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A THEORETICAL FRAMEWORK FOR DETERMINING THE APPROPRIATE LEVEL OF SUBSIDY IN ECONOMY

A framework has been developed for determining the subsidy that, in the long run, serves to equalize the per capita income shares across income classes. The framework characterizes the income dynamics by the Markov process and uses the principle of maximum entropy for selecting among alternative subsidy schemes. The study provides a means to forecast the per capita income shares at any instant of time and serves as an objective tool to decide on the appropriate level of subsidy.

Keywords: *entropy, Markov process, per capita income, subsidy*

1. Introduction

We consider an economy where frictional unemployment persists and a tax-subsidy scheme is to be implemented. We assume that the initial number of income earners is known and that the presence of frictional unemployment limits the search for a better paid job. We assume that the government strives to achieve a balance in the fiscal operations within the economy. We also assume that taxes and subsidies are the only instruments regulating the economy. Further, we assume that the tax bases and tax rates are given and that the tax rates are sufficiently fair that tax non-compliance is insignificant. The population of income earners is stratified into three states according to their earnings. These states are: low-income earners, middle-income earners and high-income earners [19]. The stratification is a reflection of income inequality in the economy. We use the per capita income shares as a proxy for

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income inequality and as the mean income [6]. We describe the dynamics of this phenomenon using the per capita income shares [12]. We adopt a quartile-based definition to classify income [1], rather than using natural breaks in the income data or the poverty line. This is because such a quartile-based definition is more objective. We intend to achieve a twofold objective, viz.: to derive a closed-form continuous-time representation for the per capita income shares and to develop a yardstick for selecting a subsidy plan among alternative subsidy schemes. The continuous-time representation of the per capita income shares is an econometric model that enables policy makers to forecast the state of the transient per capita income shares at any instant of time. We accomplish this task by formulating a continuous-time Markov chain under this tax-subsidy setting. The continuous-time Markov chain model is analogous to a birth-and--death process [9], wherein the rates at which births and deaths occur are defined to be the subsidy and the income loss through income tax, respectively, per financial year. Taxes and subsidies are economic regulatory instruments [18]. The yardstick for choosing an optimal subsidy plan that will achieve uniformity in the per capita income shares in the long-run is based on the principle of maximum entropy. The concept of entropy is well-known in the literature [8, 16, 21, 23]. The entropy of a k state system is given by

$$H = -c\sum_{i=1}^{k} p_i \ln p_i \tag{1}$$

where p_i is the probability that the system is in state i and c is a constant. Details on the derivation and the properties of the entropy function are found in [21]. Entropy has found applications in the measurement of income inequality[2] [11, 17]. In this study, we adopt equation (1) to achieve this goal. We believe that our method constitutes a plausible tool to describe the dynamics of income inequality so as to enable policy makers to make projections within a tax-subsidy setting regarding per capita income shares, and also enables them to determine an optimal subsidy for the economy. These are useful economic contributions. Thus the danger inherent in first trying-out a subsidy scheme is avoided. Our approach is not only of interest to mathematical economists, but also to politicians and policy makers, as it satisfies the requirement of using objective methods in decision making rather than relying on subjective means, such as personal experience or intuition.

[2]In such applications, the entropy formula is either based on Eq. (1) or expressed in terms of its maximum value as $\ln k - \left(-\sum_{i=1}^{k} p_i \ln p_i\right) = \sum_{i=1}^{k} p_i \ln kp_i$, as $c = 1$. This expression is known as redundancy.

2. Related work

One of the targets of any economy is the equitable distribution of income [10]. To achieve this target, attempts have been made by governments to bridge the income inequality gap among the various strata of the population through tax-subsidy policies [2]. Nonetheless, income inequality continues to persist. For this reason, it has become a subject of interest [11, 20]. Boadway and Wildasin [3] presented conditions under which it is useful to include public provision and subsidy in a policy mix and stated that taxation of wages can be used to deal with inequalities arising from random shocks to industries, regions, or occupations. Tuckman and Brosch [22] considered a model of income share in which all income earners are divided into *n* equal sized groups. They proposed that policies which increase income from labor have an equalizing effect on income distribution. We remark here that the approach in [22] would pose some challenges when the number of income earners is a prime number.

Several methods have been applied to describe the dynamics of income inequality. Bressler [4] identified methods such as the Lorenz–Gini approach and the Paglin approach as descriptive measures of income inequality. These two descriptive approaches differ in that average income is used in the Paglin approach, whereas actual income is used in the Lorenz–Gini approach. Levine and Singer [13] obtained a closed-form integral expression for the area under the Lorenz curve and then determined the effect of taxes on the income distribution and the resultant effect on income inequality. Esteban [7] noted that the descriptive task of fitting density functions to empirical data on income is not very inspiring.

