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BANZHAF VALUE FOR GAMES ANALYZING VOTING WITH ROTATION*

The voting procedure has been presented with rotation scheme used by the Governing Council of the European Central Bank as it enlarges to accommodate new members of the economic and monetary union. The main game theoretical approaches have been presented elsewhere. That paper considered the Shapley value computed in accordance with these approaches. The Banzhaf value has been analysed and the results compared with the results for the Shapley value.

Keywords: *voting, European Central Bank, Shapley value, Banzhaf value*

1. Introduction

In this paper, we analyze voting with rotation, a voting procedure which is planned to be used in the European Central Bank. The idea for such a procedure results from the predicted difficulties of voting with an enlarged number of voters and guarantees voting power to the economically most important countries.

In Sosnowska's paper [12], the Shapley value was analyzed. Some unexpected results were observed. As there exists no single method of computing power indices for voting with rotation, several methods were used. It was shown that various methods led to different results. This paper generalizes the results obtained in [12] to the Banzhaf value. The same examples as in [12] have been analyzed. The results are compared with those obtained for the Shapley value.

The paper is constructed as follows. In Section 2, the voting procedure with rotation, used by the European Central Bank, is presented. The game theory approach to

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*This paper is dedicated in remembrance to Janusz Beksiak.

this voting procedure is presented in Section 3. In Section 4, examples are analyzed. In Section 5, properties of the methods considered are presented. Conclusions are formulated in Section 6.

2. Voting procedure with rotation used in the European Central Bank

As of 21st March 2003, the European Council introduced voting with rotation into the statute of the European Central Bank (ECB). This presentation of the ECB voting system is based on the ECB Monthly Bulletin [4]. A short outline can be found in [12]. Several parts of this outline are used in this paper.

The ECB Governing Council (GC) consists of the Executive Board of the ECB (EB) and the governors of the national central banks (NBC) of the countries that have adopted the Euro as their currency. The EB comprises 6 members: the President, the Vice-President and 4 other members. All members are appointed by the European Council acting by a qualified majority. Each member of the GC has one vote. The GC decides by simple majority with the President having the casting vote in the case of a tie. The number of members of the GC who have voting rights at any given time was limited to 21. The members of the EB have permanent voting rights. So, the number of governors with voting rights could not exceed 15. This limit was raised to 18 in 2008. Latvia was the 18th country to join the Euro zone on January 1, 2014. The number of countries in the Euro zone will exceed 18 when Lithuania joins the zone.

When the number of governors exceeds 18, they will be allocated into two groups on the basis of a ranking determined by a composite economic indicator. The first group will consist of the first five governors according to that ranking. According to the present situation, the governors from Germany, France, Italy, Spain and the Netherlands form this group. The second group will consist of all the other governors. The first group will have 4 voting rights, the second group – 11.

As of the date on which the number of governors exceeds 21, the governors will be divided into 3 groups. The first group (consisting of 5 governors) will also have 4 votes. The second group will consist of half of the total number of the governors, rounded up to the nearest full number. It will have 8 votes. The third group will consist of the remaining governors. It will have 3 votes.

3. A game theoretical analysis of the ECB voting with rotation

The standard way of modeling voting is by using cooperative game theory. Usually, a game is defined and power indices or values are computed. We shall use the Shapley value [10], the Shapley–Shubik index [11], the Banzhaf value and the normalized Banzhaf value [1]. Coalitional structures (defined in many ways) in cases the where cooperation between specific players seem natural are considered.

Let us recall definitions given in [9]. The non-empty and finite set N comprises n players. Subsets of N are called coalitions. A pair $G = (N, v)$ is a cooperative game when $v{:}2^N \to R$, such that the value v ascribed to the empty set is zero. The function v is called the characteristic function of a game. A game is convex when $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$. A game is called monotonic when $S \subseteq T$ implies $v(S) \leq v(T)$. A convex game where the characteristic function takes nonnegative values is a monotonic game.

