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DETERMINING MODELS OF INFLUENCE

We consider a model of opinion formation based on aggregation functions. Each player modifies his opinion by arbitrarily aggregating the current opinion of all players. A player is influential on another player if the opinion of the first one matters to the latter. Generalization of an influential player to a coalition whose opinion matters to a player is called an influential coalition. Influential players (coalitions) can be graphically represented by the graph (hypergraph) of influence, and convergence analysis is based on properties of the hypergraphs of influence. In the paper, we focus on the practical issues of applicability of the model w.r.t. a standard framework for opinion formation driven by Markov chain theory. For a qualitative analysis of convergence, knowing the aggregation functions of the players is not required, one only needs to know the set of influential coalitions for each player. We propose simple algorithms that permit us to fully determine the influential coalitions. We distinguish three cases: a symmetric decomposable model, an anonymous model, and a general model.

Keywords: social network, opinion formation, aggregation function, influential coalition, algorithm

1. Introduction. Dynamic models of opinion formation

Models of opinion formation are widely studied in psychology, sociology, economics, mathematics, computer sciences, among others; for overviews, see, e.g., [32, 1]. A seminal model of opinion formation and imitation was introduced in [16]. In that model, individuals in a society start with initial opinions on a subject. The interaction patterns are described by a stochastic matrix, whose entry in row j and column k represents the weight that player j places on the current belief of player k in forming j's belief for the next period. These beliefs are updated over time.

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While Degroot assumes that players update their opinion by taking weighted averages of the opinions of all players [16], Grabisch and Rusinowska investigated a model of opinion formation in which players update their beliefs according to arbitrary aggregation functions [27]. Foerster et al. study a model of opinion formation in which ordered weighted averages are used in the process of updating information [21]. In this paper, we consider the model of influence based on aggregation functions [27] and discuss practical issues of applying this model w.r.t. the standard framework for opinion formation driven by Markov chain theory. For a full qualitative analysis of the convergence of opinions, i.e., determining all the terminal classes (without their probabilities), it is sufficient to identify influential coalitions, which can be easily obtained by interviewing the agents. The aim of this paper is to show that a full qualitative analysis of convergence is feasible in practical situations. We introduce simple algorithms that permit us to fully determine the influential coalitions in three cases: the symmetric decomposable model (influential coalitions reduce to individuals), the anonymous model (only the number of agents matters, not their identity), and the general model. We show how clues on convergence can be obtained in a simple way, even without determining the reduced transition matrix.

There exists a vast literature that presents other variations and extensions of the DeGroot model. We briefly recall some of them. In particular, Jackson [32] and Golub and Jackson [26] investigate a model, in which players update their beliefs by repeatedly taking weighted averages of their neighbors' opinions. According to these authors, one of the issues regarding the DeGroot framework concerns necessary and sufficient conditions for convergence of the social influence matrix and reaching a consensus (see additionally [9]). Jackson also examines the speed of convergence of beliefs [32], and Golub and Jackson analyze, in the context of the DeGroot model, whether consensus beliefs are correct, i.e., whether beliefs converge to the right probability, expectation, etc [26]. The authors consider a sequence of societies, where each society is strongly connected and convergent in opinions, and described by its updating matrix. In each social network of the sequence, the belief of each player converges to the consensus limit belief. There is a true state of nature, and the sequence of networks is wise if the consensus limit belief converges in probability to the true state as the number of societies grows.

Several other generalizations of the DeGroot model can be found in the literature, e.g., models in which the updating of beliefs can vary in time and circumstances (see e.g., [17, 35, 37, 23, 24]). In the model described by Demarzo et al., players in a network try to estimate some unknown parameter [17]. This model allows updating to vary over time, i.e., a player may place more or less weight on his own belief over time. The authors study the case of multidimensional opinions, in which each player has a vector of beliefs. They show that, in fact, individuals' opinions can often be well approximated by a one-dimensional line, where a player's position on the line determines his position on all issues. Friedkin and Johnsen study a similar framework, in which social attitudes depend on the attitudes of neighbors and evolve over time [23, 24]. In their model, players start with initial attitudes and then mix in some of their neighbors' recent attitudes with their starting attitudes.