Jana et al. [11] employed the maximum entropy approach to determine the optimal reduction in income inequality through taxation. Chakravarty [6] provided a social welfare function that mimics the entropy measure of inequality based on the mean income of the population. Cappellari and Jenkins [5] modelled income transitions between two states – poor and not poor – using a first-order, discrete-time Markov model. McCall [15] also used the discrete-time Markov model to study earnings mobility for those of working age. The states of the system were categorized into three groups: low earnings covered by social security, non-low earnings covered by social security and uncovered earnings (i.e., individuals not covered by social security). Lillard and Willis [14] used an econometric model to study earnings mobility, using the poverty line as a benchmark to distinguish between states, and then compare the results with those obtained by a Markov chain model of income mobility.

It is important to mention that this study differs from the discrete-time approach presented elsewhere [5, 15], since it models the evolution of per capita income shares as a continuous-time process. We derive ways to represent the dynamics of income inequality as a discrete-state, continuous-time Markov process.

3. Methodology

Let us consider an economy in which people are to be classified according to their income (measured in units of the local currency). Let Ω be a set of incomes in the economy at the baseline moment with sample points, ω_r, $r=1, \ldots, |\Omega|$. Let $\{A_i\}_{i=1}^3$ be partitions of Ω such that: $A_i \cap A_j = \phi$, $i \neq j$ and $\cup_{i=1}^3 A_i = \Omega$. The quartile-based classification of per capita income shares on Ω is defined as:

$$\omega_r \le Q_1 \Rightarrow \omega_r \in A_1, \quad Q_1 < \omega_r < Q_3 \Rightarrow \omega_r \in A_2 \text{ and } \quad \omega_r \ge Q_3 \Rightarrow \omega_r \in A_3$$

where Q_ρ is the ρth quartile of Ω, $\rho = 1, 2, 3$. We define the partitions, $\{A_i\}_{i=1}^3$, as follows: A_1 is the low-income class, A_2 is the middle-income class and A_3 is the high-income class. Denote the number of income earners by $|\Omega|$. The per capita income share of the ith class at the baseline moment is the proportion

$$p_i(0) = \frac{\sum_{\forall \omega_r \in A_i} \omega_r}{|A_i| \sum_{\forall j} |A_j|^{-1} \sum_{\forall \omega_r \in A_j} \omega_r}, \quad j, i \in \{1, 2, 3\} \tag{2}$$

where $p_i(0)$ is the per capita income for class i from the mean income, $|A_i|$ is the number of income earners in class i. The choice of $p_i(0)$ is in line with the Paglin approach [4, 6]. We use a quartile-based classification of income to ensure that the economic regulatory instruments of taxes and subsidies are not geared towards targeting a specific individual.

We create a suitable platform for income redistribution, based on these two economic regulatory instruments, among the classes of income earners using the Markov chain framework. We consider a tax-subsidy plan for a financial year. We assume a scenario where the low-income earners are exempted from income tax, while the high-income earners are excluded from subsidy. So the low-income earners benefit from subsidy without income tax, the middle-income earners pay income tax and enjoy subsidy, and the high-income earners pay income tax, but do not benefit from subsidy. It is possible to exclude high-income earners from subsidy, since income inequality is associated with income segregation [19]. We assume that the possible income changes in class i may either lead to an increase through subsidy to an income level in the next higher class $i + 1$, or a decrease as a result of income tax to an income level in class $i-1$ just below it, provided $i-1, i, i+1 \in \{1, 2, 3\}$. Let v_i denote the rate at which the process (an income earner) leaves class i. Let p_{ij} be the time-homogeneous transition probability from class i to j resulting from the economic regulatory instruments. We assume that at any moment in an interval $(t, t+\delta t)$ of the financial year, a subsidy is time-invariant and non-differentiated between the beneficiaries, and that the income loss through income tax is known. The transient per capita income share of class i is given as the probability of being in class i at time t given the tax-subsidy plan. We denote this income share as $p_i(t, \alpha|\beta_2, \beta_3)$. α is the subsidy (i.e., the amount spent on subsidy per year) and β_2 and β_3 are the annual budgeted revenues from the income taxes levied on class 2 and class 3, respectively. Considering the probability of transitions in the interval $(t, t+\delta t)$, based on the birth-and-death process [9], the transient income per capita share of class i at time $t+\delta t$ is given by

$$p_1(t+\delta t, \alpha|\beta_2, \beta_3) = p_1(t, \alpha|\beta_2, \beta_3)(1-v_1 p_{12}\delta t) + p_2(t, \alpha|\beta_2, \beta_3)v_2 p_{21}\delta t$$

$$p_2(t+\delta t, \alpha|\beta_2, \beta_3) = p_2(t, \alpha|\beta_2, \beta_3)(1-v_2 p_{23}\delta t - v_2 p_{21}\delta t)$$

$$+ p_1(t, \alpha|\beta_2, \beta_3)v_1 p_{12}\delta t + p_3(t, \alpha|\beta_2, \beta_3)v_3 p_{32}\delta t$$

$$p_3(t+\delta t, \alpha|\beta_2, \beta_3) = p_3(t, \alpha|\beta_2, \beta_3)(1-v_3 p_{32}\delta t) + p_2(t, \alpha|\beta_2, \beta_3)v_2 p_{23}\delta t$$