The Shapley value (Sh) is defined by the following formula:

$$Sh_i(G) = \sum_{S \subseteq N, i \notin S} \left(\frac{1}{n!}\right) s!(n-s-1)![v(S \cup \{i\}) - v(S)], \qquad \text{where } s = \#S$$

The Banzhaf (B) value is defined by the following formula:

$$B_i(G) = \frac{1}{2^{n-1}} \sum_{S \subseteq N, i \notin S} [v(S \cup \{i\}) - v(S)]$$

The Banzhaf value is not effective. Its normalization is called the normalized Banzhaf value (BN).

$$BN_i(G) = \frac{BN_i}{BN_1 + \ldots + BN_n}$$

Games such that the characteristic function only takes the values 0 and 1 are called simple games. A coalition S is called a winning coalition when $v(S) = 1$.

When the Shapley [Banzhaf] value is restricted to simple games, it is called the Shapley–Shubik [Bazhaf] index (SS, [B]).

Some game theory specialists such as Owen [9] deal only with convex games. Others such as Mesterton [8] also deal with games which are not convex games. As Machover [7] writes, mathematically, the Shapley value can also be defined for such games. However, there may be some difficulties related to the interpretation of such a value.

The Shapley value and the Banzhaf value measure the power of players considered from different points of view.

Some special simple games are very often used for modeling voting. They are called weighted voting games. Player i has the weight w_i. There is a threshold t such that a coalition S is a winning coalition whenever

$$\sum_{i \in S} w_t > t$$

Now we shall present and compare the properties of some, chosen as being the most important, propositions for measuring power in the ECB when voting with rotation is applied. We consider the game theoretic aspects of such a procedure. Game theoretic approaches were analyzed in some earlier papers by Ulrich [13], Belke and Styczyńska [3], Kosior, Rozkrut and Toroj [6], Belke and von Schnurbein [2], as well as Sosnowska [12]. We recall some definitions, which will be used in our analysis.

In Ulrich's paper [13], an intertemporal cooperative game (ICG) is defined by using intertemporal voting shares, which are the probabilities of a given member having a vote, computed for each group as the number of voting rights per member of that group. The intertemporal voting share of a player is their weight. A weighted voting game is played.

Kosior et al. [6] analyze a generalization of the Shapley–Shubik index for voting procedures with rotation, which is constructed as the weighted sum of the Shapley –Shubik indices computed for all the possible auxiliary games implied by the voting procedure with rotation used. The weight of a particular auxiliary game is the probability of it being played at a randomly chosen time. An auxiliary game is a weighted voting game where each player has weight 1 and the threshold is 50% of the number of players. We shall call this generalization the average Shapley–Shubik index (*ASS*; other authors call it the Shapley–Shubik index). The number of possible auxiliary games in the case of 27 countries is 8960. This corresponds to the least common multiple of the cycles of rotation of various groups among the NCB governors and the EB members. In this paper, we use this construction to define the average Banzhaf value (*AB*) and the average normalized Banzhaf value (*A*(*BN*)).

Sosnowska [12] considered a new concept of the Shapley value for voting with rotation. She considers the weighted value of coalition (*WVC*). As in the averaging method, we consider all the possible auxiliary games. Then we construct a new game on the whole set of players where the value of a coalition is the weighted value of that coalition in the auxiliary games considered. The weight of an auxiliary game is the probability of it being the one played at a randomly chosen time. For coalitions which cannot be constructed in such a way, their value is a parameter (usually 0), with the exception of the grand coalition, which has value 1 or another parameter. The Shapley value was computed for such a game. In this paper, we also compute the Banzhaf value.

In the next section, we shall study two examples of voting with rotation and compare the results.

4. Various methods of defining values. Examples

The examples analyzed in this section are the same as those considered in [12]. The results concerning the Shapley value originated from the same paper.

