B A D A N I A O P E R A C Y J N E I D E C Y Z J E

Nr 1 2009

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LONG MEMORY OF VOLATILITY MEASURES IN TIME SERIES

The authors analyse relations between the long memory parameter of conditional variance and estimates of the long memory in squared residuals in FIGARCH models. The investigations are performed by means of simulations FIGARCH(0, *d*, 0) and FIGARCH(1, *d*, 1) models for selected parameters. Simulation results suggest, that estimates of the conditional variance long memory and the long memory in squared residuals can considerable differ. Moreover, only for small d positive relationship between the long memory estimates of squared residuals and the fractional integration parameter d of FIGARCH model can be observed.

Keywords: *FIGARCH, long memory, simulations*

1. Introduction

Conditional volatility models are applied very often in investigations of stock exchange time series. GARCH models are the most important class of these models. There are significant issues in this area concerning stock exchange rates and foreign exchange rates. The heteroscedasticity of variance in such financial time series is well known, and documented in numerous empirical contributions. Daily returns exhibit heteroscedasticity of variance, but in general they do not show significant autocorrelation (Lamoureux and Lastrapes [30], [31], Blume et al. [8]). However, many empirically oriented contributions confirm significant autocorrelation in the time series of square returns (comp. e.g. Ding et al. [18], Bollerslev and Mikkelsen [12], Breidt et al. [13], Bollerslev and Jubiński [11]). Moreover, this autocorrelation is statistically significant even for large time lags and the ACF decays very slowly at a hyperbolic rate. This type of autocorrelation behavior is characteristic of time series with long memory. Squared returns are

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a well known and often applied measure of volatility and they act as estimators of conditional variance. From a theoretical point of view, the presence of long memory in the time series of squared returns is a reason for serious problems with the application of GARCH models. In the case of financial time series with long memory, GARCH models are less adequate than FIGARCH models (Fractionally Integrated GARCH, see e.g. Bollerslev and Mikkelsen [12]). This class of models was introduced in 1996 (Baillie et al. [10]), although an earlier formulation can be found one year before in a contribution by Robinson [41]. In spite of numerous applications of FIGARCH models in research and theoretical investigations on the financial market (e.g. Bollerslev and Mikkelsen [12], Chung [15], Davidson [17], Giraitis et al. [22], Karanasos et al. [27]), many questions remain unresolved and need explanation. This applies particularly to measures of long memory by means of FIGARCH models.

From a theoretical point of view, a FIGARCH model for returns can be identified with ARFIMA model for square returns. In both types of model the most important parameter is the long memory parameter d, which describes the behavior of data over a long time span. This parameter is present in the fractional difference operator $(1 - L)^d$, which is applied to the series of squared returns. A natural question therefore arises as to the comparability and reasons for possible differences between estimators values of long memory parameter d estimated on the basis of the same time series by means of both kinds of model. This question is strongly connected with the relation, from the point of view of the long memory feature, between conditional variance and squared returns or alternatively absolute returns.

The main goal of this paper is a comparison of estimators of long memory for both the conditional variance and square residuals using simulations based on FIGARCH models. This comparison allows us to indicate differences concerning the measures of long memory in squared residuals of FIGARCH models. Research will be performed for the most popular types of FIGARCH model, FIGARCH(0, d, 0) and FIGARCH(1, d, 1).

The main properties of long memory, its estimation and FIGARCH models will be overviewed in the next section. In section 3 the simulation method and description of the main results are presented. The most important conclusions are summarized in the last chapter.

2. Long memory and FIGARCH models

2.1. Long memory

To allow for seasonal variation in riverflow, the capacity of a reservoir must be such that it can allow for fluctuations in the supply of water above the dam, while still maintaining a relatively constant flow of water below the dam. Since dam construction costs are immense, the importance of getting the reservoir capacity right, so as to meet long term storage needs, is apparent. This problem led Hurst [26] to the rescaled range (R/S) statistic. Hurst's R/S statistic is the range of partial sums of deviations of the time series from the mean, rescaled by its standard deviation. The models that have been developed by Hurst and other hydrologists are called long-memory models. Many aspects of this concept were generalized by Mandelbrot and Van Ness [37] and Mandelbrot [36]. Hosking [25] and Granger and Joyeux [23] also made important contributions to research on long memory. Recently, scientists have also defined ARIMA models with a fractional order of integration. A very important research goal was methods for long memory estimation. Apart from the maximum likelihood estimation of the ARFIMA model (Granger, and Joyeux [23], Sowell [44]), other widely used long memory estimation methods are R/S analysis (Lo [32]), log-periodogram regression (Geweke and Porter-Hudak [21]) and local Whittle estimation (Künsch [29], Lobato [33]).

