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PROBABILISTIC MODELS FOR ANALYZING THE AVAILABILITY AND PROFIT OF A DETERIORATING 2-OUT-OF-4 SYSTEM

This paper deals with modelling and evaluating the availability and profit of a linear consecutive 2-out-of-4 system exposed to three consecutive stages of deterioration before failure. The system will pass through three consecutive stages of deterioration: slow, medium and fast before failure. The failure and repair times are assumed to be exponentially distributed. Explicit expressions for the system availability, the probability of a repairman being busy due to the failure of a unit, or due to the replacement of failed units, and the profit function have been derived using a probabilistic approach. The impacts of the failure and repair rate on system availability and profit have been investigated. The results of this paper will enhance system performance and ensure the timely execution of appropriate maintenance and improvement, and thus is a major tool for decision making, planning and optimisation.

Keywords: *deterioration, linear consecutive, profit, availability*

1. Introduction

There are systems containing three/four units in which the functioning of two/three units is sufficient to ensure the functioning of the entire system. Examples of such systems are 2-out-of-3, 2-out-of-4, or 3-out-of-4 redundant systems. Such systems have a wide range of applications in the real world, especially in industry. Furthermore, a communication system with three transmitters can be cited as a good example of such a system. Due to their importance in industry and design, such systems have received attention from many researchers (see, for instance, [1, 3, 4] and the references therein).

During operation, the strengths of systems gradually deteriorated, until some failure due to deterioration, or other types of failure. As the age of equipment increases, the

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equipment slowly deteriorates, correspondingly. In many manufacturing situation, the condition of the system has significant impact on the quantity and quality of the unit produced. Most systems are subjected to random deterioration, which can results in unexpected failures and have a disastrous effect on safety and the economy. It is therefore important to find a way to slow down the deterioration rate, and to prolong the service life of equipment. Modelling system deterioration is important because it will assist in diagnosing the best time to carry out a preventive maintenance. The concept of deterioration, as well as its impact on reliability and measures of system have been introduced by several authors [2, 5].

The problem considered in this paper is different from the works presented in [7, 8]. The contribution of this paper is twofold. The first goal is, to develop explicit expressions describing system availability, busy period and the profit function. The second is to perform a parametric investigation of various system parameters with system availability and the profit function, as well as to capture their effect on availability and the profit function. The rest of the paper is organized as follows. Section 2 gives a description of the system and states of the system. Section 3 deals with derivation of the models. The results of our numerical simulations are presented and discussed in Section 4. The paper is concluded in Section 5.

2. Description and states of the system

We consider a linear consecutive 2-out-of-4 repairable system with three modes: normal, deterioration and failure (Table 1). The deterioration mode consists of three consecutive stages: slow, medium and fast. It is assumed that the system transits from slow to fast deterioration via medium deterioration at the rate δ_i, $i = 1, 2, 3$. It is also assumed that primary units (units in operation) never fail simultaneously.

Table 1. System transition table

	S_0	S_1	S_2	S_3	S_4	S_5	S_6
S_0		δ_1			β		
S_1			δ_2		β		
S_2				δ_3		β	
S_3							β
S_4	α	α				β	
S_5			α		α		β
S_6	η						

Whenever one of the primary units fails at the failure rate β, it is sent for repair, which occurs at the rate α and the appropriate unit is then switched back on. The system fails when three units fail consecutively and then the system is replaced by a new one at the rate η.

3. Derivation of availability, busy period and profit

Let $P_i(t)$ be the probability that the system is in state *i* at time *t*. Using the approach adopted in [6], the corresponding set of differential equations obtained from Table 1 are:

$$\frac{d}{dt}P_0(t) = -\left(\delta_1 + \beta\right)P_0(t) + \alpha P_4(t) + \eta P_6(t)$$

$$\frac{d}{dt}P_1(t) = -\left(\delta_2 + \beta\right)P_1(t) + \delta_1 P_0(t) + \alpha P_4(t)$$

$$\frac{d}{dt}P_2(t) = -\left(\delta_3 + \beta\right)P_2(t) + \delta_2 P_1(t) + \alpha P_5(t)$$

$$\frac{d}{dt}P_3(t) = -\beta P_3(t) + \delta_3 P_2(t)$$

$$\frac{d}{dt}P_4(t) = -\left(2\alpha + \beta\right)P_4(t) + \beta P_0(t) + \beta P_1(t) + \alpha P_5(t)$$

$$\frac{d}{dt}P_5(t) = -\left(2\alpha + \beta\right)P_5(t) + \beta P_2(t) + \beta P_4(t)$$

$$\frac{d}{dt}P_6(t) = -\eta P_6(t) + \beta P_3(t) + \beta P_5(t) \tag{1}$$

with initial conditions

$$P_k(0) = \begin{cases} 1, & k = 0 \\ 0, & k = 1, 2, ..., 6 \end{cases} \tag{2}$$