Also, other works in sociology related to influence are worth mentioning, e.g., the eigenvector-like notions of centrality and prestige [33, 10, 11], and models of social influence and persuasion by French [22] and Harary [29] (see also [39]). A sociological model of interactions on networks is also presented by Conlisk [13] (see also [14, 15, 36]), who introduces interactive Markov chains, in which every entry in a state vector at each time represents the fraction of the population with some attribute. The matrix depends on the current state vector, i.e., the current social structure is taken into account to model how sociological dynamics evolve. Threshold models of collective behavior are discussed by Granovetter [28]. In these models, agents have two alternatives and the costs and benefits of each depend on how many other agents choose which alternative. The author focuses on the effect of individual thresholds (i.e., the proportion or number of others that make their decision before a given agent) on collective behavior, discusses an equilibrium in a process occurring over time and the stability of equilibrium outcomes. Another model of influence is studied by Asavathiratham [2] and Asavathiratham et al. [3]. This model consists of a network of nodes, each with a status evolving over time. The evolution of status acts according to an internal Markov chain, but the transition probabilities depend not only on the current status of the node, but also on the statuses of the neighboring nodes.

More research on interaction is presented by Hu and Shapley [31, 30]. The authors apply the command structure of [38] to model players' interactions using simple games. For each player, boss sets and approval sets are introduced, and based on these sets, a simple game called the command game for a player is built. Hu and Shapley introduce an authority distribution over an organization and the (stochastic) power transition matrix, in which the entry in row j and column k is interpreted as agent j's power transfered to k [30]. The authority equilibrium equation is defined. In [30], multi-step commands are considered, where commands can be implemented through command channels.

There is also a vast literature on learning in the context of social networks; see e.g. [6, 18–20, 4, 5, 25, 12, 7]. In general, in models of social learning, agents observe choices over time and update their beliefs accordingly, which is different from models where choices depend on the influence of others.

The paper is organized as follows. Section 2 presents fundamental material on models of influence based on aggregation functions, as well as establishing notation and terminology, and recalls an essential result, which is the basis for determining the qualitative part of the model of influence. Section 3 addresses the problem of determining a model of influence in practice and focuses on determining its qualitative part, which is sufficient for a qualitative analysis of convergence. Section 4 gives some concluding remarks.

2. A model of influence based on aggregation functions

In this section, we recapitulate a model of influence based on aggregation functions [27]. Consider a set $N := \{1, ..., n\}$ of players that have to make a yes-no decision on a certain issue. Each player has an initial opinion, which may change due to mutual interaction (influence) between players. By $b_{S,T}$ we denote the probability that the set S of yes-voters becomes T after one step of influence. We assume that the process of influence may iterate, and therefore obtain a stochastic process of influence, depicting the evolution of the coalition of yes-players in time. We assume that the process is Markovian ($b_{S,T}$ depends on S and T, but not on the whole history) and stationary ($b_{S,T}$ is constant over time). The states of this finite Markovian process are all subsets $S \subseteq N$ representing the set of yes-players, and we also have the transition matrix $\mathbf{B} := [b_{S,T}]_{S,T \subseteq N}$, which is a $2^n \times 2^n$ row-stochastic matrix.

For a qualitative description of the convergence of the process, it is sufficient to know the reduced matrix $\tilde{\mathbf{B}}$ given by

$$\tilde{b}_{S,T} = \begin{cases} 1, & \text{if } b_{S,T} > 0 \\ 0, & \text{otherwise} \end{cases}$$

and equivalently represented by the transition graph $\Gamma = (2^N, E)$, where E is the set of arcs, its vertices are all possible coalitions, and the arc (S, T) from state S to state T exists if and only if $\tilde{b}_{S,T} = 1$.

Definition 1. An n-place aggregation function is a mapping $A: [0, 1]^n \to [0, 1]$ satisfying

(i) $A(0, ..., 0) = 0$, $A(1, ..., 1) = 1$ (boundary conditions),

(ii) If $\mathbf{x} \le \mathbf{x}'$ then $A(\mathbf{x}) \le A(\mathbf{x}')$ (nondecreasingness).

To each player $i \in N$ we associate an aggregation function A_i, which specifies the way player i modifies his opinion based on the opinions of the other players. Let $\mathbf{A} := (A_1, ..., A_n)$ denote the vector of aggregation functions. We compute $\mathbf{A}(1_S)$ $= (A_1(1_S), ..., A_n(1_S))$, where 1_S is the characteristic vector of S, and $A_i(1_S)$ indicates the probability of player i saying yes at the next step when the set of agents presently saying yes is S. We assume that these probabilities are independent over the set of agents. Hence the probability of transition from the yes-coalition S to the yes-coalition T is given by

$$b_{S,T} = \prod_{i \in T} A_i(1_S) \prod_{i \notin T} (1 - A_i(1_S)). \tag{1}$$

A detailed study of convergence under this model is provided in [27]. It is shown, in particular, that three types of terminal class[2] can exist: singletons, cycles, and regular terminal classes. The first case occurs when a class is reduced to a single coalition (called the terminal state). The second one is the case where no convergence occurs because the process endlessly cycles over a sequence of coalitions, and the last case occurs when the class is a Boolean lattice of the form $\{S \in 2^N \mid K \subseteq S \subseteq L\}$ for some sets $K, L.$ In any case, N and $\varnothing$ are terminal states (called trivial terminal states).