Example 1

There are 4 players. Only 3 players take part in each vote. The player who does not vote, alternates evenly over the set {1, 2, 3, 4}. Hence, four different 3-player games are played. Each voter in the auxiliary games has one vote. In all these 3-player games the Shapley–Shubik index of each voter equals 1/3.

The intertemporal cooperative game (ICG) method. The method is based on Ulrich's paper [13]. Each player has a share of 3/4. Hence, we consider the weighted voting game (3/4, 3/4, 3/4, 3/4). The sum of weights equals 3. The winning coalitions are those in which the sum of the weights of the players is greater than 1.5. So, all 3-player coalitions and the grand 4-player coalition form the set of winning coalitions. Each Shapley–Shubik index equals 1/4. Similarly, each Banzhaf index equals 3/8 and each normalized Banzhaf index equals 1/4.

Averaging method. The method is based on Kosior et al. [6]. In each auxiliary 3-player game the Shapley–Shubik index equals 1/3. The probability of each game is 1/4. Each player has voting rights in 3 auxiliary games. So, the weighted sum of the Shapley–Shubik indices is equal to $3\times(1/4)\times(1/3) = 1/4$ for each player. Analogously, we compute the average sum of the Banzhaf indices (the average Banzhaf value – AB) and the normalized Banzhaf indices (the average normalized Banzhaf value – $A(BN)$). For each i, $AB_i = 3/8$, $A(BN)_i = 1/4$. The average Banzhaf index is not efficient, so it can be normalized. After normalization, we obtain the normalized average Banzhaf value ($(AB)N$). For each i, $(AB)N_i = 1/4$. The next section demonstrates that the following equality always holds: $(AB)N = A(BN)$.

Weighted value of a coalition (WVC). The method is based on a paper by Sosnowska [12]. We calculate the weighted value of a coalition. We consider all the possible systems of voting rights. For each system M, we construct the set of auxiliary games $G(M)$. Hence, in our example, we construct a 4-player game where the value of a coalition is computed as the weighted sum of the values of this coalition in the auxiliary 3-player games considered. The weight of a game is the probability of it being the one played at a random time. This probability equals 1/4 for each auxiliary game.

Let us consider 3-player auxiliary games. All these games are constructed in the same way.

$v_M(A) = 1$ when $\#A \geq 2$ and $v_M(A) = 0$ when $\#A < 2$.

Let w be the characteristic function of the 4-player game based on the weighted values of coalitions (WVC).

Each 3-player coalition can occur in only one auxiliary game. So, $v_w(A) = 1\times(1/4)\times1 = 1/4$ when $\#A = 3$.

Each 2-player coalition can occur in two auxiliary games. So, $v_w(A) = 2\times(1/4)\times1 = 1/2$ when $\#A = 2$. $v_w(A) = 0$ when $\#A < 2$. Hence, $v_w(A) = 0$ when $\#A = 1$.

It is impossible to define the value of the grand coalition in such a way. So we define $v_w(A) = a$ for A such that $\#A = 4$.

We have thus obtained a non-simple, non-monotonic and improper game.

In our game it was shown that the Shapley value is equal to $a/4$ for each player. When $a = 1$, this is the same result as obtained using the ICG and averaging methods. The Banzhaf value equals $1/8(a - 1/2)$ for each i, the normalized Banzhaf value equals 1/4 for each i. We can present the above results in the following tables.

Table 1. Example 1. Shapley–Shubik value computed by various methods

	ICG	Averaging	WVC
$I = 1, 2, 3, 4$	1/4	1/4	$a/4$

Source: Sosnowska [12].

We can see that when $a = 1$ for the WVC all the results are identical.

Table 2. Example 1. Banzhaf value and normalized Banzhaf value computed by various methods

	ICG	Averaging	WVC
B_i	3/8	3/8	$1/8(a - 1/2)$
BN_i	1/4	1/4	1/4
$A(BN)_i$		1/4	

Source: author's work.

Now, let us study an example where voting rights are not symmetric.