The differential equations in (1) can be written in the matrix form as

$$P' = TP(t) \tag{3}$$

where

$$\mathbf{T} = \begin{pmatrix} -(\delta_1+\beta) & 0 & 0 & 0 & \alpha & 0 & \eta \\ \delta_1 & -(\delta_2+\beta) & 0 & 0 & \alpha & 0 & 0 \\ 0 & \delta_2 & -(\delta_3+\beta) & 0 & 0 & \alpha & 0 \\ 0 & 0 & \delta_3 & -\beta & 0 & 0 & 0 \\ \beta & \beta & 0 & 0 & -(2\alpha+\beta) & \alpha & 0 \\ 0 & 0 & \beta & 0 & \beta & -(2\alpha+\beta) & 0 \\ 0 & 0 & 0 & \beta & 0 & \beta & -\eta \end{pmatrix}$$

The differential equations in (1) can be expressed as

$$\begin{pmatrix} P_0' \\ P_1' \\ P_2' \\ P_3' \\ P_4' \\ P_5' \\ P_6' \end{pmatrix} = \begin{pmatrix} -(\delta_1+\beta) & 0 & 0 & 0 & \alpha & 0 & \eta \\ \delta_1 & -(\delta_2+\beta) & 0 & 0 & \alpha & 0 & 0 \\ 0 & \delta_2 & -(\delta_3+\beta) & 0 & 0 & \alpha & 0 \\ 0 & 0 & \delta_3 & -\beta & 0 & 0 & 0 \\ \beta & \beta & 0 & 0 & -(2\alpha+\beta) & \alpha & 0 \\ 0 & 0 & \beta & 0 & \beta & -(2\alpha+\beta) & 0 \\ 0 & 0 & 0 & \beta & 0 & \beta & -\eta \end{pmatrix} \begin{pmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ P_3(t) \\ P_4(t) \\ P_5(t) \\ P_6(t) \end{pmatrix}$$

The steady state probability of system availability can be obtained from the solutions for $P_i(t)$, $i = 0, 1, 2, \ldots, 6$. States 0–5 are the only working states of the system in Table 2, thus the steady state probability of the system availability $A_V(\infty)$ at time t is

$$A_V(\infty) = P_0(\infty) + P_1(\infty) + P_2(\infty) + P_3(\infty) + P_4(\infty) + P_5(\infty) \tag{4}$$

States 4–6 are the only states in Table 2 where the repairman is busy repairing a failed unit, thus the steady state probability of a busy period due to the failure of a unit $B_{R1}(\infty)$ at time t is

$$B_{R1}(\infty) = P_4(\infty) + P_5(\infty) + P_6(\infty) \tag{5}$$

State 6 is the only state in Table 2 where the repairman is busy replacing failed units 1, 2 and 3, thus the steady state probability of a busy period due to the replacement of failed units $B_{R2}(\infty)$ at time t is

$$B_{R2}(\infty) = P_6(\infty) \tag{6}$$

Table 2. State of the system

S_0	Initial state, units 1 and 2 are working, units 3 and 4 are in a standby mode. The system is working.
S_1	The system is in the slow deterioration stage and is working.
S_2	The system is in the medium deterioration stage and is working.
S_3	The system is in the fast deterioration stage and is working.
S_4	Units 2, and 3 are working, unit 4 is in a standby mode, unit 1 is down and under repair. The system is in slow deterioration state and is working.
S_5	Units 3 and 4 are working; units 1 and 2 down and under repair. The system is in the medium deterioration state and is working.
S_6	Units 1, 2 and 3 are down are being replaced by new and identical ones. Unit 4 is functional. The system is in a state of major deterioration and has failed.