We emphasize two particular aggregation functions. The first one is the well-known weighted arithmetic mean (WAM), defined by

$$\mathrm{WAM}_w(x_1, ..., x_n) = \sum_{i=1}^{n} w_i x_i$$

where $w = (w_1, ..., w_n)$ is a weight vector, i.e., $w \in [0, 1]^n$ with the property $\sum_{i=1}^{n} w_i = 1.$ Weighted arithmetic means are used in most models of opinion formation, e.g., the DeGroot model. Another noteable aggregation function is the ordered weighted arithmetic mean (OWA) [40], defined by

$$\mathrm{OWA}_w(x_1, ..., x_n) = \sum_{i=1}^{n} w_i x_{(i)}$$

where w is a weight vector, and the inputs have been arranged in decreasing order: $x_{(1)} \geq ... \geq x_{(n)}$. Note that, unlike in the case of WAM, these weights do not act on inputs, but on the rank of the inputs, so that the minimum and the maximum are particular cases, by taking $w = (0, ..., 0, 1)$ and $w = (1, 0, ..., 0),$ respectively. Applied to our context of influence where the input vectors are binary, if each agent aggregates his opinions according to an OWA, we obtain a model of anonymous influence, because each agent

[2]A class is a maximal collection of coalitions such that for any two distinct coalitions S, T in the class, there exists a sequence of transitions inside the class leading from S to T. A class is terminal if no transition to a coalition outside that class is possible.

updates his opinion according to the number of agents saying yes, not to which agents say yes. Models of anonymous influence have been studied in detail in [21].

Definition 2. Let A_i be the aggregation function of agent *i*. A nonempty coalition $S \subseteq N$ is yes-influential for *i* if

(i) $A_i(1_S) > 0$,

(ii) for all $S' \subset S$, $A_i(1_{S'}) = 0$.

Similarly, a coalition *S* is no-influential for i if

(i) $A_i(1_{N \setminus S}) < 1$,

(ii) for all $S' \subset S$, $A_i(1_{N \setminus S'}) = 1$.

We denote by $\mathcal{C}_i^{\text{yes}}$ and $\mathcal{C}_i^{\text{no}}$ the collections of yes- and no-influential coalitions for *i*. Coalition S is yes-influential for player i if, when the players in S say yes and every other player says no, i has a positive probability of saying yes (and similarly for no-influential coalitions). Moreover, S has no superfluous player. If an influential coalition is formed by only one player, then we call it an influential player. Note that these collections are never empty, since if no proper subcoalition of N were yes- or no-influential, then N would be both yes- and no-influential by Definition 2. More importantly, each such collection is an antichain in 2^N, that is, for any two distinct members of the collection S, S', $S \not\subset S'$ and $S' \not\subset S$.

Influential players can easily be represented in a directed graph. Define $G_{\mathbf{A}}^{\text{yes}}$, the graph of yes-influence, as follows: the set of nodes is the set of agents *N*, and there is an arc (j, i) from j to i if j is yes-influential on *i*. The graph of no-influence $G_{\mathbf{A}}^{\text{no}}$ is defined similarly. The representation of influential coalitions requires the more complex notion of a hypergraph.

Definition 3. We define the following concepts:

(i) A hypergraph [8] H is a pair $(N, \mathcal{E})$ with N being the set of nodes and $\mathcal{E}$ the set of hyperedges, where a hyperedge $S \in \mathcal{E}$ is a nonempty subset of *N*. If $|S| = 2$ for all $S \in \mathcal{E}$, then we have a classical graph.

(ii) A directed hypergraph on N is a pair $N, \mathcal{E}$ with $\mathcal{D}$ being the set of directed hyperedges, where a directed hyperedge is an ordered pair (S', S'') (called an hyperarc from S' to S''), with S', S'' both being nonempty.

(iii) A directed hyperpath from i to j is a sequence $i_0(S_1', S_1'')i_1(S_2', S_2'')i_2 \ldots i_{q-1}(S_q', S_q'')i_q$, where $i_0 := i, i_1, \ldots, i_{q-1}, j =: i_q$ are nodes, $(S_1', S_1''), \ldots, (S_q', S_q'')$ are hyperarcs such that $S_k' \ni i_{k-1}$ and $S_k'' \ni i_k$ for all $k = 1, \ldots, q$.