Example 2

We consider a situation where there are 4 players. Player 1 has permanent voting rights. Only 2 of players 2, 3, 4 have voting rights at any one time. The player who does not vote, alternates over this set of three players. Three different 3-player auxiliary games are considered. We analyze the same 3 methods as in Example 1.

ICG. The players' weights are the probabilities of having voting rights. The weight of player one is 1, and the weights of players 2, 3, 4 are each 2/3. So, we consider the weighted voting game (1, 2/3, 2/3, 2/3). The sum of weights is 3. The threshold is 1.5.

The characteristic function is computed as follows:

$v(S) = 1$ when $\#S \geq 3$ or ($\#S = 2$ and $1 \in S$).

$SS_1 = 1/2$, $SS_i = 1/6$ for $i = 2, 3, 4$. $B_1 = 3/4$, $B_2 = B_3 = B_4 = 1/4$. $BN_1 = 1/2$, $BN_2 = BN_3 = BN_4 = 1/6$.

Averaging method. The following triples of players can have voting rights in the au×iliary games: {1, 2, 3}, {1, 3, 4}, {1, 2, 4}. In each auxiliary 3-player game, $SS_i = 1/3$ for each voter. The probability of each game is 1/3. The probability that player 1 takes part in a 3-player auxiliary game equals 1. For the other players, this probability is equal to 2/3. So the weighted sum of the Shapley–Shubik indices for player 1 equals 3×(1/3)×(1/3) = 1/3 (number of games × probability × SS_1). For players $i = 2, 3, 4$, the weighted sum of the Shapley–Shubik indices equals 2×(1/3)×(1/3) = 2/9. The Banzhaf index for each of the three voters in each auxiliary game is equal to $B_i = 3/4$, $BN_i = 1/3$. We thus obtain the following average values $AB_1 = 3\times(1/3)\times(3/4) = 3/4$, $AB_i = 2\times(1/3)\times(3/4) = 1/2$, $i = 2, 3, 4$, $A(BN)_1 = 1/3$, $A(BN)_i = 2/9$, $i = 2, 3, 4$. $(AB)N_1 = 1/3$, $A(BN)_i = 2/9$, $i = 2, 3, 4$.

WVC. The sets of voters in the 3-player auxiliary games G_1, G_2, G_3 are {1, 2, 3}, {1, 3, 4} and {1, 2, 4}, respectively. Each auxiliary game is a weighted voting game with weights 1 for each voters and threshold 1.5. The winning coalitions are presented in Table 3.

Table 3. Example 2. Winning coalitions in 3-player games using the WVC method

Game		
G_1	G_2	G_3
{1, 2, 3}, {1, 2}, {2, 3}, {1, 3}	{1, 3, 4}, {1, 4}, {3, 4}, {1, 3}	{1, 2, 4}, {1, 2}, {1, 4}, {2, 4}

Source: Sosnowska [12].

Each auxiliary game is played with the probability of 1/3. So the weighted values of coalitions are as follows:

$v_w(\{1, 2, 3\}) = v_w(\{1, 3, 4\}) = v_w(\{1, 2, 4\}) = (1/3)\times1 = 1/3$;

$v_w(\{1, c\}) = 2\times(1/3)\times1 = 2/3$, $c = 2, 3, 4$;

$v_w(\{c, d\}) = 1\times(1/3)\times1 = 1/3$ where $c, d = 2, 3, 4$, $c \neq d$;

$v_w(\{c\}) = 0$ for $c = 1, 2, 3, 4$.

We construct the game w on the set of players {1, 2, 3, 4} where $w(S) = v_w(S)$ if S is a coalition from one of the games G_1, G_2, G_3. Some subsets of {1, 2, 3, 4} do not occur in particular auxiliary games G_i, $i = 1, 2, 3$, therefore we have to define their values. We introduce parameters a and b and define $v_w(\{1, 2, 3, 4\}) = a$ and

$v_w(\{2, 3, 4\}) = b$. The most intuitive way of defining a and b is $a = 1$ (because we are dealing with the grand coalition) and $b = 0$ (because such a coalition can never exist). We compute the Shapley value of the game $G = (\{1, 2, 3, 4\}, v_w)$.