In the steady state, the derivatives of the state probabilities become zero which enable us to compute the steady state probabilities using (3) written as

$$T_1 P(\infty) = 0 \tag{7}$$

which in matrix form becomes

$$\begin{pmatrix} -(\delta_1+\beta) & 0 & 0 & 0 & \alpha & 0 & \eta \\ \delta_1 & -(\delta_2+\beta) & 0 & 0 & \alpha & 0 & 0 \\ 0 & \delta_2 & -(\delta_3+\beta) & 0 & 0 & \alpha & 0 \\ 0 & 0 & \delta_3 & -\beta & 0 & 0 & 0 \\ \beta & \beta & 0 & 0 & -(2\alpha+\beta) & \alpha & 0 \\ 0 & 0 & \beta & 0 & \beta & -(2\alpha+\beta) & 0 \\ 0 & 0 & 0 & \beta & 0 & \beta & -\eta \end{pmatrix} \begin{pmatrix} P_0(\infty) \\ P_1(\infty) \\ P_2(\infty) \\ P_3(\infty) \\ P_4(\infty) \\ P_5(\infty) \\ P_6(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using the normalizing condition

$$P_0(\infty) + P_1(\infty) + P_2(\infty) + P_3(\infty) + P_4(\infty) + P_5(\infty) + P_6(\infty) = 1 \tag{8}$$

The solution of (7) gives the steady state probabilities. The explicit expressions for the steady state probabilities of availability and busy periods are given by

$$A_V(\infty)=\frac{N_0}{D} \tag{9}$$

$$B_{R1}(\infty)=\frac{N_1}{D} \tag{10}$$

$$B_{R2}(\infty)=\frac{N_2}{D} \tag{11}$$

$$N_0=\eta\beta(\alpha^2\beta^2+2\alpha^2\beta\delta_2+2\alpha^2\beta\delta_3+4\alpha^2\delta_2\delta_3+2\alpha\beta^3+3\alpha\beta^2\delta_2+3\alpha\beta^2\delta_3+4\alpha\beta\delta_2\delta_3$$

$$+\beta^4+\beta^3\delta_2+\beta^3\delta_3+\beta^2\delta_2\delta_3)+\beta\eta(\alpha^2\beta^2+2\alpha^2\beta\delta_3+2\alpha^2\beta\delta_1+4\alpha^2\delta_1\delta_3+\alpha\beta^3+3\alpha\beta^2\delta_1$$

$$+\alpha\beta^2\delta_3+4\alpha\beta\delta_1\delta_3+\beta^2\delta_1\delta_3)+\beta\eta(2+\beta^3\delta_1+\alpha^2\beta\delta_2+4\alpha^2\delta_1\delta_2+\alpha\beta^3+2\alpha\beta^2\delta_2+\alpha\beta^2\delta_1$$

$$+4\alpha\beta\delta_1\delta_2+\beta^2\delta_1\delta_2)+\beta^2\eta(\alpha\beta^2+\alpha\beta\delta_2+2\alpha\beta\delta_3+\alpha\beta\delta_1+2\delta_1\delta_3+2\alpha\delta_2\delta_3+\beta^3+\beta^2\delta_1$$

$$+\beta^2\delta_3+\beta^2\delta_2+\beta\delta_1\delta_3+\beta\delta_2\delta_3)+\beta^2\eta(\alpha\beta\delta_2+2\alpha\delta_1\delta_2+\beta^3+\beta^2\delta_1+\beta^2\delta_3+\beta^2\delta_1$$

$$+\beta\delta_1\delta_2+\beta\delta_2\delta_3+\beta\delta_1\delta_3)$$

$$N_1=\beta^2\eta(\alpha\beta^2+\alpha\beta\delta_2+2\alpha\beta\delta_3+\alpha\beta\delta_1+2\alpha\delta_1\delta_3+2\alpha\delta_2\delta_3+\beta^3+\beta^2\delta_1+\beta^2\delta_3+\beta^2\delta_2$$

$$+\beta\delta_1\delta_3+\beta\delta_2\delta_3)+\beta^2\eta(\alpha\beta\delta_2+2\alpha\delta_1\delta_2+\beta^3+\beta^2\delta_1+\beta^2\delta_3+\beta^2\delta_2+\beta\delta_1\delta_2+\beta\delta_2\delta_3$$

