs that have i as a member then i's PGI value is

$$h_i = \frac{m_i}{\sum_{i \in N} m_i} \qquad (1)$$

The underlying assumption is that collective decisions are about public goods and, in principle, non-exclusion and non-rivalry in consumption apply. Everybody has access to enjoy the public good or to suffer from it. The members of a MWC are decisive for a particular public good: they get what they want – otherwise they would not vote for the particular good (e.g., a "policy") that the coalition represents. Others either free-ride or suffer – in any case, they have no power in relation to this particular good[8].

The PGI has been identified with solidarity (within the members of a MWC)[9], and yet the values h_i look like shares. However, this is due to standardization: their sum is 1 (for a non-standardized version of the PGI, called the public value, see [26]).

The Deegan–Packel index considers the number of players (e.g., "parties") a MWC coalition has and divides the value of the coalition S, standardized as $v(S) = 1$, by the cardinality $|S| = s$[10]. This measure considers the value of a coalition to be a private good that is equally shared among the members of a coalition which is a minimum winning one, in order to keep shares large. In general, this does not correspond to the aggregation of preferences and the determination of a collective outcome. Therefore, this index seems to be of minor interest for the discussion in this paper.

[8]See [23]. For an axiomatization of the PGI, see [34] and [40]. [9] provides a valuable summary of the PGI literature and the history of this measure.

[9]See [19].

[10]For details, see [12]. For a discussion of this measure, taking a priori unions into account, see [5].

Given that N is the set of all players of game v, player i is a swing player if $S \subset N$ is a winning coalition and $S \backslash \{i\}$ is a losing coalition. The normalized Banzhaf index of player i counts the number of coalitions S that have i as a swing player. In Eq. (2), this number is c_i. For normalization this number is divided by the total number of swing positions that characterize the game v. A formal definition of the normalized Banzhaf index is

$$\beta_i = \frac{c_i}{\sum_{i \in N} c_i} \tag{2}$$

Felsenthal and Machover [14] conjecture that this measure represents *I*-power, capturing the impact of player i on the voting outcome (see [51] for a critical discussion). It should be relevant for the discussion of voting paradoxes.

With explicit reference to PGI, Bertini et al. [9] introduced the public help index (PHI) which has the same mathematical structure as Eqs. (1) and (2). However, it is based on the number of winning coalitions which have player i as a member, irrespective of whether i has a swing in a particular coalition S or not. This measure satisfies LM and global monotonicity. However, does it measure power? It does not refer to swing players, but to membership. Even dummy players get assigned values. Yet, a dummy player i in winning coalition S cannot reject a particular public good, generated by coalition S, if i does not like this good, but even suffers from it. A dummy player i can neither "exclude" him/herself, as the coalition offers a public good, nor can i veto a public good if it is "a bad" to him or her. In the case of PGI, those who are members of a MWC can prevent the availability of the corresponding good simply by leaving the coalition or by voting "no".

The PHI invites us to consider preferences which decide whether a public good is "privately" good or bad. It measures the potential for success which implies making a comparison of outcome and preferences. The latter is the approach proposed by this paper.

We will not go into the details of the definition of the Shapley–Shubik index here but point out that it is based on counting swing players, without restricting to MWCs, and permutations, i.e., orderings of players. There is an immediate interpretation of a permutation as expressing the ideological closeness of the players. This invites us to consider preferences in addition to power relations defined by winning coalitions.

The application of power indices is motivated by the widely shared "hypothesis" that the vote distribution is a poor proxy for a priori voting power. If this is the case, does it make sense to evaluate a power measure by means of a property that refers to the vote distribution as suggested by LM? Of course, our intuition supports LM. However, if we could trust our intuition, do we need these highly sophisticated power measures at all?[11] If we take the vote distribution w as a proxy for the power distribution then we have no problems with the concepts of monotonicity defined above. However, in the case of the voting game $v = (51; 49, 48, 3)$, this could be difficult to justify with respect to both political experiences and data.

7. How to achieve monotonicity?

Holler and Napel argued that the PGI shows non-monotonicity with respect to the vote distribution (and this confirms that the measure does not satisfy LM) if the game is not decisive and no winning coalition may exist, as in the above weighted voting game $v° = (51; 35, 20, 15, 15, 15)$, or if it improper, i.e., two winning coalition may exist the same time [27, 28]. However, there are voting games that are proper and decisive and still LM is not satisfied for the PGI. And yet, a violation of LM suggests that perhaps we should worry about the design of the decision situation. The fact that the PGI shows such violations, while other power measures do not, could be considered an asset and not a pathology. But what is the reason for PGI violating LM? How can we cure a violation of LM?