Long memory in financial time series is still the subject of intensive theoretical and empirical research. In recent years, one may notice the special interest of researchers in the long memory of return volatility expressed by the absolute values of returns or alternatively by squared returns and trading volume (eg. Baillie [3], Barkoulas and Baum [5], Barkoulas et al. [6], Bollerslev and Mikkelsen [12], Cheung and Lai [16], Ding et al. [18], Lobato and Savin [34], McKenzie and Faff [38]). The notion of long memory concerns not only univariate time series, but also long-run dependencies between two or more time series e.g. return volatility and trading volume (Bollerslev and Jubinski [11], Lobato and Velasco [35]).

A stochastic process is said to possess a long memory if there exists a long-run dependence between observations which are far away from each other in time. The exact definition of long memory can be formulated in terms of the spectral density function.

A covariance stationary stochastic process exhibits a long memory with memory parameter d when its spectral density function $f(\lambda)$ satisfies the condition

$$f(\lambda) \sim c\lambda^{-2d} \text{ when } \lambda \to 0^+, \tag{1}$$

where c is a finite constant and $d \in (-1/2; 1/2)$ and the symbol "~" means that the ratio of the left and right hand sides tends to 1. If the process fulfills this condition and $d > 0$, then the ACF decreases at a hyperbolic rate (Granger and Joyeux [23], Hosking [25], Beran [7]), i.e.

$$\rho_k \sim c_\rho k^{2d-1} \text{ when } k \to \infty .$$

As we have already mentioned, the parameter d stands for the memory of the process (time series). In particular, when $d > 0$ the spectral density function is unbounded in the neighborhood of 0 and such a stochastic process is called a long memory process.

As mentioned above, in the literature we can find different methods of long memory estimation. The most popular are semiparametric approaches, which are based on the spectral density function in the neighborhood of 0 according to condition (1). Semiparametric estimators are based on the information included in a periodogram, but for very low frequencies. This explains the observed insensitivity of these estimators with respect to different short term shocks. This does not hold true in the case of parametric estimators. One of the most popular semiparametric estimators is the GPH estimator based on log-periodogram regression defined by Geweke and Porter-Hudak [21]. This estimator was investigated in detail by Robinson [42].

Based on (1), after taking the logarithm and substituting of sample estimators, the following approximate relation can be derived:

$$\ln(I(\lambda_j)) \approx \text{const} - 2d\ln(\lambda_j),$$

where $\lambda_j = \frac{2\pi j}{T}$ for $j = 1, ..., m$ and $I(\lambda)$ is a periodogram computed on the basis of the sample $x_1, ..., x_T$. A GPH estimator can be computed from this formula by means of the least squares method, where the parameter $m = m(T)$ should satisfy the following condition:

$$\frac{1}{m} + \frac{m}{T} \to 0 \text{ when } T \to \infty.$$

In their paper Geweke and Porter-Hudak suggest $m = \sqrt{T}$, where T is the sample size. The GPH estimator is asymptotically normal for $d \in (-1/2, 1/2)$. This property was proved by Robinson [42]. In other contributions, e.g. Kim and Philips [28] and Velasco [45], consistency of the GPH estimator was established for $d \in (-1/2, 1)$ and asymptotic normality for $d \in (-1/2, 3/4)$.

This method appears in several versions. For example, Agiakloglou et al. [1] suggest replacing the constant in the regression by means of a polynomial. The goal is to reduce bias (Andrews and Guggenberger [2]). Shimotsu and Phillips [43] describe a modification which allows for short-term component. GPH estimator can also be generalized to the case of a multivariate process.

Fractionally Integrated Autoregressive and Moving Average processes (ARFIMA), introduced by Granger and Joyeux [23], are the most popular class of models satisfying condition (1).

It can be said that x_t is an ARFIMA(p, d, q) process if

$$\Phi(L)(1-L)^d(x_t - \mu) = \Theta(L)\varepsilon_t,$$

where L is the time lag operator, $\Phi(z)=1-\phi_1 z-\ldots-\phi_p z^p$ and $\Theta(z)=1+\theta_1 z+\ldots+\theta_q z^q$ are polynomials of time lags of order p and q, respectively. The roots of both polynomials should be located outside of the unit circle. Moreover, $\varepsilon_t \sim iid\ (0, \sigma^2)$, and the expression $(1-L)^d$ is defined by means of the series expansion

$$(1-L)^d=\sum_{j=0}^{\infty}\frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)}L^j , \qquad (2)$$

where Γ stands for the Euler gamma function. This means that x_t is an ARFIMA(p, d, q) process, while $(1-L)^d(x_t-\mu)$ is an ARMA(p, q) process. In the case $d = 0$, ARFIMA(p, 0, q) can be denoted by ARMA(p, q) and the corresponding time series has a short memory. This means that the correlation between consecutive observations fades out quickly and the series returns to its constant mean. For $d = 1$ we obtain an ARIMA(p, 1, q) process, also called a unit root process. Its mean, variance and covariance are non-stationary. Each element of such a series is a function of the previous value and current error. The effect of a shock cumulates over time and the series does not revert to a constant mean level. In the case when $d\in(0, 0.5)$, the autocorrelations of an ARFIMA process decay hyperbolically to zero as $k\to\infty$, which is contrary to the faster exponential decay of a stationary ARFIMA(p, 0, q) process. The sum of the absolute values of the autocorrelations diverges as the number of components tends to infinity. Such a process is said to exhibit long memory or long-range positive dependence. For $d\in(-0.5, 0)$ the process is said to have intermediate long memory (antipersistence), or long-range negative dependence. For $d\in[0.5, 1)$ the process is mean reverting, even though it is not covariance stationary, as an innovation will have a long run impact on future values of the process. When $d > -0.5$ an ARFIMA process is invertible and has Wold's representation. In the case when $d < 0.5$, the process is stationary. To summarize, for $0 < d < 0.5$, an ARFIMA process is stationary and exhibits long memory. The ML estimators of the parameters of the model are consistent and asymptotically normal (Sowell [44]). Long memory can be modeled not only by an ARFIMA model. In the case of conditional volatility, other models are necessary. These models are considered in the next section.