In matrix notation, the differential equations describing the transient per capita income shares are given as:

$$\frac{d}{dt}\mathbf{P}(t, \alpha|\beta_2, \beta_3) = \mathbf{P}(t, \alpha|\beta_2, \beta_3)\begin{bmatrix} -v_1 p_{12} & v_1 p_{12} & 0 \\ v_2 p_{21} & -(v_2 p_{23} + v_2 p_{21}) & v_2 p_{23} \\ 0 & v_3 p_{32} & -v_3 p_{32} \end{bmatrix}$$

and at $t = 0$, the vector of proportions at the baseline moment is given as:

$$\mathbf{P}(0) = \left\{ (p_1(0), p_2(0), p_3(0)) : \sum_{i=1}^{3} p_i(0) = 1, \; p_i(0) \geq 0, \; i = 1, 2, 3 \right\}$$

where

$$\frac{d}{dt}\mathbf{P}(t, \alpha|\beta_2, \beta_3) = \left[\frac{d}{dt}p_1(t, \alpha|\beta_2, \beta_3) \quad \frac{d}{dt}p_2(t, \alpha|\beta_2, \beta_3) \quad \frac{d}{dt}p_3(t, \alpha|\beta_2, \beta_3) \right]$$

is a row vector of the rates of change of the per capita income shares with respect to time for a given choice of income generated through taxation,

$$\mathbf{P}(t,\ \alpha|\beta_2\,,\ \beta_3) = [\,p_1(t,\ \alpha|\beta_2\,,\ \beta_3)\quad p_2(t,\ \alpha|\beta_2\,,\ \beta_3)\quad p_3(t,\ \alpha|\beta_2\,,\ \beta_3)]$$

is a row vector of the per capita income shares at time t for a given choice of income generated through taxation, and the matrix

$$\begin{bmatrix} -v_1 p_{12} & v_1 p_{12} & 0 \\ v_2 p_{21} & -(v_2 p_{23} + v_2 p_{21}) & v_2 p_{23} \\ 0 & v_3 p_{32} & -v_3 p_{32} \end{bmatrix}$$

is the Markov generator.

Since subsidy leads to an increase in income and income tax causes a decrease in income, we set the subsidy such that $\alpha = v_1 p_{12} = v_2 p_{23}$ and tax levels such that $\beta_i = v_i\, p_{i(i-1)}$. We do this because we assumed that the subsidy is time invariant and non-differentiated between the beneficiaries (i.e., the low- and middle-income earners). It is possible to achieve this by subsidizing amenities in areas occupied by the low- and middle-income earners as income inequality is associated with income segregation [19]. For $\beta_2 \neq \beta_3$, it means that the middle-income earners and the high-income earners do not pay equal tax. Thus we obtain the generator matrix, denoted by $\mathbf{Q}(\alpha|\beta_2\,,\ \beta_3)$, as follows:

$$\mathbf{Q}(\alpha|\beta_2\,,\ \beta_3) = \begin{bmatrix} -\alpha & \alpha & 0 \\ \beta_2 & -(\alpha + \beta_2) & \alpha \\ 0 & \beta_3 & -\beta_3 \end{bmatrix}$$

We therefore state a method for projecting the per capita income shares at any time instant in the following proposition.

Proposition 1. The transient per capita income shares are given by the vector $\mathbf{P}(t,\ \alpha|\beta_2\,,\ \beta_3)$ as follows:

$$\mathbf{P}(t,\ \alpha|\beta_2\,,\ \beta_3) = \mathbf{P}(0)\left(\frac{1}{\gamma_0\gamma_1}\mathbf{C}_0 + \frac{1}{\gamma_0(\gamma_0-\gamma_1)}\mathbf{G}_1\exp(-\gamma_0 t) + \frac{1}{\gamma_1(\gamma_1-\gamma_0)}\mathbf{G}_2\exp(-\gamma_1 t)\right)$$

where

$$\gamma_0 = \frac{1}{2}(2\alpha + \beta_2 + \beta_3 + \sqrt{(\beta_2 - \beta_3)^2 + 4\alpha\beta_2}$$

$$\gamma_1 = \frac{1}{2}(2\alpha + \beta_2 + \beta_3 - \sqrt{(\beta_2 - \beta_3)^2 + 4\alpha\beta_2})$$

$$\mathbf{C}_0 = \begin{bmatrix} \beta_2\beta_3 & \alpha\beta_3 & \alpha^2 \\ \beta_2\beta_3 & \alpha\beta_3 & \alpha^2 \\ \beta_2\beta_3 & \alpha\beta_3 & \alpha^2 \end{bmatrix}$$