$Sh_1(G) = (1/24)[6(a - b) + 3\times(4/3)] = (a - b)/4 + 1/6$.

$Sh_i(G) = (1/24)[6a - 2 + 2b - 2/3] = a/4 + b/12 - 1/18$ for $i = 2, 3, 4$.

Table 4 presents the Shapley values according to the three methods considered.

Table 4. Example 2. The Shapley values according to various methods

Player	Method		
	ICG	Averaging	WVC
1	1/2	1/3	$(a - b)/4 + 1/6$
2, 3, 4	1/6	2/9	$a/4 + b/12 - 1/18$

Source: Sosnowska [12].

We compute the Banzhaf and the normalized Banzhaf value.

$B_1(G) = (1/8)\times[(a - b) + ((1/3) - (1/3))\times 3 + ((2/3) - 0)\times 3] = (a - b + 2)/8$,

$B_i(G) = (1/8)\times[(a - (1/3)) + ((1/3) - (2/3))\times 2 + (b - (1/3)) + 2/3 + (1/3)\times 2] = (a - b)/8$, $i = 2, 3, 4$.

$$\sum_{i=1}^{4} B_i = (2a - 2b + 1)/4,\ BN_1 = \frac{a-b+2}{2(2a-2b+1)},\ BN_i = \frac{a-b}{2(2a-2b+1)}, i = 2, 3, 4.$$

Tables 5 and 6 present the Banzhaf values and the normalized Banzhaf values according to the three methods considered.

Table 5. Example 2. The Banzhaf values according to various methods

Player	Method		
	ICG	Averaging	WVC
1	3/4	3/4	$(a - b + 2)/8$
2, 3, 4	1/4	1/2	$(a - b)/8$

Source: author's work.

Table 6. Example 2. The normalized Banzhaf values according to various methods

Player	Method		
	ICG	Averaging	WVC
1	1/2	1/3	$(a - b + 2)/(2(2a - 2b + 1))$
2, 3, 4	1/6	2/9	$(a - b)/(2(2a - 2b + 1))$

Source: author's work.

This time the results from using these three different methods are different. First, we analyze the Shapley value. Comparing the results presented in Table 5, we see that for parameters satisfying $a - b < 4/3$, the index of player 1, the one with permanent voting rights, is maximized when we use the ICG method. When $a - b \leq 2/3$, the index of Player 1 is minimized when the WVC method is used. Player 1 is also stronger than any of players 2, 3, 4 according to the ICG and ASS methods and for $b \leq 2/3$ according to the WVC method.

Consider $a = 1$ and $b = 0$. In this case, player 1 is stronger than any of players 2, 3, 4 according to the WCV method. We obtain $Sh_1 = 5/12$ and $Sh_i = 7/36$ for $i = 2, 3, 4$. In this case, the results according to the various methods are compared in Table 7.

Table 7. Example 2. The Shapley values according to the WVC method for $a = 1$ and $b = 0$

Player	Method		
	ICG	Averaging	WVC
1	1/2	1/3	5/12
2, 3, 4	1/6	2/9	7/36

Source: Sosnowska [12].

We obtain different results for the Banzhaf value. When $a - b > 4$, the non-normalized index of Player 1 is maximized when the WVC method is used. For $a = 1$ and $b = 0$, the normalized index of player 1 is maximal for when either the ICG or averaging method is used.

Consider the normalized Banzhaf value. When $a - b < 1$, the index of player 1 is maximized when the WVC method is used. For $a = 1$ and $b = 0$, the index of player 1 is maximal under either the ICG or WVC method.

Table 8. Example 2. The Banzhaf values according to various methods for $a = 1$, $b = 0$

Player	Method		
	ICG	Averaging	WVC
1	3/4	3/4	3/8
2, 3, 4	1/4	1/2	1/8

Source: author's work.