2.2. FIGARCH models

The class of ARFIMA models allows us to model long memory in the case of time series which exhibit constant variance. In practice, many time series exhibit heteroscedasticity of variance. In this case, the most popular models are those belonging to the GARCH family of models. For time series which exhibit not only conditional volatility but also long memory, Fractionally Integrated GARCH models (FIGARCH), which are analogous to ARFIMA models, can be applied. Integrated EGARCH models, called FIEGARCH, are a further modification (Bollerslev et al. [10], Bollerslev and Mikkelsen [12], Cai [14], Hamilton and Susmel [24], Nelson [39], Omran and McKenzie [40]). EGARCH (FIEGARCH) models take into account not only heteroscedasticity of variance, but also asymmetric effects of positive and negative residuals on conditional variance. In the context of modeling returns, this can be interpreted as asymmetric market responses to good and bad news.

In order to construct a FIGARCH model for conditional variance, first consider a stochastic process $\{\varepsilon_t\}$, where

$$\varepsilon_t = e_t\sqrt{h_t}\ , \tag{3}$$

and the e_t are independent random variables with $E_{t-1}(e_t) = 0$ and $\mathrm{VAR}_{t-1}(e_t) = 1$, and h_t stands for conditional variance which changes over time. Usually, the ε_t are the residuals in a model for the conditional mean e.g. ARMA, ARFIMA.

In the basic ARCH(q) model suggested by Engle [19]), the conditional variance h_t is a linear function of the lagged square of residuals $\{\varepsilon_t\}$.

$$h_t = \omega + \alpha(L)\varepsilon_t^2\ , \tag{4}$$

where $\alpha(L) = \alpha_1 L + \alpha_2 L^2 + \cdots + \alpha_q L^q$ is a lag polynomial.

This basic model cannot take into account heteroscedastic effects, typically observed in the form of fat tails, clustering of volatilities, and the leverage effect. In this context, Autoregressive Conditional Heteroscedastic models, called ARCH, were later generalized by Bollerslev [9] in the form of GARCH(p, q) models. As with ARMA models, a GARCH specification of the data leads to a more parsimonious representation of the temporal dependencies and thus provides more flexibility than the linear ARCH model when applied to conditional variance. The impact on the variance of square returns, as well as past values of the variance, is modeled using the formula:

$$h_t = \omega + \alpha(L)\varepsilon_t^2 + \beta(L)h_t\ , \tag{5}$$

where $\beta(L) = \beta_1 L + \beta_2 L^2 + \cdots + \beta_p L^p$.

A GARCH(p, q) process for ε_t can be written down in the form of an ARMA(m,q) model for the squared residuals ε_t^2 as

$$[1-\alpha(L)-\beta(L)]\varepsilon_t^2 = \omega + [1-\beta(L)]\upsilon_t\ , \tag{6}$$

where $m = \max\{p, q\}$. The expression $\upsilon_t = \varepsilon_t^2 - h_t$ can be treated as a formula for innovations. Their expected value is zero and they do not exhibit autocorrelation.

Stationarity and invertibility of the $\{\varepsilon_t^2\}$ process can be assumed if all the roots of the polynomials $1-\alpha(L)-\beta(L)$ and $1-\beta(L)$ lie outside of the unit circle.

If the autoregressive lag polynomial $1-\alpha(L)-\beta(L)$, has a unit root, then model (6) becomes an ARIMA(m – 1, q) model, which can be expressed as

$$\varphi(L)(1-L)\varepsilon_t^2 = \omega + [1-\beta(L)]\upsilon_t\,, \tag{7}$$

where $\varphi(L)$ is a lag polynomial of order m – 1. Instead of a GARCH(p, q) model, we obtain an integrated GARCH model, which can be denoted as IGARCH(p, q) (comp. Engle and Bollerslev [20]).

