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Marcin ANHOLCER*

STOCHASTIC GENERALIZED TRANSPORTATION PROBLEM WITH DISCRETE DISTRIBUTION OF DEMAND

The generalized transportation problem (GTP) allows us to model situations where the amount of goods leaving the supply points is not equal to the amount delivered to the destinations (this is the case, e.g. when fragile or perishable goods are transported or complaints may occur). A model of GTP with random, discretely distributed, demand has been presented. Each problem of this type can be transformed either into the form of a convex programming problem with a piecewise linear objective function, or a mixed integer LP problem. The method of solution presented uses ideas applied in the method of stepwise analysis of variables and in the equalization method.

Keywords: *stochastic generalized transportation problem, stochastic programming, equalization method*

1. Introduction

The generalized transportation problem (GTP) arises in many real-life applications. It has the form of a classical transportation problem, with the additional assumption that the quantities of goods change during the transportation process. We assume that a uniform good is delivered from m supply points to n destination points. The amount of this good changes during the transportation process. Thus the amount x_{ij} leaving supply point i to go to destination point j, is modified by a multiplier r_{ij} (in most cases belonging to the interval (0, 1), which means a reduction in the amount of the good). The amount of the good finally delivered to destination j from the supply point i equals $r_{ij}x_{ij}$. We assume that the unit transportation costs c_{ij} are constant, demand b_j at each destination point must be satisfied, and the supply capacity a_i of each supply point must not be exceeded. The model can be thus written as follows:

*Department of Operations Research, Poznań University of Economics, al. Niepodległości 10, 61-875 Poznań, Poland, e-mail: m.anholcer@ue.poznan.pl

$$\min\left\{ f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \right\}$$

s. t.

$$\sum_{i=1}^{m} r_{ij}x_{ij} = b_j, \qquad j = 1, ..., n$$

$$\sum_{j=1}^{n} x_{ij} \le a_i, \qquad i = 1, ..., m$$

$$x_{ij} \ge 0, \qquad i = 1, ..., m, \qquad j = 1, ..., n$$

The values of the multipliers r_{ij} depend, in general, on the delivery time and distance between particular source and destination points. However, in this paper we assume that both characteristics are constant for every connection between source and destination. This is a rational assumption, as we consider a short-term horizon. In consequence, we can assume that the values of r_{ij} are also constant.

The generalized transportation problem was discussed, e.g. in [8, 9, 17]. In [19], the authors considered the transportation problem with additional constraints of the GTP type. In [7], the authors by investigating some structural properties of this kind of problem, found a relation between the complaints ratio and the complexity of the optimal distribution network.

A more general version, the generalized minimum cost flow problem, was described in [1, Chapter 15]. This is a generalization of the well-known minimum cost flow problem where we assume, in addition, that the quantity of a good changes during the flow process. In [1], one may find various examples of applications of this problem, in particular in the analysis of financial, mineral or energy networks, aircraft assignment, managing the warehousing of goods and fund flows or in land management. Other interesting applications may be found in [18]. Here the authors apply generalized flows to problems of transporting perishable products such as blood, medical and nuclear materials, food, pharmaceuticals and fast fashion apparels. Such problems were also briefly described in [10]. Various algorithms for generalized flow problems and more detailed considerations may be found, e.g. in [1, Chapter 15] as well as in [12, 13, 25].

The stochastic transportation problem (STP), was analyzed, e.g. in [2–4, 14–16, 23, 24]. This is a variant of the ordinary (linear) transportation problem with random demand. Using the Dantzig–Madansky approach, we minimize the total expected cost of transportation, deliveries, storage, etc. In [2–4, 23, 24] the equalization method, that can be applied to this kind of problem, was analyzed. In [2, 3], the convergence of this method was proved. In [14–16] other methods for solving this problem were analyzed (in particular the cross decomposition method and a variant of the Frank–Wolfe algorithm). Yet another algorithm for STP, called the forest iteration method, was described in [20].

The stochastic generalized transportation problem (SGTP) is a version of GTP closer to real-life situations than the deterministic variant. As in the case of STP, we assume that the demand at the destination points is not determined. However, we assume that we know the distribution of the demand for each destination. In [5, 6] variants of the equalization method for stochastic and nonlinear versions of GTP were presented. Convergence theorems were also proved using, e.g. the theorems presented in [11, Chapter 7]. An algorithm called the A-forest iteration method for SGTP was described in [21].

In all the above publications concerning stochastic versions of such problems, it was assumed that the demand at each destination point is continuously distributed. As far as the author knows, the only paper where the demand distribution was assumed to be discrete is [22], where the method of stepwise analysis of variables was applied to solve such a version of STP. No paper treating the SGTP with a discrete distribution of demand is known to the author.

In this paper, we present how a modified version of the equalization method may be used to solve the stochastic generalized transportation problem with a discrete distribution of demand. Some ideas from the method of stepwise analysis of variables have also been used. The following sections contain a detailed description of the problem and formulation of the model, the method of solution, the results of numerical experiments and the final conclusions.