Table 9. Example 2. The normalized Banzhaf values according to various methods for $a = 1$, $b = 0$

Player	Method		
	ICG	Averaging	WVC
1	1/2	1/3	1/2
2, 3, 4	1/6	2/9	1/6

Source: author's work.

5. Various methods of defining values. Properties

The results presented in the previous section are connected with some general properties of the WVC method. We consider the most intuitive case where we set the value of the grand coalition to be 1 and assume other coalitions which do not occur in auxiliary games have the value of 0. Our results follow from the symmetry in the construction of the rotation scheme and the anonymity properties of the Shapley and Banzhaf values.The following definitions are formulated by Sosnowska in [12].

First, let us deal with the situation presented in Example 1. Let us consider a voting system with rotation where there are n players and k voting rights, $n > k$. We assume that players are ordered in such a way that the players with voting rights in the first vote are labeled from 1 to k. Then the first voter relinquishes his voting rights to the first player who did not vote in the previous turn. So, in the second vote players with numbers from 2 to $k + 1$ have voting rights. An analogous method of relinquishing voting rights is used in each subsequent round. In the m-th vote, players with numbers m, $(m + 1)(\text{mod } n)$, ..., $(m + k - 1)(\text{mod } n)$, where $m = 1, ..., n$, have voting rights. In the last vote considered, players with numbers n, 1, ..., $k - 1$ have voting rights. We denote this system by $V(n, k)$.

The following lemma holds.

Lemma 1 [12]. In the game, $G = (N, v)$ constructed for the system $V(n, k)$, where $N = \{1, ..., n\}$ and v is constructed according to the WVC method, the Shapley values of all the players are equal. This result also holds for the Banzhaf value and the normalized Banzhaf value.

Lemma 2. In the game $G = (N, v)$ constructed for the system $v(n, k)$, where $N = \{1, ..., n\}$ and v is constructed according to the WVC method, the Banzhaf values and the normalized Banzhaf values of all the players are equal.

Proof. The proof is the same as the proof of Lemma 1 and uses the anonymity property of the Banzhaf value [5].□

Let us study the situation presented in Example 2. It is a generalization of the situation in Example 1, and Lemma 3 is a generalization of Lemma 1.

Let the set of players N be divided into q disjoint groups of players N_i, each with k_i voting rights, $i = 1, ..., q$. $n_i = \#N_i$. We assume that players are ordered and

$N_1 = \{1, ..., n_1\}$, $N_2 = \{n_1 + 1, ..., n_1 + n_2\}$, ..., $N_i = \{n_1 + \dots + n_{i-1} + 1, ..., n_1 + ... + n_{i-1} + n_i\}$.

Let us denote $m_0 = 0$, $m_i = n_1 + ... + n_i$, $i = 1, ..., q$. Then $N_i = \{m_{i-1} + r: r = 1, ..., n_i\}$, $i = 1, ..., q$. In the following, we shall use the convention $m(\text{mod } m) = m$.

Players in group N_i have k_i voting rights, where $k_i \leq n_i$. $k = k_1 + ... + k_n$. For $k_i < n_i$, the rotation scheme works as follows: We assume that in the first round the first k_i players in N_i vote, $i = 1, ..., q$. These are players $m_{i-1} + r$, $r = 1, ..., k_i$. Players $m_{i-1} + r$, $r = k_i + 1, ..., n_i$ do not have voting rights in the first round. In each successive round, the first player in the list of those with voting rights relinquishes his vote to the first player in the list of those without voting rights. Both of these players are transferred to the end of the list opposite the list he has just left.