In order to take into account the long-run dependencies between the squared residuals ε_t^2, as well as the analogy with ARFIMA models, fractionally integrated GARCH models, called FIGARCH, can be defined. This can be performed by replacing the difference operator (1 – L) in the IGARCH model by an operator of fractional integration $(1 - L)^d$ given by formula (2). In this case, a FIGARCH(p, d, q) model can be written as (comp. Baillie et al. [4]):

$$\varphi(L)(1-L)^d\varepsilon_t^2 = \omega + [1-\beta(L)]\upsilon_t\,, \tag{8}$$

where $0 < d < 1$ and all the roots of the polynomials $\phi(L)$ and $1-\beta(L)$ lie outside the unit circle. It follows from (8) that the FIGARCH model is derived from an ARFIMA model for the time series of squared residuals, ε_t^2.

This specification of a FIGARCH(p, d, q) model suffers from some drawbacks. According to Chung [15], the relations of this specification with ARFIMA models for the conditional mean are not perfect. In particular, the nature of the constant ω is quite different to that of the constant μ in ARFIMA models. The reason for this situation is that the fractional integration operator exhibits an impact on μ, while being irrelevant to ω. Moreover, for a given unconditional variance σ^2, the parameter ω in expression (8) should be equal to zero independently of the value of σ^2.

In order to avoid this disadvantage, Chung [15] suggested another form of FIGARCH model:

$$\varphi(L)(1-L)^d(\varepsilon_t^2 - \omega) = [1-\beta(L)]\upsilon_t\,. \tag{9}$$

In this case, the operator of fractional integration has an impact on the constant ω. This value plays the same role as the constant μ in an ARFIMA model, although its interpretation is still not completely clear. The analysis of a FIGARCH model in the form of an ARFIMA model for the squared residuals raises a question with respect to estimators of the long memory: what is the relation between long memory in squared residuals ε_t^2 and the parameter of fractional integration d in a FIGARCH model? It follows from (8) and (9) that these two values coincide. However, as we will demonstrate in a later part of our paper, this assumption is not true.

Taking into account formulas (8) and (9), FIGARCH models for the conditional variance can be written in the following forms:

$$[1-\beta(L)]h_t = \omega + \{1-\beta(L)-\varphi(L)(1-L)^d\}\varepsilon_t^2 , \tag{10}$$

and

$$[1-\beta(L)]h_t = [1-\beta(L)]\varepsilon_t^2 - \varphi(L)(1-L)^d(\varepsilon_t^2-\omega) . \tag{11}$$

In this context the following questions arise: What is the relation between the long memory of conditional variance and the parameter of fractional integration *d?* What is the dependence between the long memory of conditional variance and squared residuals.

At present, these questions can only be partially answered from a theoretical point of view. Karanasos et al. [27] derived the form of the autocorrelation function for the time series of squared residuals ε_t^2 for model (9) when $0 < d < 0.5$. This ACF coincides with the ACF of an ARFIMA(p, d, q) model. This means in particular that the parameter of fractional integration d in a FIGARCH model is equal to the long memory parameter d of the squared residuals. The authors claim that both parameterizations (8) and (9) have the same correlation properties. Therefore, the derived formulas should also hold true for FIGARCH models in the version suggested by Baillie et al. [4]. According to our simulation results, which will be described in the next section, the value of d does not in fact describe the long memory in squared residuals, as should be expected from Karanasos et al. [27]. The probable reason for this is the false assumption made by the author that the residuals $\upsilon_t = \varepsilon_t^2 - h_t$ are uncorrelated, as in the case of a FIGARCH model. However, according to Davidson [17], this assumption is a purely formal one, and model (8) cannot be treated as an ARFIMA model for squared residuals.

FIGARCH models were derived in order to use the long memory property observed in the squared residuals of time series models for the mean to describe the conditional variance. On one hand, it is widely accepted in the literature that a FIGARCH model can be defined in terms of an ARFIMA model for squared residuals, while on the other hand, some consider this assumption to be false. Therefore, an investigation focused on the long memory properties of FIGARCH models is very much required.

In the next chapter we present the simulation procedure for FIGARCH models and describe the results of the investigations on long memory in detail.

3. Simulation results

Based on formula (10), the FIGARCH(1, d, 1) model is of the form

$$\varepsilon_t = e_t\sqrt{h_t}$$

$$(1-\beta L)h_t = \omega + (1-\beta L)\varepsilon_t^2 - (1-\varphi L)(1-L)^d \varepsilon_t^2 .$$

Using the binomial expansion of the fractional integration operator $(1 - L)^d$ given by (2), the above equation of the conditional variance can be rearranged as follows:

$$h_t = \omega + \beta h_{t-1}(1-e_{t-1}^2) - \sum_{j=0}^{\infty} \pi_j(d) h_{t-j} e_{t-j}^2 + \varphi \sum_{j=0}^{\infty} \pi_j(d) h_{t-1-j} e_{t-1-j}^2 .$$

However, to make the simulation procedure feasible, the infinite sums in the above formula must be approximated by finite sums of length $t + M$, for each $t = 1, ..., T$ and a given pre-sample of length M. For each set of parameters, 1000 realizations of a FIGARCH process of length $T = 500$, 1000 and 1500[1] are generated with M equal to 2000. The simulation procedure is as follows:

1. For $t = -2000, ..., T$ the elements e_t are independently generated according to the standard normal distribution.

2. For $t = 1, ..., T$ the conditional variances h_t are computed from the formula:

$$h_t = \omega + \beta h_{t-1}(1-e_{t-1}^2) - \sum_{j=0}^{t+M} \pi_j(d) h_{t-j} e_{t-j}^2 + \varphi \sum_{j=0}^{t+M} \pi_j(d) h_{t-1-j} e_{t-1-j}^2 ,$$

where the elements e_{-2000}, ..., e_0 are used as presample values, the $\pi_j(d)$ are coefficients in the expansion of the fractional difference operator $(1-L)^d$. The presample values h_{2000}, ..., h_0 of the conditional variance are assumed to be equal to 1.