The more popular power measures, i.e., the Shapley–Shubik index or the Banzhaf index satisfy LM and thus do not indicate any particularity if the game is neither decisive nor proper. Yet, these measures also show a violation of LM if we consider a priori unions and the equal probability of permutations and coalitions, respectively, does no longer apply[12]. This suggests that a deviation of the equal probability of coalitions causes a violation of LM. Note since the PGI considers MWC only, this is formally equivalent to put a zero weight on coalitions that have surplus players. Is this the ("technical") reason why the PGI may show non-monotonicity?

Instead of accepting the violation of LM, we may ask which decision situations guarantee monotonic results for the PGI. An answer to this question may help to design adequate voting bodies. Obviously, the PGI satisfies LM for unanimity games, dictator games and symmetric games. The latter are games that give equal power to each voter; in fact, unanimity games are a subset of symmetric games. Note that for these types of games the PGI is identical with the normalized Banzhaf index.

Holler et al. analyse alternative constraints on the number of players and other properties of the decision situations [33]. For example, it is obvious that LM will not be violated by any of the known power measures, including PGI, if there are n voters and $n - 2$ voters are dummies. It is, however, less obvious that LM is also satisfied for the

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[11]See [25] and [29] for this argument.

[12]See [3] for examples of voting games with a priori unions and [6] as well as [5] for a discussion. Closely related are voting games with incompatible players, see [5] for the PGI and [7] for Banzhaf index.

PGI if one constrains the set of games so that there are only $n – 4$ dummies, i.e., voting games with 4 players or less satisfy LM.

This illustrates the approach that has been labelled constrained monotonicity [33]. We can think of many set of constraints that guarantee LM for PGI. A hypothesis that needs further research is that the PGI does not show a violation of LM if the voting game is decisive and proper and the number of decision makers is lower than 6. The idea of restricting the set of games such that LM applies for PGI has been further elaborated in [4] in the form of weighted monotonicity of power. The core of this concept is to give the various MWC weights such that the modified PGI satisfies monotonicity. Then we get PGI-monotonicity. These considerations are relevant for all power indices if we drop the equal probability assumption and, for example, take the possibility of a priori unions into account.

Following the approach chosen in [33], Freixas and Kurz [19] distil a subset of (weighted) voting games which guarantee LM for the PGI and the Deegan–Packel index. They show that LM is satisfied for this measure if the voting game is uniform, i.e., if all MWC have the same number of members. This is a sufficient condition, but it is not necessary as Freixas and Kurz demonstrate using the game example $v^* = (3; 2, 1, 1, 1)$. This game is not uniform but satisfies LM (the game is proper and decisive). There are three MWC of two members and one with three members. The PGI values are $h(v^*) = (3/9, 2/9, 2/9, 2/9)$. They are identical to the values of the normalized Banzhaf index for this voting game. As already mentioned, a core result of [33] is that the PGI satisfies LM is the number of voters is four or less – so uniformity is not needed to guarantee LM for voting game v^*.

Widgrén [51] proved the following linear relationship that relates the normalized Banzhaf index and the PGI[13].

$$\beta_i = (1 - \gamma)h_i + \gamma\varepsilon_i \tag{3}$$

where

$$\varepsilon_i = \frac{\bar{c}_i}{\sum_{i\in N} \bar{c}_i} \quad \text{and} \quad \gamma = \frac{\sum_{i\in N} \bar{c}_i}{\sum_{i\in N} c_i}$$

Here, c_i represents the number of coalitions that contain player i as a swing player and $\bar{c}_i$ represents the number of coalitions which have a swing player i but are not minimum winning. Loosely speaking, the coalitions represented by $\bar{c}_i$ are the source of the

[13]Widgrén [51] uses the symbols θ_i for the PGI and C_i for the set of coalitions that contain i as a swing player. Correspondingly, c_i is the number of elements of C_i.

difference between the normalized Banzhaf index, β_i, and the PGI, h_i. Can we identify the corresponding factors in Eq. (3) as the cause for the violation of LM that characterizes the PGI, but not the Banzhaf? Can we see from the properties of this factor whether the PGI will indicate a violation for a particular game, or not? These questions have not been answered so far, but it is immediate from Eq. (3) that the PGI satisfies LM for unanimity games, dictator games and symmetric games. For these games $\gamma = 0$ and the PGI equals the normalized Banzhaf index (which satisfies LM for all voting games).