Applying this scheme, in round s, the first voter in this round, player $m_{i-1} + s(\mathrm{mod}\ n_i)$ relinquishes his voting rights to player $m_{i-1} + (s + k_i)(\mathrm{mod}\ n_i)$. So, in round s, the set of players in N_i with voting rights is $N_i^s = \{[m_{i-1} + (s + t)(\mathrm{mod}\ n_i)](: t = 0, ..., k_i - 1\}$, where $s = 1, ..., K$, and K is the lowest common multiple of the numbers $n_1, ..., n_q$. We define K in such a way because we need to consider all the possible auxiliary games resulting from this scheme. We shall denote such a system of voting rights by $v(n_1, ..., n_q, k_1, ..., k_q)$.

For the system $V(n_1, ..., n_q, k_1, ..., k_q)$, let us define the auxiliary game $G^s = (N^s, v^s)$, $s = 1, ..., K$, where $N^s = \bigcup_{i=1}^{q} N_i^s$ and v^s is defined as follows:

For $T \subseteq N^s$, $v(T) = 1$ if $\#T > k/2$ and $v(T) = 0$ otherwise. The family of auxiliary games G^s is constructed from all the games which may be played using the rotation scheme. The probability of each game is $1/K$. Now we define the game $G = (N, v)$, where $v(T) = 1/K \sum_{s:T \subseteq N^s} v^s(T)$ if there exists s such that $T \subseteq N^s$, $v(T) = 1$ when $T = N$ and $v(T) = 0$ otherwise. $v(T)$ is equal to the expected value of v^s in the case where it is possible to compute $v^s(T)$, 1 for the grand coalition N and 0 for other coalitions when the expected value cannot be computed.

The following lemmas hold.

Lemma 3 [12]. Let us consider a system of voting rights $V(n_1, ..., n_q, k_1, ..., k_q)$ and $h, j \in N_i$. Then $Sh_h(G) = Sh_j(G)$.

Lemma 4. Let us consider a system of voting rights $V(n_1, ..., n_q, k_1, ..., k_q)$ and $h, j \in N_i$. Then $B_i(G) = B_j(G)$, $BN_i(G) = BN_j(G)$.

Proof. The same construction as in the proof of Lemma 13 and the anonymity property of the Banzhaf value are used. □

These lemmas are also true for other methods of deriving the power indices. Let us consider ICG. All players from the same group have the same weights, so their Shapley values are the same. For the same reason, their Banzhaf values are the same. The equality of the normalized Banzhaf values follows from the anonymity property. So, the following lemma holds.

Lemma 5. Let us consider the system of voting rights $V(n_1, ..., n_q, k_1, ..., k_q)$ and the ICG method, $h, j \in N_i$. Then $Sh_h(G) = Sh_j(G)$, $B_i(G) = B_j(G)$, $BN_i(G) = BN_j(G)$.

The averaging method also possesses the above property. The following lemma holds.

Lemma 6. Let us consider the system of voting rights $V(n_1, ..., n_q, k_1, ..., k_q)$ and the averaging method. $h, j \in N_i$. Then $ASS_h(G) = ASS_j(G)$, $AB_i(G) = AB_j(G)$, $A(BN)_i(G) = A(BN)_j(G)$.

Proof. Let us consider the auxiliary game G. It is a weighted majority game with $k = k_1 + \ldots + k_q$ voters, each with weight 1, and threshold $E(k/2) + 1$. So, $Sh_h(G) = Sh_j(G)$, $B_h(G) = B_j(G)$ and $BN_h(G) = BN_j(G)$. The average value for player j is computed according to the following formula: probability of game×number of auxiliary games with player j×value of player j in auxiliary game G. The probability of each auxiliary game is $1/K$, the number of auxiliary games where player j has voting rights is the same as the number of auxiliary games where player h has voting rights if h and j are from the same N_i. The Shapley values of all the voters in auxiliary game G are the same. So, the Banzhaf values for players in the same group are equal, as are the normalized Banzhaf values. □

As mentioned in Section 4, $A(BN) = (AB)$N. We shall prove this fact.

Lemma 7. For the averaging method $\mathrm{A}(BN)_j = (\mathrm{AB})N_j$ for all $j \in N$.