$$\mathbf{G}_1 = \begin{bmatrix} (\beta_3 - \gamma_0)(\alpha + \beta_2 - \gamma_0) - \alpha\beta_3 & \alpha(\beta_3 - \gamma_0) & \alpha^2 \\ \beta_2(\beta_3 - \gamma_0) & (\alpha - \gamma_0)(\beta_3 - \gamma_0) & \alpha(\alpha - \gamma_0) \\ \beta_2\beta_3 & \beta_3(\alpha - \gamma_0) & (\alpha - \gamma_0)(\alpha + \beta_2 - \gamma_0) - \alpha\beta_2 \end{bmatrix}$$

and

$$\mathbf{G}_2 = \begin{bmatrix} (\beta_3 - \gamma_1)(\alpha + \beta_2 - \gamma_1) - \alpha\beta_3 & \alpha(\beta_3 - \gamma_1) & \alpha^2 \\ \beta_2(\beta_3 - \gamma_1) & (\alpha - \gamma_1)(\beta_3 - \gamma_1) & \alpha(\alpha - \gamma_1) \\ \beta_2\beta_3 & \beta_3(\alpha - \gamma_1) & (\alpha - \gamma_1)(\alpha + \beta_2 - \gamma_1) - \alpha\beta_2 \end{bmatrix}$$

Proof. Taking the *s* transform of $\mathbf{P}(t,\ \alpha \,|\, \beta_2,\ \beta_3)$,

$$M_{\mathbf{P}(t,\alpha|\beta_2,\beta_3)}(s) = \int_0^\infty \exp(-st)\mathbf{P}(t,\ \alpha \,|\, \beta_2,\ \beta_3)dt$$

we have

$$M_{\mathbf{P}(t,\alpha|\beta_2,\beta_3)}(s) = \mathbf{P}(0)\begin{bmatrix} s+\alpha & -\alpha & 0 \\ -\beta_2 & s+(\alpha+\beta_2) & -\alpha \\ 0 & -\beta_3 & s+\beta_3 \end{bmatrix}^{-1} \tag{3}$$

where $M_{\mathbf{P}(t,\alpha|\beta_2,\beta_3)}(s)$ is the $s-$transform of $\mathbf{P}(t,\ \alpha|\beta_2,\ \beta_3)$. The determinant of the matrix $\begin{bmatrix} s+\alpha & -\alpha & 0 \\ -\beta_2 & s+(\alpha+\beta_2) & -\alpha \\ 0 & -\beta_3 & s+\beta_3 \end{bmatrix}$, denoted by $|Q(s)|$, is given by

$$|Q(s)| = s(s^2+(2\alpha+\beta_2+\beta_3))s+\alpha^2+\alpha\beta_3+\beta_2\beta_3 = s(s+\gamma_0)(s+\gamma_1)$$

Therefore,

$$M_{\mathbf{P}(t,\alpha|\beta_2,\beta_3)}(s)\frac{1}{|Q(s)|}\mathbf{P}(0)$$

$$\times\begin{bmatrix} (s+\beta_3)(s+\alpha+\beta_2)-\alpha\beta_3 & \alpha(s+\beta_3) & \alpha^2 \\ \beta_2(s+\beta_3) & (s+\alpha)(s+\beta_3) & \alpha(s+\alpha) \\ \beta_2\beta_3 & \beta_3(s+\alpha) & (s+\alpha)(s+\alpha+\beta_2)-\alpha\beta_2 \end{bmatrix}$$

Using the method of partial fractions, we obtain

$$M_{\mathbf{P}(t,\alpha|\beta_2,\beta_3)}(s) = \mathbf{P}(0)\left(\frac{1}{\gamma_0\gamma_1 s}\mathbf{C}_0 + \frac{1}{\gamma_0(\gamma_0-\gamma_1)(\gamma_0+s)}\mathbf{G}_1 + \frac{1}{\gamma_1(\gamma_1-\gamma_0)(\gamma_1+s)}\mathbf{G}_2\right) \quad (4)$$

We thus express Eq. (4) as

$$M_{\mathbf{P}(t,\alpha|\beta_2,\beta_3)}(s)$$
$$= \mathbf{P}(0)\times\int_0^\infty \exp(-st)\left(\frac{1}{\gamma_0\gamma_1}\mathbf{C}_0 + \frac{1}{\gamma_0(\gamma_0-\gamma_1)}\mathbf{G}_1\exp(-\gamma_0 t) + \frac{1}{\gamma_1(\gamma_1-\gamma_0)}\mathbf{G}_2\exp(-\gamma_1 t)\right)dt$$

Therefore, the transient per capita income shares are given in matrix-vector form as

$$\mathbf{P}(t,\ \alpha|\beta_2,\ \beta_3)$$
$$= \mathbf{P}(0)\left(\frac{1}{\gamma_0\gamma_1}\mathbf{C}_0 + \frac{1}{\gamma_0(\gamma_0-\gamma_1)}\mathbf{G}_1\exp(-\gamma_0 t) + \frac{1}{\gamma_1(\gamma_1-\gamma_0)}\mathbf{G}_2\exp(-\gamma_1 t)\right) \quad (5)$$

This completes the proof.