Freixas and Kurz introduce a class of new indices that result out of convex combinations of PGI and Banzhaf index choosing weights such that LM is satisfied [19]. The resulting indices are more solidary than the Banzhaf index and perhaps a better *yardstick for doing a fair division of a public good*. It turns out that there are *costs* [of obtaining LM inasmuch as] *the achievable new indices satisfying LM are closer to the Banzhaf index than to the Public Good Index* [19]. If we reconsider that LM is implied by the dominance relation and thus by the desirability relation, proposed in [18] to define *reasonable power measures*, then it seems that there is not much space to take care of solidarity. However, this discussion needs, first of all, definitions of solidarity, fairness, etc. Still, the class of new indices, elaborated in [19] prepares the ground and raises relevant questions.

The approach of [19] is somewhat related to project of strict proportional representation through randomizing over several decision rules applied to a particular vote distribution such that the weighted power indices are identical to the seat distribution (see [24, 8, 49] for this approach[14]). Then, LM is of course guaranteed. However, in this exercise the convex combination implied by the randomization was over indices of one kind, only. Combinations of indices of different proveniences were not considered and no indication of more or less solidarity could be derived.

We conclude this section with two questions that we could not answer to our satisfaction: How is the Freixas–Kurz model [19] of combining Banzhaf index and PGI related to Widgrén's Eq. (3) above? Do the weights, proposed by Freixas and Kurz, specify PGI-monotonicity?

8. Non-monotonicity of voting and of voting power. A research project

If we abstract from preferences and focus on a priori voting power in the case of the no-show paradox, illustrated in Table 1, then we see that the power of the group of 47 voters carries over to the group of 2 voters. The latter has no power if the 47 voters

[14]Related, [35] presents an "average representation of games" such that the resulting power distributions are proportional to the distribution of weights. Again, LM is guaranteed.

vote while the group of 47 voters have a power of 1/3 if measured by the standard power indices. However, if the 47 voters abstain then the power of the group of 2 voters is 1/3 while the power of the group of 47 voters is zero. Since the members of the group of 2 voters have identical preferences as the members of the group of 47 voters, the group of 2 voters is an adequate representation of the group of 47 voters.

Of course, we get the same result, including the no-show paradox, as long as the active representatives of the preference ordering B $\succ$ C $\succ$A are not numerous enough to make B an alternative in the second round competing with A. Given the votes and preferences for the groups of 26 voters and 25 voters, this is the case when the active representatives of the preferences have less than 25 votes. On the other hand, the active representatives have to number more than 1 vote so that alternative C gets a majority of votes in the second round when competing with A. Thus Table 1 represents an extreme case highlighting a paradoxical effect of no-show.

There is a non-monotonicity in this example that relates votes, power and preferences. By forgoing the casting of votes and thus reducing the voting power to zero, the group of 47 voters gets a better outcome, captured by the winning alternative, than by making use of its votes and therefore exercising voting power. However, abstaining is also making use of voting power as obviously it has an impact on the outcome and corresponds to the agents will[15]. Note that in the discussion of the paradoxes in Chapters 2–4, coalition formation was not considered: sincere voting was assumed and voters voted according to their preferences. However, the no-show paradox proposes that all the members of a group of voters abstain, i.e., behave like a bloc. Moreover, the paradox relies on that the members of another group, characterized by the same preference ordering as the abstaining group, vote. The coordination between the two groups is not discussed. It seems implicit that the two groups behave like a coalition, splitting votes, in order to avoid there least preferred alternative.

However, if we generalize this assumption then we enter the realm of power indices. If so, we can discuss whether the no-show paradox corresponds to the paradox of the quarreling members. This paradox is based on the assumption that coalitions are impossible that contain members that have a "quarrel". The paradox prevails if the sum of power values of two quarreling members X and Y is larger than the power value of the union of X and Y, i.e., $\pi(X) + \pi(Y) > \pi(X \cup Y)$[16]. This result can be observed for any standard measure of voting power.

But are we allowed to add up the power values? If the preferences of A and B are identical, this seems to be justifiable – as in the case of the groups of 47 voters and 2 voters in preference profile given in Table 1. We conclude: The potential to coordinate

[15]This relates to Max Weber's definition of power: *In general, we understand by "power" the chance of a man or of a number of men to realize their own will in a communal action even against the resistance of others who are participating in the action* [50].

[16]For the paradox of quarreling members, see [36] and [11].

votes within a bloc generates extra voting power. The case of the no-show paradox illustrated in Table 1 gives substance to the paradox of quarreling members.

If there is coordination such that the splitting up is successful, then of course we could also think that, given a quota of 51 votes in the first round, the members of the group of 47 voters vote strategically. Instead of abstaining, they could transfer 24 and 23 of their votes for A and C, respectively, in the first round, supporting a run-off of A and C in the second round. In the second round all 47 members could vote for C, further augmenting the voting share of the winning alternative C.