3. For $t = 1, ..., T$ the realizations of the process are computed from the formula $\varepsilon_t = e_t \sqrt{h_t}$.

3.1. Simulation results for a FIGARCH(0, *d*, 0) model

As a first step, simulations of the simplest class of processes, FIGARCH(0, *d*, 0), were conducted. A FIGARCH(0, *d*, 0) model written as an ARFIMA model for the squares of residuals is of the form:

$$(1-L)^d \varepsilon_t^2 = \omega + \upsilon_t ,$$

[1] In the case of stock return data, these lengths of time series correspond to periods of length approx. 2, 4 and 6 years, respectively. In this paper only results for $T = 1500$ are presented. The remaining results are available upon request.

where the conditional variance is given by:

$$h_t = \omega + \sum_{j=0}^{\infty} \pi_j(d) h_{t-j} e_{t-j}^2 .$$

In order to simulate the FIGARCH(0, d, 0) processes, the following set of parameters is assumed to be constant: $\omega = 0.5$, $\beta = 0$, $\varphi = 0$, $\sigma^2 = 1$, while the integration parameter d ranges from 0.05 to 0.95. For each realization of a given FIGARCH(0, d, 0) process generated, the parameters of long memory for the residual squares ε_t^2 and the conditional variances h_t are estimated using the GPH method. Descriptive statistics for the 1000 estimates of long memory computed for each value of the fractional difference parameter d are presented in Tables 1 and 2.

Table 1. Descriptive statistics for GPH estimates of the long memory for squared residuals ε_t^2 computed from 1000 realizations of a FIGARCH(0, d, 0) process

d	Estimates of long memory for ε_t^2				
	min.	1st Quartile	median	3rd Quartile	max
0.05	–0.169	0.003	0.045	0.089	0.240
0.10	–0.130	0.052	0.097	0.145	0.401
0.15	–0.154	0.095	0.147	0.190	0.397
0.20	–0.007	0.138	0.190	0.242	0.515
0.25	–0.026	0.172	0.223	0.281	0.657
0.30	–0.053	0.196	0.253	0.320	0.652
0.35	0.035	0.214	0.276	0.338	0.618
0.40	0.036	0.233	0.284	0.351	0.742
0.45	0.030	0.227	0.283	0.356	0.734
0.50	0.055	0.229	0.290	0.365	0.764
0.55	–0.007	0.225	0.289	0.360	0.784
0.60	0.032	0.220	0.287	0.359	0.959
0.65	0.020	0.196	0.259	0.341	0.855
0.70	–0.021	0.182	0.255	0.333	0.910
0.75	–0.017	0.162	0.228	0.314	1.117
0.80	–0.051	0.144	0.198	0.275	0.808
0.85	–0.050	0.120	0.178	0.250	0.869
0.90	–0.075	0.092	0.142	0.216	0.862
0.95	–0.105	0.060	0.109	0.175	0.690

Table 2. Descriptive statistics for GPH estimates of long memory of the conditional variances h_t computed from 1000 realizations of a FIGARCH(0, *d*, 0) process

d	Estimates of long memory for h_t				
	min.	1st Quartile	median	3rd Quartile	max
0.05	0.093	0.293	0.341	0.384	0.530
0.10	0.157	0.325	0.367	0.419	0.669
0.15	0.146	0.350	0.398	0.446	0.652
0.20	0.223	0.371	0.420	0.475	0.736
0.25	0.188	0.380	0.433	0.491	0.859
0.30	0.126	0.385	0.438	0.505	0.840
0.35	0.162	0.380	0.443	0.504	0.782
0.40	0.149	0.379	0.433	0.497	0.888
0.45	0.058	0.356	0.413	0.484	0.864
0.50	0.173	0.344	0.403	0.478	0.877
0.55	0.104	0.323	0.386	0.459	0.881
0.60	0.121	0.304	0.371	0.443	1.043
0.65	0.112	0.266	0.330	0.411	0.926
0.70	0.031	0.240	0.313	0.391	0.968
0.75	0.030	0.209	0.275	0.361	1.163
0.80	–0.015	0.179	0.233	0.311	0.844
0.85	–0.032	0.145	0.204	0.276	0.895
0.90	–0.059	0.108	0.158	0.232	0.878
0.95	–0.086	0.068	0.117	0.183	0.698

In order to make the dependence of the estimates of long memory on the true value of the difference parameter *d* in the FIGARCH(0, *d*, 0) process more visible, the results from the above tables are once again presented in Figure 1. The lines represent the 3rd quartile, median and 1st quartile of these estimates for the squares of innovations (thin lines) and the series of conditional variances (thick lines).