We define the hypergraphs $H_{\mathbf{A}}^{\text{yes}}, H_{\mathbf{A}}^{\text{no}}$ of yes-influence and no-influence as follows: for $H_{\mathbf{A}}^{\text{yes}}$, the set of nodes is N, and there is a hyperarc $(C, \{i\})$ for each $C \in \mathcal{C}_i^{\text{yes}}$ (similarly for $H_{\mathbf{A}}^{\text{no}}$).

Grabisch and Rusinowska [27] prove that the hypergraphs $H_{\mathbf{A}}^{\text{yes}}, H_{\mathbf{A}}^{\text{no}}$ (equivalently, the collections $\mathcal{C}_i^{\text{yes}}$ and $\mathcal{C}_i^{\text{no}}$ for all $i \in N$) are equivalent to the reduced matrix $\tilde{\mathbf{B}}$, and therefore contain the entire qualitative description of convergence.

Theorem 1. Consider an influence process $\mathbf{B}$ based on the aggregation functions $\mathbf{A}$. Then $\tilde{\mathbf{B}}$ can be reconstructed from the hypergraphs $H_{\mathbf{A}}^{\text{yes}}$ and $H_{\mathbf{A}}^{\text{no}}$ as follows: for any $S, T \in 2^N$, $\tilde{b}_{S,T} = 1$ if and only if

1. For each $i \in T$, there exists a nonempty $S_i' \subseteq S$ such that S_i' is yes-influential on *i*, i.e., $S_i' \in \mathcal{C}_i^{\text{yes}}$,

2. For each $i \notin T$, there exists a nonempty S_i'' such that $S_i'' \cap S = \varnothing$ and S_i'' is no-influential on *i*, i.e., $S_i'' \in \mathcal{C}_i^{\text{no}}$.

In particular, $\tilde{b}_{\varnothing,T} = 0$ for all $T \neq \varnothing$, $b_{\varnothing,\varnothing} = 1$, and $\tilde{b}_{N,T} = 0$ for all $T \neq N$, $b_{N,N} = 1$. Recall that (1) is valid only if the probabilities of saying yes are independent over the set of agents. Therefore, non-independence in this sense makes the determination of the transition matrix difficult. However, $\tilde{\mathbf{B}}$ is insensitive to possible correlation between agents, because $\tilde{b}_{S,T} = 1$ if and only if $A_i(1_S) > 0$ for every $i \in T$ and $A_i(1_S) < 1$ for every $i \notin T$, regardless of the correlation between agents.

3. Determination of the model

An important issue concerns the determination of a model of influence of the above type in a practical situation. This implies that we are making essentially two assumptions:

1. Each agent aggregates the opinion of all the other agents to form his opinion in the next step.
2. The aggregation function is monotonically increasing.

The latter assumption implies that anti-conformist behaviors (i.e., the more individuals say yes, the more I am inclined to say no) cannot be modeled in this framework.

3.1. General considerations

Complete determination of the model amounts to identifying either the transition matrix **B** or all the aggregation functions $A_1, \ldots, A_n$ (assuming the absence of correlation). Considering the size of the matrix $\mathbf{B}$ ($2^n \times 2^n$), statistical determination of $\mathbf{B}$ seems to be nearly impossible, unless a huge number of observations are made. As for the determination of the aggregation functions, the situation is even worse, since questioning agents about their aggregation functions (type, parameters) appears to be quite unrealistic. We know from Section 2 that knowledge of the reduced matrix $\tilde{\mathbf{B}}$ is enough to obtain a qualitative description of the convergence of the model, which is insensitive to possible correlations between agents. Moreover, knowledge of $\tilde{\mathbf{B}}$ (size 2^{2n}) is equivalent by Theorem 1 to knowledge of the collections of all yes- and no-influential coalitions of the size at most $2n\begin{pmatrix} n \\ \left\lfloor \frac{n}{2} \right\rfloor \end{pmatrix}$, which is, in turn, equivalent to knowledge of the hypergraphs of yes- and no-influence. In some favorable cases (e.g., the WAM model), the hypergraphs reduce to ordinary graphs. This immediately indicates two ways of identifying the (qualitative part of the) model: either by observation of the transitions, i.e., the evolution of the coalition of the yes agents, or by interviewing the agents. In the first case, observing a transition from S to T yields $\tilde{b}_{S,T} = 1.$ In the second case, interviews permit us to determine influential coalitions or graphs of influence.