An abstention of 47 votes in a committee of 100 does not look good. On the other hand, one might argue that the voting of 24 members of the 47 group in favor of alternative A does not seem to be realistic. Nor is the splitting of votes of voters with preference orderings $B \succ C \succ A$ convincing. Moreover, the basic assumption of the no-show paradox is that the other group of voters do not abstain and vote sincerely. Is this realistic?

Such peculiarities could be observed when Urho Kekkonen was elected to be President in Finland in 1956. In the second round (out of a three-round run-off voting procedure), a sufficiently large share of representatives of the socialist-communist party SKDL casted their votes for their least preferred candidate K.A. Fagerholm and the others voted for Kekkonen, just to make sure that the Condorcet winner J.K. Paasikivi will not participate in the decisive run-off voting round. The other parties voted for their most preferred candidates. As a result Kekkonen was elected in the third round. He and all other candidates would have been defeated by Paasikivi in pairwise comparison [37].

Kekkonen was president of Finland for the next 25 years. The no-show paradox in Table 1 and the case of Kekkonen's election clearly demonstrate that we cannot rely on preferences only, if we want to forecast the outcome of voting. In the two cases, "intention" matters, but also the pure numbers of votes and the voting procedure given by the quota and the sequence of rounds. Power indices are taking care of the latter.

Recent party politics and coalition formation in the German Bundestag can also serve as a nice illustration of the interrelationship between a priori voting power, determined by the seat shares and the decision rules, and party preferences, representing political ideologies. In the election of September 22, 2013, the CDU/CSU lost its partner FDP from the previous government as the Liberals' vote share was below the 5% threshold necessary for entering the Bundestag. The forming of a coalition government was cumbersome, but also necessary because even the 311 seats of the CDU/CSU were not sufficient for a majority government as the total number of seats were 631. The second strongest faction was formed by Social Democrats (SPD) which controlled 193 seats, while the Green Party and the left-wing party "Die Linke" were only represented by 63 and 64 seats, respectively. Given the seat distribution (311, 193, 63, 64) the set of MWC is obvious, but some of them were not feasible because of *unsurmountable differences in the party preferences*. The CDU/CSU bargained with the Green Party, but even this coalition turned out to be infeasible because of the diversity of political opinions. Alternatively, a red-red-green coalition of the SPD, Die Linke, and Green Party was discussed in the public, but less so among politicians, because the SPD leadership made clear that the party's political position does not allow to cooperate with Die Linke in a coalition. In the end, the Große Koalition of CDU/CSU and SPD was formed.

Already the day after there were party members of the CDU/CSU and the Greens that suggested that the two parties revise their policy positions so that they can form coalitions in the future. Also to prepare for this, the two parties formed coalition governments in several regions, i.e., Bundesländer. The argument for this inclination for convergence is based on numbers, i.e., on a priori voting power. On the one hand, the CDU/CSU wants to have an alternative to the SPD as coalition partner, on the other, it is not so sure that the incompatibility of SPD and Die Linke is of permanence. There could be a day when a red-red-green coalition will be feasible. And then the CDU/CSU must be in a position to make the Green Party an offer that this party cannot reject, but which is also consistent with its own perspective.

These considerations were driven by numbers. This should not be surprising since the power to govern is encapsulated in number of seats and majority quota. Now the considerations have to be revised, and the revision already started, because it is expected that a new player entered the playfield, the right-wing party AfD, with an expected share of 10–12%. Moreover, perhaps even the Liberals might come back to the Bundestag, outrunning the 5% threshold. Both possibilities will have immediate numerical effects, but of course the more exciting dimension of coalitions is embedded in party preferences.

To sum up: Taking care of the preferences of the voters gives a justification for treating group of voters as bloc that it interacts with other blocs of voters forming winning coalitions. The power of such a bloc can then be measured by one of the indices, depending which index seems more appropriate for the underlying problem: is it the division of a cake (or political spoils) or on the specification of a public good? A related question is: Can power index theories contribute to the analysis of the aggregation of preferences? Note this question does not imply that we should consider preferences when measuring voting power[17]. But the units that enter the voting power analysis (factions of party representatives, voting blocs, etc.) are the result of the preferences of their members. Nevertheless, this does not necessarily mean that the internal aggregation problem is solved and the units vote in accordance to these preferences.

More specifically, we can ask whether there is a relationship between the paradoxes in the aggregation of preferences and the non-monotonicities we find in the application

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[17]This refers to extensive discussion that is summarized in [41] and [10] which reflect the two controversial views on the *possibility and impossibility of a preference-based power index*. See also [30] for further details.

of power indices. Again, this not to say that we have to consider preferences when measuring power. But we cannot speak of the success of a vote without considering preferences, and success, the matching of outcome and preferences, is what most voters want.

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