$$+\beta\delta_1\delta_3)+\beta(\beta^5+\alpha^2\beta^2\delta_2+2\alpha^2\beta\delta_2\delta_3+2\alpha^2\beta\delta_1\delta_2+4\alpha^2\delta_1\delta_2\delta_3+\alpha\beta^4+2\alpha\beta^3\delta_2$$

$$+2\alpha\beta^3\delta_3+\alpha\beta^3\delta_1+3\alpha\beta^2\delta_2\delta_3+3\alpha\beta^2\delta_1\delta_2+2\alpha\beta^2\delta_1\delta_3+4\alpha\beta\delta_1\delta_2\delta_3+\beta^4\delta_2+\beta^4\delta_3$$

$$+\beta^4\delta_1+\beta^3\delta_1\delta_2+\beta^3\delta_2\delta_3+\beta^3\delta_1\delta_3+\beta^2\delta_1\delta_2\delta_3)$$

$$N_2=\beta(\alpha^2\beta^2\delta+2\alpha^2\beta\delta_2\delta_3+2\alpha^2\beta\delta_1\delta_2+4\alpha^2\delta_1\delta_2\delta_3+\alpha\beta^4+2\alpha\beta^3\delta_2+2\alpha\beta^3\delta_3+\alpha\beta^3\delta_1$$

$$+3\alpha\beta^2\delta_2\delta_3+3\alpha\beta^2\delta_1\delta_2+2\alpha\beta^2\delta_1\delta_3+4\alpha\beta\delta_1\delta_2\delta_3+\beta^5+\beta^4\delta_2+\beta^4\delta_3+\beta^4\delta_1+\beta^3\delta_1\delta_2$$

$$+\beta^3\delta_2\delta_3+\beta^3\delta_1\delta_3+\beta^2\delta_1\delta_2\delta_3)$$

$$D = 3\beta^4\eta\delta_2 + \beta^3\delta_1\delta_2\delta_3 + \beta^4\delta_1\delta_2 + \beta^4\delta_2\delta_3 + \beta^5\delta_2 + 3\alpha\beta^3\delta_1\delta_2 + 3\alpha\beta^3\delta_2\delta_3 + 2\alpha\beta^4\delta_2$$

$$+ 4\alpha\beta^2\delta_1\delta_2\delta_3 + \alpha^2\beta^3\delta_2 + \beta^4\delta_1\delta_3 + 3\beta^4\eta\delta_3 + 3\beta^3\eta\delta_1 + \beta^6 + 3\beta^3\eta\delta_2\delta_3 + 2\beta^3\eta\delta_1\delta_2$$

$$+ 3\beta^3\eta\delta_1\delta_3 + 2\alpha^2\beta^2\delta_2\delta_3 + \beta^5\delta_3 + 4\alpha^2\beta^2\eta\delta_2 + 2\alpha^2\beta^2\delta_1\delta_2 + 2\alpha^2\beta^2\eta\delta_1 + 4\alpha^2\beta^2\eta\delta_3$$

$$+ 7\alpha\beta^3\eta\delta_2 + 2\alpha\beta^3\delta_1\delta_3 + 7\alpha\beta^3\mu\delta_3 + 5\alpha\beta^3\eta\delta_1 + 3\beta^5\eta + \beta^5\delta_1 + \beta^2\eta\delta_1\delta_2\delta_3 + \alpha\beta^5$$

$$+ 2\alpha^2\beta^3\eta + \alpha\beta^4\delta_1 + 2\alpha\beta^4\delta_3 + 5\alpha\beta^5\eta + 6\alpha^2\beta\eta\delta_2\delta_3 + 4\alpha^2\beta\delta_1\delta_2\delta_3 + 4\alpha^2\beta\eta\delta_1\delta_2$$

$$+ 4\alpha^2\beta\eta\delta_1\delta_3 + 4\alpha^2\eta\delta_1\delta_2\delta_3 + 6\alpha\beta^2\delta_1\delta_2 + 8\alpha\beta^2\eta\delta_2\delta_3 + 7\alpha\beta^2\eta\delta_1\delta_3 + 4\alpha\beta\eta\delta_1\delta_2\delta_3$$

The system/units are subject to corrective maintenance/replacement on failure as can be observed in states 4, 5 and 6. From Table 1, the repairman is busy performing corrective maintenance action on the units due to failure in states 4, 5 and 6 or replacing the entire system in state 6. Let C_0, C_1 and C_2 be the revenue generated when the system is in working state (it is assumed that no income is obtained when in the failed state), and the cost of a unit of repair time (corrective maintenance), and the cost of replacement time, respectively. The expected total profit per unit time obtained by the system in the steady state is

$$PF = C_0 A_V(\infty) - C_1 B_{R1}(\infty) - C_2 B_{R2}(\infty) \tag{12}$$

where PF is the profit obtained bythe system.