In the remaining part of this section, we mainly focus on the second approach. Concerning the first one, we only mention an important fact. The underlying assumptions of the model mean that the reduced matrix $\tilde{\mathbf{B}}$ is not arbitrary and has specific properties. Recall that $\tilde{b}_{S,T} = 1$ if and only if for all $i \in T,$ $A_i(1_S) > 0$ and for all $i \notin T,$ $A_i(1_S) < 1.$ This implies the following fact:

Fact 1. For a given $S \subseteq N,$ $S \neq \varnothing, N,$ the candidates transitions are all sets of the form $T = K \cup L,$ where

$$K = \{i \in N \mid A_i(1_S) = 1\}$$

$$L \subseteq \{i \in N \mid 0 < A_i(1_S) < 1\}$$

In other words, $\bigcap T$, the intersection of all possible transitions from S yields the set $K = \{i \in N \mid A_i(1_S) = 1\},$ while $N \setminus \bigcup T$ yields $K' = \{i \in N \mid A_i(1_S) = 0\}.$ When S

increases, K increases, while K' decreases. This fact permits us to detect, when $\tilde{\mathbf{B}}$ is constructed from observations, possible deviations from the model (e.g., presence of anti-conformists).

3.2. Determination of influential coalitions

We may distinguish three cases, according to the type of underlying model:

1. WAM model (symmetric decomposable model): all aggregation functions are weighted arithmetic means.
2. OWA model (anonymous model): all aggregation functions are ordered weighted averages.
3. General model (no special assumptions).

The symmetric decomposable model. The case of the WAM model is particularly simple and has been studied in depth in [27]. It has been proved to be equivalent to a symmetric decomposable model. An aggregation model is decomposable if for every agent $i \in N$, every yes- and no-influential coalition for agent i is a singleton. Now, an aggregation model is symmetric if a yes-influential coalition for i is also no-influential for i and vice versa, for every $i \in N$. Note that the first property implies that the hypergraphs of yes- and no-influence reduce to ordinary graphs, while the second property implies that the two graphs are identical, and therefore the whole (qualitative) model is represented by a single graph representing influence. This makes interviewing agents particularly simple: it suffices to ask to every agent whom he asks for advice. Then, i asks j for advice is translated into the graph representing influence by an arc from j to i.

We applied this technique to a real case [27], namely the manager network of Krackhardt [34]. The agents are the 21 managers of a small manufacturing firm in the USA, and the network is obtained as follows: each agent k is asked if he/she thinks that agent i asks agent j for advice. An arc from j to i is placed in the graph representing influence if a majority of agents think that i asks j for advice. From the graph, and due to the properties of symmetric decomposable models, many conclusions can be easily drawn on the convergence of the model. In particular, it is possible to detect the presence of regular terminal classes (Theorem 8 in [27]). There is also a simple criterion to determine if there is no regular terminal class: it suffices that for each agent i, there is an arc in the influence graph from cl(i) to every agent outside cl(i), where cl(i), the closure of i, is the set of agents who can reach i by a path in the influence graph.

The anonymous model. According to the OWA model, agents do not change their opinion due to particular individuals but due to the number of individuals saying yes.

Therefore, in general, these are not decomposable models, and one needs to determine influential coalitions as in the general case. However, because according to these models the players are anonymous, a collection $\mathcal{C}_i^{\text{yes}}$ or $\mathcal{C}_i^{\text{no}}$ is composed of all sets of a given size s, $1 \le s \le n$, and this is characteristic of an anonymous model. Therefore, under the assumption of anonymity, it suffices to ask to every agent i the following questions:

Q1. Suppose that your opinion on some question is yes. What is the minimal number of agents saying no that may make you change your opinion?

Q2. Suppose that your opinion on some question is no. What is the minimal number of agents saying yes that may make you change your opinion?

Assuming that the answers are respectively s and s', it follows that

$$\mathcal{C}_i^{\text{no}} = \{S \in 2^N \,\|\, S| = s\}, \qquad \mathcal{C}_i^{\text{yes}} = \{S \in 2^N \,\|\, S| = s'\}$$

Now, it is easy to see that given s, s' for agent i, one can get the form of the weight vector w in the aggregation function OWA_w of agent i (Proposition 2 in [21]):

$$w = (\underbrace{00\cdots 0}_{s'-1 \text{ zeros}}, \bullet\bullet\cdots\bullet, \underbrace{00\cdots 0}_{s-1 \text{ zeros}})$$

where $\bullet$ indicates any nonzero weight. In particular, all agents are yes-influential (no--influential) on i if and only if $w_1 > 0$ ($w_n > 0$).

As for convergence under this model, it is shown in [21] (Proposition 3) that no cycle can occur but the two other types of terminal classes may occur. Terminal states are easily detected as follows: S of size s is a terminal state if and only if for every $i \in S$, the size of a no-influential coalition is at least $n-s+1$, and for every $i \notin S$, the size of a yes-influential coalition is at least $s+1$. The absence of regular terminal classes can also be characterized only through influential coalitions but this condition is more complex (see Corollary 3 in the aforementioned paper).