It can be easily seen that $\mathbf{P}(t,\ \alpha|\beta_2,\ \beta_3)\mathbf{e} = 1$ for all $p_i(t,\ \alpha|\beta_2,\ \beta_3) \geq 0,$ $i = 1,\ 2,\ 3$, where $\mathbf{e}$ is a 3×1 vector of ones. This is because

$$\frac{1}{\gamma_0\gamma_1}\mathbf{C}_0\mathbf{e} = \mathbf{e}$$

and the row sums of matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ are respectively equal to zero as

$$\begin{aligned}\alpha^2 + \alpha\beta_3 + \beta_2\beta_3 - (2\alpha + \beta_2 + \beta_3)\gamma_0 + \gamma_0^2 &= (\alpha^2 + \alpha\beta_3 + \beta_2\beta_3) - \frac{1}{2}\Big((2\alpha + \beta_2 + \beta_3)^2 \\ &+ (2\alpha + \beta_2 + \beta_3)\sqrt{(\beta_2 - \beta_3)^2 + 4\alpha\beta_2}\Big) + \frac{1}{4}\Big((2\alpha + \beta_2 + \beta_3)^2 + 2(2\alpha + \beta_2 + \beta_3) \\ &\times \sqrt{(\beta_2 - \beta_3)^2 + 4\alpha\beta_2} + (2\alpha + \beta_2 + \beta_3)^2 - 4(\alpha^2 + \alpha\beta_3 + \beta_2\beta_3)\Big) = 0\end{aligned}$$

and

$$\begin{aligned}\alpha^2 + \alpha\beta_3 + \beta_2\beta_3 - (2\alpha + \beta_2 + \beta_3)\gamma_1 + \gamma_1^2 &= (\alpha^2 + \alpha\beta_3 + \beta_2\beta_3) - \frac{1}{2}\Big((2\alpha + \beta_2 + \beta_3)^2 \\ &- (2\alpha + \beta_2 + \beta_3)\sqrt{(\beta_2 - \beta_3)^2 + 4\alpha\beta_2}\Big) + \frac{1}{4}\Big((2\alpha + \beta_2 + \beta_3)^2 - 2(2\alpha + \beta_2 + \beta_3) \\ &\times \sqrt{(\beta_2 - \beta_3)^2 + 4\alpha\beta_2} + (2\alpha + \beta_2 + \beta_3)^2 - 4(\alpha^2 + \alpha\beta_3 + \beta_2\beta_3)\Big) = 0\end{aligned}$$

Next, we consider the long-run situation. In order to derive the entropy measure for the per capita income shares in the long-run, it is obvious that we need to obtain the steady-state solution $\lim_{t\to\infty}\mathbf{P}(t,\ \alpha|\beta_2,\ \beta_3)$. From Eq. (5), we have

$$\lim_{t\to\infty}\mathbf{P}(t,\ \alpha|\beta_2,\ \beta_3) = \frac{1}{\gamma_0\gamma_1}\mathbf{P}(0)\begin{bmatrix}\beta_2\beta_3 & \alpha\beta_3 & \alpha^2 \\ \beta_2\beta_3 & \alpha\beta_3 & \alpha^2 \\ \beta_2\beta_3 & \alpha\beta_3 & \alpha^2\end{bmatrix} \tag{6}$$

Now, for each income class *i*, we have

$$0 \leq \lim_{t\to\infty} p_i(t,\ \alpha|\beta_2,\ \beta_3) \leq 1.$$

It follows that

$$0 \le -\sum_{i=1}^{3} \lim_{t\to\infty} p_i(t,\ \alpha \,|\, \beta_2,\ \beta_3) \ln \lim_{t\to\infty} p_i(t,\ \alpha \,|\, \beta_2,\ \beta_3) \le \ln 3$$

So, if we define $c = \dfrac{1}{\ln 3}$, the entropy in Eq. (1) becomes

$$H = -\frac{1}{\ln 3}\sum_{i=1}^{3} \lim_{t\to\infty} p_i(t,\ \alpha \,|\, \beta_2,\ \beta_3) \ln \lim_{t\to\infty} p_i(t,\ \alpha \,|\, \beta_2,\ \beta_3) \tag{7}$$

where $0 \le H \le 1$. The upper bound for H is attained when

$$\lim_{t\to\infty} p_1(t,\ \alpha \,|\, \beta_2\ ,\beta_3) = \lim_{t\to\infty} p_2(t,\ \alpha \,|\, \beta_2,\ \beta_3) = \lim_{t\to\infty} p_3(t,\ \alpha \,|\, \beta_2,\ \beta_3) = \frac{1}{3}$$

This represents the case where equal per capita income shares among the income classes in the economy is achieved at the steady-state. The lower bound is attained when

$$\lim_{t\to\infty} p_1(t,\ \alpha \,|\, \beta_2,\ \beta_3) = \lim_{t\to\infty} p_2(t,\ \alpha \,|\, \beta_2,\ \beta_3) = 0, \qquad \lim_{t\to\infty} p_3(t,\ \alpha \,|\, \beta_2,\ \beta_3) = 1$$

This situation would exist when only the high-income class earn income reaches the steady-state. For intermediate values, the entropy measures the extent of income inequality which is present at the steady-state. We now make the following proposition.