4. Numerical examples

The following set of parameters are fixed throughout the simulations for consistency: $\delta_1 = 0.3$, $\delta_2 = 0.8$, $\eta = 0.1$, $\alpha = 0.93$, $\beta = 0.1$, $C_0 = 20\,000$, $C_1 = 100$, $C_2 = 250$, $0.02 \le \delta_3 \le 0.08$. Fast deterioration has implication regarding reliability measures such as system availability and profit. Furthermore, it is common knowledge that fast deterioration can reduce system performance and will ultimately lead to random failure.

Figures 1 and 3 display the trends of system availability and profit against the repair rate α for various values of fast deterioration rate. It is evident from the figures that the system availability and profit are increasing in α for each of the values of fast the deterioration rate. The gaps between the curves in the figures become smaller as the fast deterioration rate increases. Moreover, as the fast deterioration rate increases, the system availability and profit decrease slightly, this means that the system availability and profit are sensitive to the fast deterioration rate. This sensitivity analysis implies that major maintenance should be invoked to minimize the fast deterioration rate in order to maximize the system availability and profit.

On the other hand, system availability and profit will also be affected by other parameter such as failure rate. Thus, the higher the unit failure rate is, the lower the system availability, production output as well the as profit generated.

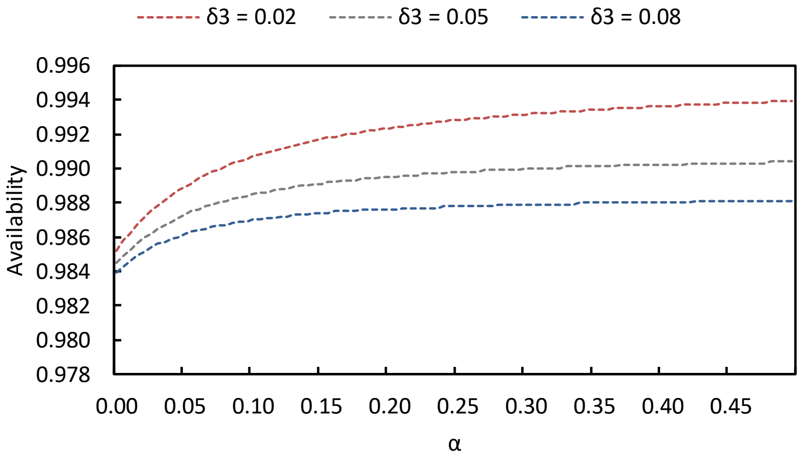

Fig. 1. Dependencies of the system availability on α for various fast deterioration rates δ_3

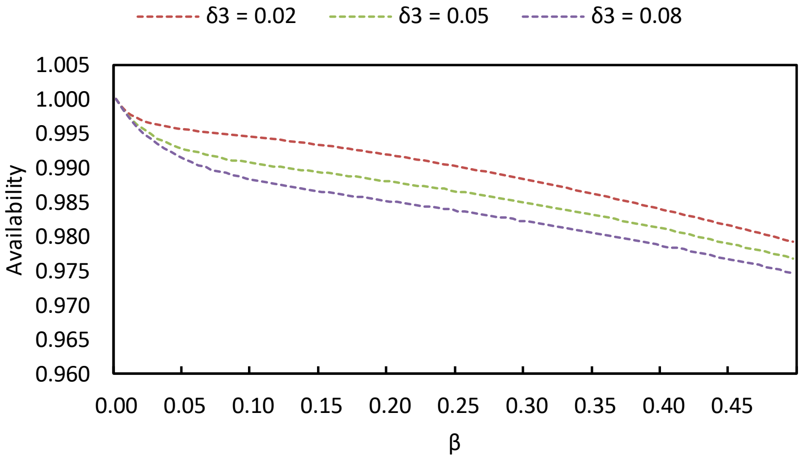

Fig. 2. Dependencies of the system availability on β for various fast deterioration rates δ_3