The general model. We now address the general case, where no special assumption is made on the model, except the following: we assume that each agent is yes- and no--influential on himself, which means that $A_i(1_i) > 0$, $A_i(1_{N\setminus i}) < 1$ (in other words, the agent trusts his opinion to a nonnull extent). This induces some simplification in the algorithm, but it would not be difficult to generalize it, in order to overcome this limitation.

Interview for Agent i

0. Set $\mathcal{C}_i^{\text{yes}} = \{\{i\}\}$, $\mathcal{C}_i^{\text{no}} = \{\{i\}\}$, $\mathcal{N}_i^{\text{yes}} = \mathcal{N}_i^{\text{no}} = 2^{N\setminus i}$

% $\mathcal{N}_i^{\text{yes}}, \mathcal{N}_i^{\text{no}}$ are the sets of candidate coalitions.

1. For each agent $j \in N$, $j \neq i$, ask[3]:

Q. Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if Agent j says no?

If the answer is positive:

- add $\{j\}$ to C_i^{no}, and discard $\{j\}$ and all sets containing j from N_i^{no}.
- If $N_i^{\text{no}} = \varnothing$, STOP (GO TO STEP 3).

Otherwise, discard $\{j\}$ from N_i^{no}.

2. For $\ell = 2$ to $n-1$, do:

2.1. Define $\mathcal{S} = \{S \in N_i^{\text{no}} : |S| = \ell\}$.

2.2. Ask Q: Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if one of the coalitions in $\mathcal{S}$ says no? In the case of affirmative answer, for which ones?

For every set S answered, do:

- add S to C_i^{no} and discard all supersets of S from N_i^{no}
- If $N_i^{\text{no}} = \varnothing$ or if $|C_i^{\text{no}}| = \begin{pmatrix} n \\ \lfloor \frac{n}{2} \rfloor \end{pmatrix}$, STOP (GO TO STEP 3).

2.3. Set $N_i^{\text{no}} \leftarrow N_i^{\text{no}} \setminus \mathcal{S}$.

3. Exactly as in Steps 1 and 2 for C_i^{yes}, Question 1 becomes: Suppose that your opinion on some question is no. Would you be inclined to change your opinion if Agent *j* says yes, etc.?

We give some examples.

Example 1. (braces are omitted when denoting coalitions) Consider $N = \{1, 2, 3, 4, 5\}$. We detail the process of interviewing Agent 1.

1. We have $C_1^{\text{no}} = \{1\}$. We take agent 2.

Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if Agent 2 says no?

Answer: Yes. Hence, $C_1^{\text{no}} = \{1, 2\}$, and $N_1^{\text{no}} = \{3, 4, 5, 34, 35, 45, 345\}$.

2. Agent 3.

Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if Agent 3 says no?

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[3]As in Step 2, it is possible to gather all these questions into a single one: Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if one of the agents in $N \setminus i$ says no? In the case of an affirmative answer, for which ones?

Answer: No. Thus $N_1^{\text{no}} = \{4, 5, 34, 35, 45, 345\}$.

3. Agent 4.

Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if Agent 4 says no?

Answer: No. Thus $N_1^{\text{no}} = \{5, 34, 35, 45, 345\}$.

4. Agent 5.

Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if Agent 5 says no?

Answer: No. Thus $N_1^{\text{no}} = \{34, 35, 45, 345\}$.

5. Coalitions of size 2.

Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if one of the coalitions in $\{34, 35, 45\}$ says no? In the case of an affirmative answer, for which ones?

Answer: Yes, 34. Thus $C_i^{\text{no}} = \{1, 2, 34\}$, and $N_1^{\text{no}} = \{35, 45\}$. It follows that $N_1^{\text{no}} = \varnothing$, since all coalitions of size 2 are discarded. STOP.

Finally, $C_1^{\text{no}} = \{1, 2, 34\}$. We do the same for C_1^{yes}.

1. For all individual agents.

Suppose that your opinion on some question is no. Would you be inclined to change your opinion if one of the agents 2, 3, 4, 5 says yes? In the case of an affirmative answer, for which ones?

Answer: No. Thus $N_1^{\text{yes}} = \{23, 24, 25, 34, 35, 45, 234, 235, 245, 345, 2345\}$.

2. Coalitions of size 2.

Suppose that your opinion on some question is no. Would you be inclined to change your opinion if one of the coalitions in $\{23, 24, 25, 34, 35, 45\}$ says yes? In the case of an affirmative answer, for which ones?

Answer: No. Thus $N_1^{\text{yes}} = \{234, 235, 245, 345, 2345\}$.