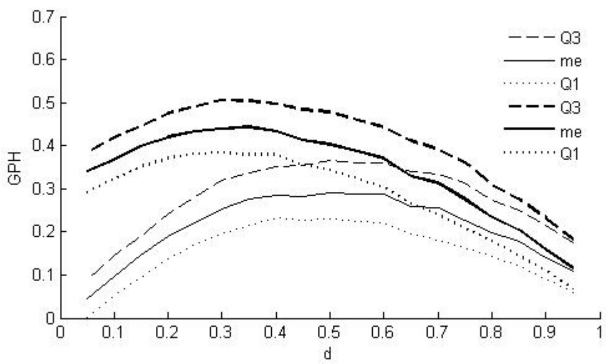

Fig. 1. Descriptive statistics (3rd quartile – Q3, median – me and 1st quartile – Q1) of estimates of long memory for squared innovations ε_t^2 (thin lower lines) and conditional variance h_t (thick upper lines) based on 1000 realizations of the FIGARCH(0, *d*, 0) process

From the results of the FIGARCH(0, *d*, 0) simulations presented, it follows that the estimates of long memory for conditional variance have higher values than the estimates of long memory for squared innovations. This difference is especially pronounced for low values of the difference parameter *d*. When *d* increases, the series of conditional variances and squared innovations reveal similar long memory. This observation is also confirmed by tests for the existence of a common long memory. Moreover, for $d < 0.4$ the estimates of long memory for the series of conditional variances tend to be greater than the value of the difference parameter *d* for the FIGARCH process simulated. On the other hand, for $d < 0.25$ the median estimates of long memory for squared innovations are close to the true values of *d*. When the value of *d* increases (especially for $d > 0.5$), the estimates of long memory decrease for both the conditional variance and squared innovations. Thus, it follows that, *d* describes the long memory for squared residuals in the FIGARCH(0, *d*, 0) model for low values of the difference parameter *d*. This contradicts the results obtained by Karanasos et al. [27] and, as mentioned above, means that the main condition which allows us to write a FIGARCH model as an ARFIMA model for squares is not fulfilled. Hence, it raises a question about the degree to which the conditional variance of a FIGARCH model captures the long memory for ε_t^2. In the case of financial markets, this is a question regarding how the conditional variance of stock returns reflects the autocorrelation of squared returns.

3.2. Results of FIGARCH(1, *d*, 1) simulation

In the previous subsection only one particular type of FIGARCH models was analyzed. Hence, it seems essential to check whether the relation between the long memory for ε_t^2 and the fractional difference parameter *d* described there is common to the family of FIGARCH models. Thus, in this subsection an analogous examination of FIGARCH(1, *d*, 1) models is conducted. To make them as general as possible, the constraints on the parameters of FIGARCH(1, *d*, 1) models suggested by Chung [15]:

$$0 \le \varphi \le \beta \le d < 1$$

are assumed. This set of parameters is complementary to that proposed by Baillie et al. [4] and ensures the non-negativity of the conditional variance. Simulations were performed for all the sets of parameters allowed under the above constraints when parameters took values in the set {0, 0.1, 0.2, ..., 0.9}[2]. Thus, e.g. for $d = 0.8$ the possible pairs of values of the parameters β and φ are:

[2] According to the definition of a FIGARCH model, it is assumed that $d > 0$.

$\beta = 0 \qquad \varphi = 0$

$\beta = 0.1 \qquad \varphi = 0; \qquad \varphi = 0.1$

...

$\beta = 0.8 \qquad \varphi = 0; \qquad \varphi = 0.1; \qquad \varphi = 0.2; \ldots \varphi = 0.8.$