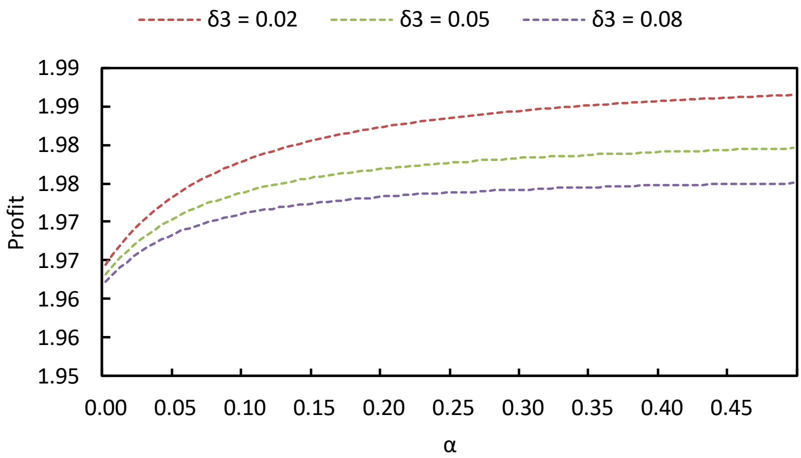

Fig. 3. Dependencies of the profit on α for various fast deterioration rates δ_3

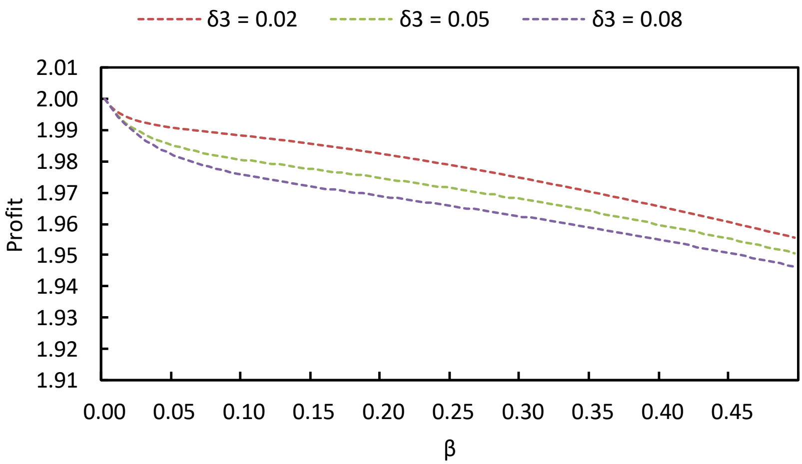

Fig. 4. Dependencies of the profit on β for various fast deterioration rates δ_3

Figures 2 and 4 show the behaviour of system availability and profit against the failure rate β for various values of the fast deterioration rate. It is clear from these figures that the system availability and profit are decreasing in β. The gaps between the curves in Figs. 2 and 4 are closer to each other than in Figs. 1 and 3 as the fast deterioration rate increases. Moreover, as the fast deterioration rate increases, system availability and profit also decreases, which means that system availability and profit are sensitive to the fast deterioration rate. This sensitivity analysis implies that replacement of failed units/the entire system should be invoked to improve and maximize system availability, production output as well as profit.

4. Conclusion

A linear consecutive 2-out-of-4 repairable system has been considered with three consecutive stages of deterioration before failure. Explicit expressions are given for the system availability, the probability of being in busy period and the profit function. Analysis of the effect of various system parameters on the system availability, and the profit function was performed. These are the main contribution of the paper. On the basis of the graphical results obtained in Figs. 1–4 for a particular case, it is suggested that the availability and profit of a system can be improved by taking more redundant unit in cold standby, invoking preventive maintenance prior to deterioration and increasing the unit repair rate. Maintenance managers, reliability engineers and system designers are faced with the effects of competition and market globalisation on maintenance systems to improve efficiency and reduce operational costs. The models developed in this paper are found to be highly beneficial to engineers, maintenance managers, system designers and plant management in the appropriate analysis of maintenance policy and procedures, as well as the evaluation of performance and the safety of the system as a whole. The results derived in this paper could be applied in practical fields by making suitable modification and extensions. Further studies on such subjects would be expected.

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