3. Coalitions of size 3.

Suppose that your opinion on some question is no. Would you be inclined to change your opinion if one of the coalitions in $\{234, 235, 245, 345\}$ says yes? In the case of an affirmative answer, for which ones?

Answer: Yes: 234, 235, 245. Thus $N_1^{\text{yes}} = \varnothing$, STOP.

Finally, $C_1^{\text{yes}} = \{1, 234, 235, 245, 345\}$.

We now give another example to illustrate how the reduced transition matrix $\tilde{\mathbf{B}}$ can be obtained from the influential coalitions using Theorem 1. To this end, we suppose that the above algorithm has been applied to each agent, in order to obtain all influential coalitions. The condition that every agent is self-influential permits us to simplify the application of the theorem to determine each term $\tilde{b}_{S,T}$. Indeed, the following facts are easy to show.

Fact 2. Suppose that $\{i\} \in \mathcal{C}_i^{\text{yes}}$ and $\{i\} \in \mathcal{C}_i^{\text{no}}$ for every $i \in N$. It follows that:

1. $\tilde{b}_{S,S} = 1$ for every $S \in 2^N$.

2. If $T \subseteq S$, condition (1) in Theorem 1 is always sufficient to check that $\tilde{b}_{S,T} = 1$, moreover, if $\tilde{b}_{S,T} = 0$, then $\tilde{b}_{S,T'} = 0$ for every $T' \subset T$. Similarly, if $\tilde{b}_{S,T} = 1$, then $\tilde{b}_{S,T'} = 1$ for every $T \subset T' \subseteq S$.

3. If $T \supseteq S$, condition (2) in Theorem 1 is always sufficient to check that $\tilde{b}_{S,T} = 1$. Moreover, if $\tilde{b}_{S,T} = 0$, then $\tilde{b}_{S,T'} = 0$ for every $T' \supset T$; similarly, if $\tilde{b}_{S,T} = 1$, then $\tilde{b}_{S,T'} = 1$ for every $S \subseteq T' \subset T$.

Example 2. Consider a society $N = \{1, 2, 3, 4\}$ of 4 agents. Suppose that the following collections have been obtained (braces are omitted when denoting coalitions):

$$\mathcal{C}_1^{\text{no}} = \{1, 2, 34\}, \qquad \mathcal{C}_1^{\text{yes}} = \{1, 234\}$$

$$\mathcal{C}_2^{\text{no}} = \{2, 34\}, \qquad \mathcal{C}_2^{\text{yes}} = \{2, 134\}$$

$$\mathcal{C}_3^{\text{no}} = \{2, 3\}, \qquad \mathcal{C}_3^{\text{yes}} = \{3, 12\}$$

$$\mathcal{C}_4^{\text{no}} = \{12, 4\}, \qquad \mathcal{C}_4^{\text{yes}} = \{4\}$$

Observe that agent 4 is stubborn when he supports yes (no influence is possible when agent 4 thinks yes).

Let us apply Theorem 1. Using Fact 2, one easily finds that $\tilde{b}_{S,T} = 1$ only for the following S, T (braces omitted):

$$S = 1: \qquad T = \varnothing, 1$$

$$S = 2: \qquad T = \varnothing, 2$$

$$S = 3: \qquad T = \varnothing, 3$$

$$S = 4: \qquad T = \varnothing, 4$$

$$S = 12: \qquad T = \varnothing, 1, 2, 3, 12, 13, 23, 123$$

$$S = 13: \qquad T = \varnothing, 1, 3, 13$$

$$S = 14: \qquad T = 4, 14$$

$$S = 23: \qquad T = 23$$

$$S = 24: \qquad T = 24$$

$$S = 34: \qquad T = \varnothing, 3, 4, 34$$

$$S = 123: \qquad T = 123$$

$$S = 124: \qquad T = 124, 1234$$

$$S = 134: \qquad T = 4, 14, 24, 34, 124, 134, 234, 1234$$

$$S = 234: \qquad T = 234, 1234.$$

We detail the case $S = 12$ for illustrative purposes. We can see from condition (2) of Theorem 1 that $T = \varnothing$ is possible (i.e., $\tilde{b}_{S,T} = 1$). Indeed, for $i = 1, 2, 3, 4,$ there exists a set in $\mathcal{C}_i^{\text{no}}$ which is disjoint from 12. Thus, by Fact 2.2, it follows that $T = 1, 2, 12$ are also possible. Now, for $T = 13, 23$ both conditions of Theorem 1 must be checked, while for $T = 123,$ only condition (1) has to be checked. Lastly, observe that all the remaining sets contain 4. Thus condition (1) of Theorem 1 is never satisfied, since there is no $S' \in \mathcal{C}_4^{\text{yes}}$ which is included in 12.