Proposition 2. In this tax-subsidy framework, uniform per capita income shares are attained in the long-run when and only when the taxes from each of the middle- and high-income classes are respectively equal to the subsidy.

Proof. When the taxes from each of the middle- and high-income classes are equal to the subsidy, then

$$\frac{\beta_2 + \beta_3}{2} = \alpha \tag{8}$$

We determine the subsidy α in such a way that the entropy index

$$H = -\frac{1}{\ln 3}\sum_{i=1}^{3} \lim_{t\to\infty} p_i(t,\ \alpha \,|\, \beta_2,\ \beta_3) \ln \lim_{t\to\infty} p_i(t,\ \alpha \,|\, \beta_2,\ \beta_3)$$

is maximized subject to

$$\sum_{i=1}^{3} \lim_{t\to\infty} p_i(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) = 1$$

$$\lim_{t\to\infty} p_i(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) \geq 0$$

This is attained when

$$\lim_{t\to\infty} p_1(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) = \lim_{t\to\infty} p_2(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) = \lim_{t\to\infty} p_3(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) = \frac{1}{3}$$

Thus the long-run equitable per capita income shares are achieved when

$$\frac{1}{\gamma_0\gamma_1}\mathbf{P}(0)\begin{bmatrix} \beta_2\beta_3 & \alpha\beta_3 & \alpha^2 \\ \beta_2\beta_3 & \alpha\beta_3 & \alpha^2 \\ \beta_2\beta_3 & \alpha\beta_3 & \alpha^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \tag{9}$$

Solving the system given by Eq. (9), we obtain $\beta_2 = \alpha$. Similarly, from Eq. (8), $\alpha = \beta_3$. Conversely, when the baseline trend of income inequality persists, i.e.,

$$\lim_{t\to\infty} p_1(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) < \lim_{t\to\infty} p_2(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) < \lim_{t\to\infty} p_3(t,\ \alpha \,|\, \beta_2\,,\ \beta_3)$$

then $\beta_2\beta_3 < \alpha\beta_3 < \alpha^2$. Therefore, $\beta_2 < \alpha$ and $\beta_3 < \alpha$. Again when $\beta_2 > \alpha$ and $\beta_3 > \alpha$, then

$$\lim_{t\to\infty} p_1(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) > \lim_{t\to\infty} p_2(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) > \lim_{t\to\infty} p_3(t,\ \alpha \,|\, \beta_2\,,\ \beta_3)$$

In either of the converse scenarios when $\beta_2 \neq \alpha$ or $\beta_3 \neq \alpha$, there are uneven per capita income shares. This is a contradiction. Hence, $\beta_2 = \beta_3 = \alpha$. This completes the proof.

In practice, when the following monotone pattern,

$$\lim_{t\to\infty} p_1(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) > \lim_{t\to\infty} p_2(t,\ \alpha \,|\, \beta_2\,,\ \beta_3) > \lim_{t\to\infty} p_3(t,\ \alpha \,|\, \beta_2\,,\ \beta_3)$$

holds, it indicates excessive taxation of both the middle- and the high-income earners, so much so that the low-income earners are ultimately enriched through subsidy.

Proposition 2 therefore reveals that uniform per capita income shares are attainable when the total loss of income through taxation equals the total gains through subsidy. This reflects balance in the fiscal operations. The yardstick for choosing an optimal subsidy plan that tends towards achieving uniformity in the per capita income shares in the long-run is based on the principle of maximum entropy. This yardstick is given in the corollary below.

Corollary. Given a family of subsidies, $\sum_{\alpha}$, the policy option is determined by selecting $\alpha^* \in \sum_{\alpha}$ where:

$$\min_{\forall \alpha}\left(\sum_{i=1}^{3}\left|p_i(t,\ \alpha|\beta_2,\beta_3)-\frac{1}{3}\right|^2\right)^{1/2}=\left(\sum_{i=1}^{3}\left|p_i(t,\alpha^*|\beta_2,\ \beta_3)-\frac{1}{3}\right|^2\right)^{1/2}$$

4. Simulation

We illustrate the use of our approach on the basis of the following hypothetical example. Suppose the government of a region is considering a tax-subsidy policy in a financial year wherein the low-income earners are excluded from income tax and both the low- and the middle-income earners benefit from a subsidy re-investment program. The government budget statement for the region requires that 6 million naira will be collected as income tax from the middle-income earners, while 8.4 million naira will be collected from the high-income earners. Based on evidence obtained so far, there is no clear cut distinction among the income earners in the region. The current tax bases (in million Naira) for 111 income earners in the region are as follows:

1.3	0.2	1.5	1.4	2.1	0.9	3.3	2.5	1.5	2.1	2.5	1.9	1.7	0.9
3.1	0.7	0.6	1.7	4.8	1.5	4.4	8.0	2.4	0.0	4.0	1.7	2.5	2.4
1.9	1.2	4.0	5.2	3.3	3.4	2.3	1.1	9.5	4.3	2.8	2.4	1.6	3.4
3.2	2.2	3.2	2.1	11.0	2.7	3.9	2.9	9.9	1.6	3.5	3.1	5.0	2.3
3.5	2.3	1.8	4.1	3.2	3.2	2.3	3.9	2.6	2.3	1.3	2.4	2.8	1.0
2.3	6.8	1.7	3.2	3.1	4.3	1.4	3.5	4.0	4.3	4.1	1.4	2.3	2.5
2.0	3.7	2.8	3.1	3.0	1.8	2.3	3.7	3.7	4.0	1.9	2.1	3.7	1.6
3.6	2.6	2.3	1.9	2.7	2.2	4.9	2.5	2.7	1.6	2.0	9.1	2.6	

For the subsidy re-investment program, the government seeks a budgetary allocation that will bridge the income inequality gap in the future. The task is to distinguish between the income earners so as to guide the government on the implementation of the tax-subsidy policy and to decide on the budgetary allocation for the subsidy re-investment program for the financial year.

For the initial analysis, we partition the income earners into three classes using the quartiles. We carry out all our computations in the MATLAB environment. We compute the quartiles as $Q_1 = 1.9$, $Q_2 = 2.5$ and $Q_3 = 3.5$ (in million naira). The incomes are therefore classified as:

1.3	0.2	1.5	1.4	0.9	1.5	1.9	1.7	0.9	0.7	0.6		
1.7	1.5	0	1.7	1.9	1.2	1.1	1.6	1.6	1.8	1.6	$\in A_1$	$\lvert A_1 \rvert = 31$;
1.3	1.0	1.7	1.4	1.4	1.8	1.9	1.6	1.9				

2.1	3.3	2.5	2.1	2.5	3.1	2.6	2.4	2.5	2.4	3.3		
3.4	2.3	2.8	2.4	3.4	3.2	2.2	3.2	2	2.1	2.7		
2.9	3.1	2.3	2.3	3.2	3.2	2.3	2.6	2.3	2.4	2.8	$\in A_2$	$\lvert A_2 \rvert = 50$
2.3	3.2	3.1	2.7	2.3	2.5	2	2.8	3.1	3	2.3		
2.1	2.6	2.3	2.7	2.2	2.5							

4.9	4.8	4.4	8	4	9.1	4	5.2	9.5	4.3	11		
3.9	9.9	3.5	5	3.5	4.1	3.9	6.8	4.3	3.5	3.6	$\in A_3$	$\lvert A_3 \rvert = 30$
4	4.3	4.1	3.7	3.7	3.7	4	3.7					

The sizes of the income classes are unequal because the low income earners consist of those amongst the 25% of lowest earners, while the upper 25% are high income earners. Thus, the largest group of income earners is the middle income earners, in which the income lies between the 25% and the 75% quartiles of the income distribution. This partitioning is in line with quartile-based income classification. The schematic representation of the partitioning of the income data is shown in Fig. 1.