Proof. Let us consider the family of auxiliary games $G^s = (N^s, v^s)$, $s = 1, ..., K$, as in the proof of Lemma 4 and use the same arguments. All the auxiliary games are k-person weighted voting games with each voter having weight 1 and threshold $E(k/2) + 1$. So, in every auxiliary game all the voters are symmetrical and have the same Banzhaf value and normalized Banzhaf value. All auxiliary games G^s are the same from a game theoretical point of view, so their Banzhaf values and normalized Banzhaf values are the same. Let b denote the Banzhaf value and b' – the normalized Banzhaf value. $b' = 1/k$ because the sum of the normalized Banzhaf values in each auxiliary game G^s is equal to 1.

Let $j \in N_i$. $\#\{s: j \in N^s\}$ is the same for all j from N^s. We denote $c_i = \#\{s: j \in N^s\}$. This is the number of auxiliary games G^s in which player j plays

$$(AB)_j = \left(\frac{1}{K}\right)c_i b, \qquad \sum_{j=1}^{n} AB_j = \left(\frac{1}{K}\right) b \sum_{i=1}^{q} \sum_{j \in N_i} c_i = \left(\frac{1}{K}\right) b \sum_{i=1}^{q} n_i c_i \text{ because}$$

$$c_i = \left(\frac{K}{n_i}\right) k_i, \quad n_i c_i = K k_i \quad \text{and} \quad \sum_{j=1}^{n} (AB)_j = \left(\frac{1}{K}\right) b \sum_{i=1}^{q} K k_i = b \sum_{i=1}^{q} k_i = bk$$

Thus, $N(AB)_j = \dfrac{AB_j}{\sum_{j=1}^{n} AB_j} = \left(\dfrac{1}{K}\right)\dfrac{c_i b}{(bk)} = \left(\dfrac{1}{K}\right)\left(\dfrac{c_i}{k}\right) = \dfrac{c_i}{kK}.$

$$A(BN)_j = \left(\frac{1}{K}\right)c_i b' = \left(\frac{1}{K}\right)c_i\left(\frac{1}{k}\right) = c_i(Kk) = N(AB)_j. \ \square$$

Let us consider the following example to see that the Banzhaf values and the normalized Banzhaf values can be different according to the ICG and WVC methods.

Example 3

$N = \{1, 2, 3\}$, $N_1 = \{1\}$, $N_2 = \{2, 3\}$, $q_1 = 1$, $q_2 = 1$. According to the ICG, we are dealing with a weighted voting game (1, 1/2, 1/2). $B_1 = 3/4$, $B_2 = B_3 = 1/4$, $BN_1 = 3/5$, $BN_2 = BN_3 = 1/5$.

According to the WVC method, 2 auxiliary games are considered with $N^1 = \{1, 2\}$ and $N^2 = \{1, 3\}$, each occurring with the probability of 1/2. So we obtain the following characteristic function v: $v(\{1, 2, 3\}) = a$, $v(\{2, 3\}) = b$, $v(\{1, 2\}) = v(\{1, 3)\}) = 1/2$ and 0 for other coalitions. $B_1 = (a - b + 1)/4$, $B_2 = B_3 = (a + b)/4$, $BN_1 = (a - b + 1)/(3a + b + 1)$, $BN_2 = BN_3 = (a + b)/(3a + b + 1)$. For $a = 1$, $b = 0$, we get $B_1 = 1/2$, $B_2 = B_3 = 1/4$, $BN_1 = 1/2$, $BN_2 = BN_3 = 1/4$.

6. Conclusions

It has been demonstrated that various methods of computing the values of games modeling voting with rotation (power indices in the case of simple games) lead to different results. Although it may not be particularly interesting from a mathematical point of view, it may be important for analysts who deal with predictions. Predictions made using different methods cannot be compared. Different definitions of values also give different results. Both the values considered here, the Shapley and Banzhaf values, used for modeling voting with rotation have similar mathematical properties.

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