One can check that Fact 1 is satisfied. Observe that this approach is very useful to identify quickly all the possible *T*s: it suffices to find the minimal one (K) and the maximal one ($K \cup L$). The corresponding transition graph Γ is given in Fig. 1. It is seen that, apart from the trivial terminal classes, 23, 24 and 123 are terminal states. There is no regular nor cyclic class.

We now show that it is possible to get conclusions on convergence without computing $\tilde{\mathbf{B}}$, by solely examining the hypergraphs, thanks to results presented in [27]. To this end, we need the notion of an ingoing hyperarc. We say that a coalition S has an ingoing hyperarc (T', T'') in hypergraph H if $T' \subseteq N \setminus S$ and $T'' \subseteq S$ (and vice versa for an outgoing hyperarc).

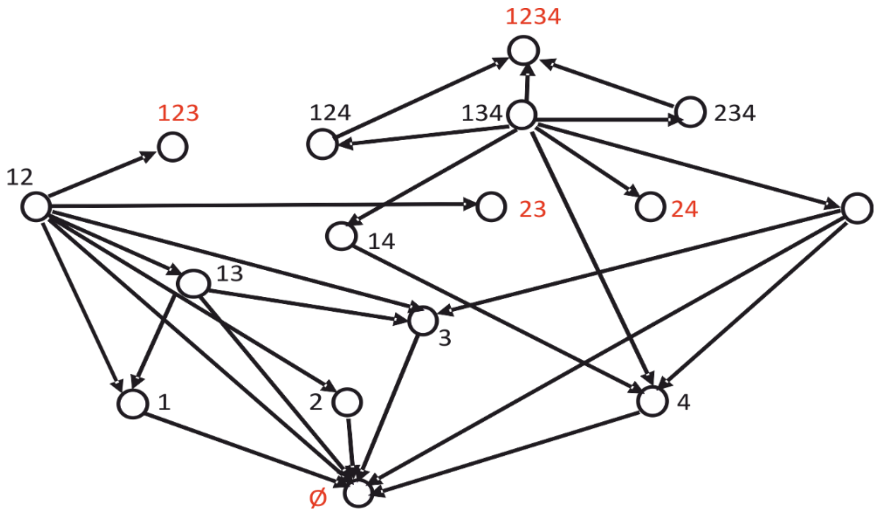

Fig. 1. Transition graph (loops are omitted)

Now, Theorem 3 in the aforementioned paper establishes that a nonempty $S \neq N$ is a terminal state if and only if S has no ingoing arc in the hypergraph $(\hat{H}^{\text{yes}}_{\mathbf{A}})^* \cup \hat{H}^{\text{no}}_{\mathbf{A}}$, where $()^*$ indicates that the hyperarcs have been inverted, and $\hat{H}$ indicates that only normal hyperarcs are considered[4]. This result can be translated in terms of influential collections as follows:

Fact 3. A nonempty $S \neq N$ is a terminal state if and only if

3.1. For every $i \notin S$, there is no $T \in \mathcal{C}_i^{\text{yes}}$ such that $T \subseteq S$.

3.2. For every $i \in S$, there is no $T \in \mathcal{C}_i^{\text{no}}$ such that $T \cap S = \varnothing$.

Applying this fact to Example 1, we indeed find that the only terminal states are 23, 24 and 123. For example, 23 is a terminal state, because none of 1, 234 are subsets of 23 (condition (1) for $i = 1$), 4 is not a subset of 12 (condition (1) for $i = 4$), none of 2, 34 are disjoint from 23 (condition (2) for $i = 2$), and none of 2, 3 are disjoint from 23 (condition (2) for $i = 3$).

The advantage of Fact 3 is that it is not necessary to find all S, T such that $\tilde{b}_{S,T} = 1$ (i.e., it is not necessary to know the transition graph) to check whether a given coalition is a terminal state (or to find all of them).

[4] A hyperarc (T', T'') is normal if $T' \cap T'' = \varnothing$. Note that due to our assumption that every player is self-influential, all hyperarcs are normal.

4. Concluding remarks

We have shown how, in a practical situation, one can determine a model of influence based on aggregation functions. Exact determination of such a model, yielding the type and parameters of the aggregation function of each agent, appears to be out of reach without using complex procedures. What we show is that, on the contrary, it is easy to obtain the qualitative part of the model, which permits a full qualitative analysis of the convergence of opinions, that is, to determine all terminal classes. This is sufficient to predict whether a consensus will occur or, on the contrary, society will become polarized, or a cycle will appear, etc. Simple criteria are available to detect terminal states or the presence of regular terminal classes, without even determining the reduced transition matrix. We believe that this study will make the use of models of influence based on aggregation functions more familiar and easier to use.

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