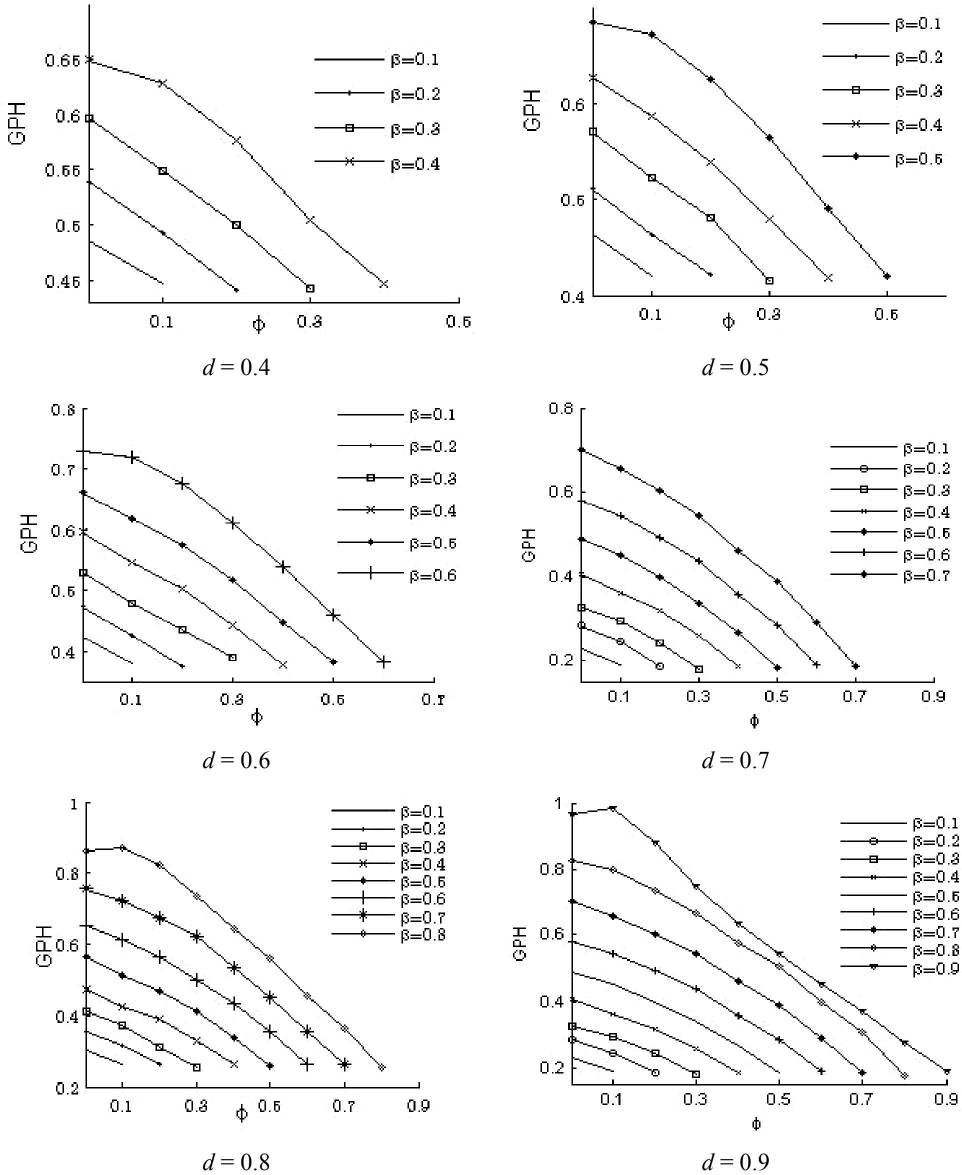

Fig. 2. Estimates of long memory for the conditional variance for different FIGARCH(1, d, 1) models

As a result, 219 different FIGARCH(1, d, 1) models were considered. It is impossible, therefore, to report the results of the conducted simulations in a table as in the case of the previous FIGARCH(0, d, 0) model, which is why they must be presented graphically. In Figure 2, the means of the estimates of long memory for the conditional variance are shown. Each panel considers FIGARCH models with a given parameter d, while each line illustrates estimates of the long memory for models with a given parameter β and the appropriate values of the parameter φ on the abscisse axis.

From the simulations conducted, the following conclusion can be made: For any given values of β and the fractional difference parameter d in a FIGARCH(1, d, 1) model, if φ increases, then estimates of the long memory for the conditional variance h_t decrease. On the other hand, for a given d and φ when β increases, the estimates of long memory increase too. However, for β and φ fixed, when d increases (from d = 0.4), the estimates of long memory for h_t decrease. The last conclusion can be easily made from Figure 3, where the means of the estimates of long memory are illustrated for φ = 0 and different values of β.

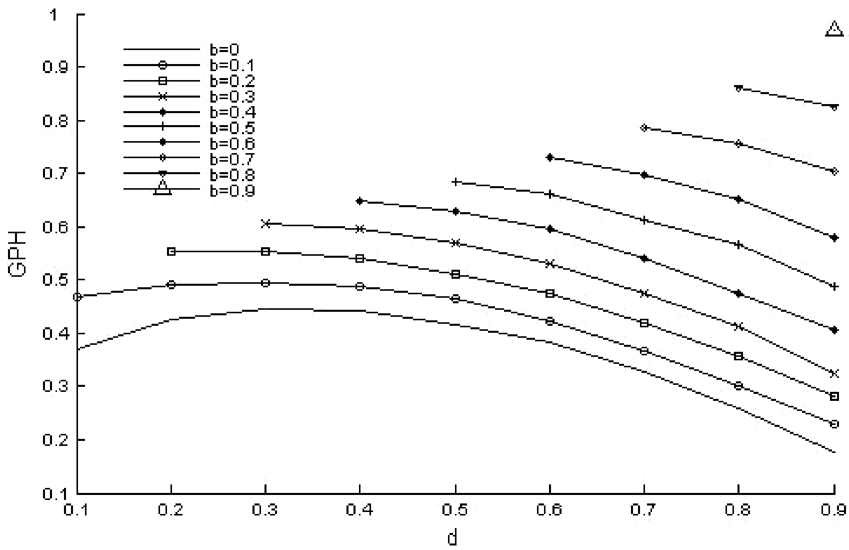

Fig. 3. Means of the estimates of long memory for the conditional variance for FIGARCH(1, d, 1) models with φ = 0. The difference parameter d is on the X axis

Analogous relations between the mean estimates of long memory and the parameters of the FIGARCH model hold for squared residuals.