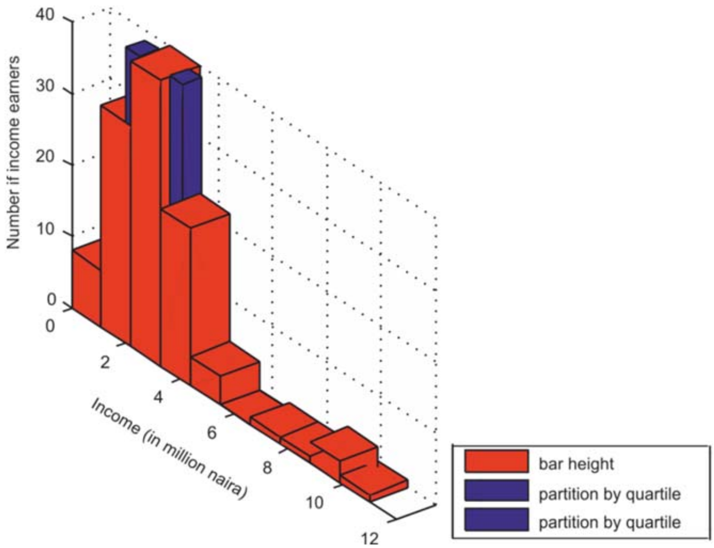

Fig. 1. Pictorial view of the class of income earners

Using this approach, the income earners to be affected by the government policy are clearly distinguished. Then we use Eq. (2) to compute the per capita income shares at the baseline moment. We obtain: $\mathbf{P}(0) = [0.1503 \quad 0.2900 \quad 0.5597]$. $\mathbf{P}(0)$ indicates that, on average, the income of one high-income earner exceeds the sum of that of an income earner in the middle-income class and another one in the low-income class. Thus, income inequality exists in the region. Notice that $\beta_2 \neq \beta_3$, thus we cannot have $\alpha = \beta_2 = \beta_3$. To determine the subsidy plan, we consider the following possible values for α : $\alpha = \beta_2 = 6$, $\alpha = \beta_3 = 8.4$, $\alpha = (\beta_2 + \beta_3)/2 = 7.2$ (in million naira) and then employ the corollary. For a financial year, we consider the time scale on a monthly basis so that $t = \frac{1}{12}, \frac{2}{12}, ..., 1$ (in years). We compute the entropy values by applying Eq. (7), as well as the Euclidean distance norm defined in the corollary, for the dynamics of the per capita income shares resulting from the classification. The results are shown in Figs. 2 and 3.

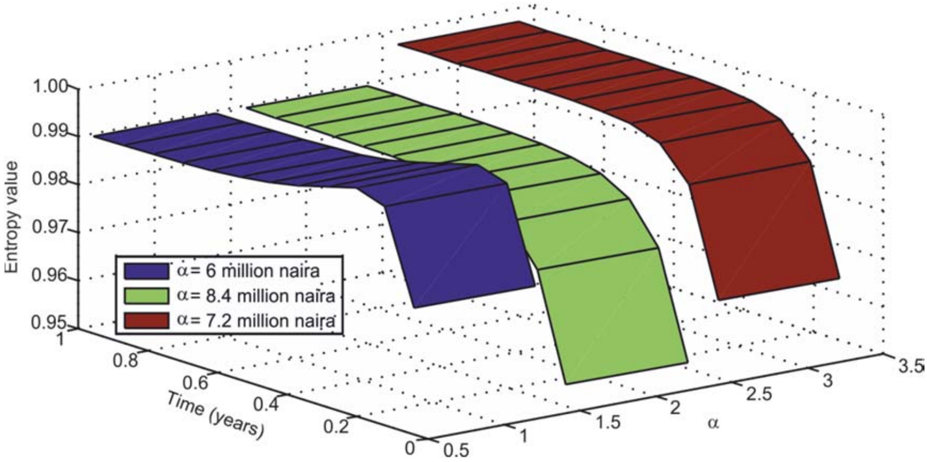

Fig. 2. Entropy value of the per capita income shares
corresponding to various values of α for the financial year

The figures show that allocating 7.2 million naira to the subsidy in the financial year is the best option. This is because it gives the highest entropy value and the minimum Euclidean distance norm throughout the financial year compared to the other options considered (except in the first quarter of the year when a subsidy budget of 6 million naira gives the lowest values of the Euclidean distance norm). Therefore, this is a practical way to determine subsidy, rather than a mere subjective allocation. Nonetheless, perfect equality may not exist in the economy, even when 7.2 million naira is allocated to the subsidy. This is because the mean per capita income shares are used. Furthermore, the mean per capita income shares do not have a uniform distribution, as the maximum entropy value of $H = 1$ is not attained.

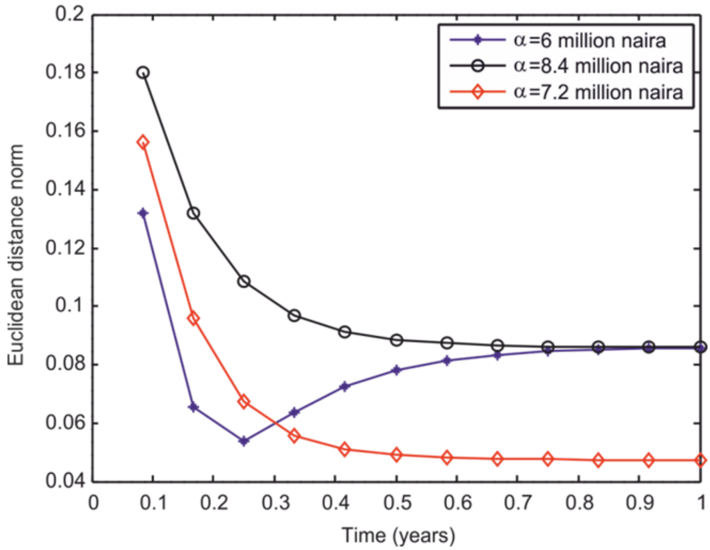

Fig. 3. Euclidean distance from the uniform distribution corresponding to various values of α for the financial year

5. Conclusions

This study complements the existing literature on mathematical economics by providing an insight into the classification of income earners by quartiles. Our approach to the dynamics of income inequality is novel, as we provide a new representation for the evolution of per capita income shares based on a tax-subsidy plan within the framework of a continuous-time Markov chain. We have provided a guide to the determination of the subsidy budget based on the concept of the entropy of income inequality used in economic theory. However, further work may be undertaken so as to incorporate additional macroeconomic indicators and political factors that may affect the use of economic regulatory instruments in the redistribution of income.

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