4. Concluding remarks

The simulations conducted for the FIGARCH(0, d, 0) models allowed a comparison of the estimators of long memory for the conditional variance and squared residuals. We established that the first are in general greater than the second. As the parameter of long memory, d, in the FIGARCH process increases, the estimators of long memory for the conditional variance and for squared residuals tend to exhibit approximately the same values. This observation was confirmed by means of testing for the equality of the mean estimate of long memory. It follows from the simulations that for $d < 0.4$ the estimators of long memory for the conditional variance are higher than the estimators of fractional integration for a FIGARCH process, and in addition, the estimator of long memory for squared residuals is a virtually unbiased estimator of the fractional integration parameter. An increase in the fractional integration parameter d in the FIGARCH process (to $d > 0.4$) implies a decrease in the parameters of long memory for both the conditional variance and the time series of squared residuals.

Simulation results for the FIGARCH(1, d, 1) model have led us to the conclusion that for given values of the fractional integration parameter d and β, an increase in the parameter φ is accompanied by a decrease in the parameter of long memory for conditional variance. For given values of the fractional integration parameter d and φ, an increase in the parameter β implies a simultaneous increase in the parameter of long memory for the conditional variance in the models investigated. For given parameters β and φ, an increase in the value of the parameter d (from an initial value of $d = 0.4$) causes a decrease in the parameter of long memory for conditional variance. Analogous results were obtained for the long memory for the squared residuals.

These simulations show that the estimators of long memory for conditional variance and for squared residuals can exhibit significant differences. Moreover, their means are significantly different from the fractional integration parameter in FIGARCH models. From the simulation results, we may conclude that the ARFIMA form of GARCH models is a matter only of formal notation, and that there is probably no significant relation between the values of the fractional integration parameters in these models.

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Długa pamięć miar zmienności w szeregach czasowych

W artykule przedstawiono wyniki porównania estymatorów długiej pamięci warunkowej wariancji oraz estymatorów długiej pamięci kwadratów reszt modeli FIGARCH. Badanie zostało przeprowadzone na podstawie symulacji modeli FIGARCH(0, d, 0) oraz FIGARCH(1, d, 1). W przypadku modelu FIGARCH(0, d, 0) okazało się, że estymatory długiej pamięci warunkowej wariancji przyjmują na ogół wyższe wartości niż estymatory długiej pamięci kwadratów reszt. Uzyskane wyniki wskazują ponadto, że wraz ze wzrostem wartości parametru d ułamkowej integracji procesu FIGARCH estymatory długiej pamięci warunkowej wariancji oraz kwadratów reszt przyjmują bardzo zbliżone wartości. Potwierdzają to wyniki testowania istnienia wspólnej długiej pamięci. Z badań symulacyjnych wynika, że dla $d < 0{,}4$ estymatory długiej pamięci kwadratów reszt są zbliżone do wartości parametru d ułamkowej integracji

w procesie FIGARCH. Wraz ze wzrostem wartości parametru *d* ułamkowej integracji procesu FIGARCH (dla $d > 0,4$) zmniejszają się wartości estymatorów długiej pamięci zarówno warunkowej wariancji, jak i kwadratów reszt.

Badania symulacyjne dla modelu FIGARCH(1, *d*, 1) pozwalają na stwierdzenie, że dla ustalonej wartości parametru ułamkowej integracji *d* oraz dla ustalonego parametru β wraz ze wzrostem wartości parametru φ maleje wartość estymatora długiej pamięci warunkowej wariancji tego modelu. Z kolei dla wybranej wartości parametru ułamkowej integracji *d* oraz dla danego parametru φ wzrost wartości parametru β pociąga za sobą wzrost wartości estymatora długiej pamięci warunkowej wariancji analizowanego modelu. Ponadto dla ustalonych parametrów β i φ wzrostowi wartości parametru *d* (począwszy od $d = 0,4$) towarzyszy zmniejszanie się długiej pamięci warunkowej wariancji.

Wyniki symulacji sugerują, że estymatory długiej pamięci warunkowej wariancji oraz estymatory długiej pamięci kwadratów reszt mogą się na ogół znacznie różnić. Ponadto różnią się one od wartości parametru ułamkowej integracji modelu FIGARCH. Oznacza to w szczególności, że postać ARFIMA modelu FIGARCH jest tylko zapisem formalnym i raczej nie istnieje zależność pomiędzy wartościami parametrów ułamkowej integracji w obu modelach.

Słowa kluczowe: *FIGARCH, długa pamięć, symulacje*