2. Formulation of the problem

In the stochastic variant of the problem, the demand at each destination point is a random variable with known distribution. In this paper, we focus on the case where for each j the variable B_j representing the demand at point j has a discrete distribution:

$$\Pr\left(B_j = b_j^s\right) = p_j^s, \qquad s = 1, \ldots, S_j$$

where S_j denotes the number of possible values of demand at point j. For convenience (without loss of generality) we assume that the values b_j^s are sorted in increasing order:

$$b_j^{s+1} > b_j^s, \qquad s = 1, \ldots, S_j - 1$$

In addition we assume that the following obvious condition holds:

$$\sum_{s=1}^{S_j} p_j^s = 1$$

For each destination point, the unit surplus cost $s_j^{(1)}$ is known. This has to be paid if the amount of the good delivered to destination j is greater than the declared demand. This can be considered, e.g. as the unit storage cost. Also, the unit shortage cost $s_j^{(2)}$ is known. This is supposed to be paid, in turn, when the amount of the good delivered to destination j is less than the declared demand. This value is usually identified with the additional cost that must be paid in order to find the missing amount of a good in emergency mode.

The function of expected additional cost related to destination point j is given by the formula

$$f_j\left(x_j\right) = s_j^{(1)} \sum_{t<x_j} \left(x_j - t\right) \Pr\left(B_j = t\right) + s_j^{(2)} \sum_{t>x_j} \left(t - x_j\right) \Pr\left(B_j = t\right)$$

which can be rewritten in the following equivalent form

$$f_j\left(x_j\right) = s_j^{(2)} \left(E\left(X_j\right) - x_j\right) + \left(s_j^{(1)} + s_j^{(2)}\right) \int_0^{x_j} \Phi_j\left(t\right) dt$$

where Φ_j is the cumulative distribution of the demand at the destination point j.

Finally, the problem under discussion takes the form:

$$\min\left\{ f\left(x\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} f_j\left(x_j\right) \right\}$$

s. t.

$$\sum_{i=1}^{m} r_{ij} x_{ij} = x_j, \qquad j = 1, \ldots, n$$

$$\sum_{j=1}^{n} x_{ij} \le a_i, \qquad i = 1, \ldots, m$$

$$x_{ij} \ge 0, \qquad i = 1, \ldots, m, \qquad j = 1, \ldots, n$$

Because of the fact that the variables B_j have discrete distributions, their cumulative distributions are non-decreasing step functions (piecewise linear) and so the functions f_j are piecewise linear.

Moreover, each function f_j is convex and increasing on the interval $\left(b_j^1, \infty\right)$. The derivative of such a function is described by the formula

$$f_j'\left(x_j\right) = -s_j^{(2)} + \left(s_j^{(1)} + s_j^{(2)}\right)\Phi_j\left(x_j\right)$$

Each of these derivatives is a piecewise constant function, defined on the open intervals $\left(0, b_j^1\right), \left(b_j^s, b_j^{s+1}\right), s = 1, ..., S_j - 1$ and $\left(b_j^{s+1}, \infty\right)$. Observe that these derivatives are continuous (even constant) on these intervals. At the points $b_j^s, s = 1, ..., S_j - 1$, the derivatives are not defined, which could suggest that the use of gradient methods is impossible. However, due to the properties of the equalization method (the partial derivatives are used independently in a predictable way), we are able to modify the method by using one-sided derivatives:

$$f_{-j}'\left(x_j\right) = \lim_{x \to x_j^-}\left(-s_j^{(2)} + \left(s_j^{(1)} + s_j^{(2)}\right)\Phi_j\left(x\right)\right)$$

$$f_{+j}'\left(x_j\right) = \lim_{x \to x_j^+}\left(-s_j^{(2)} + \left(s_j^{(1)} + s_j^{(2)}\right)\Phi_j\left(x\right)\right)$$

Observe that in the case of the problem discussed, the following condition always holds:

$$f_{-j}'\left(x_j\right) \le f_{+j}'\left(x_j\right)$$

where the inequality is sharp only at the points $b_j^s, s = 1, ..., S_j$.

Left-sided derivatives will be considered when the influence of a decrease in the amount of a good on the objective function is analyzed. Right-sided derivatives will be used when an increase in amount is considered. After introducing these changes, we can use a modified variant of the equalization method. Variants of this algorithm for stochastic and nonlinear GTP have been presented in [5] and [6], together with convergence proofs.

3. Method of solution

In order to solve the problem using the Equalization Method, we introduce the slack variables $x_{i,n+1}$. We set $c_{i,n+1}=0, r_{i,n+1}=1$ for i = 1, ..., m and $f_{n+1}\left(x_{n+1}\right)\equiv 0$. Then the problem takes the following form:

$$\min\left\{f\left(x\right)=\sum_{i=1}^{m}\sum_{j=1}^{n+1}c_{ij}x_{ij}+\sum_{j=1}^{n+1}f_j\left(x_j\right)\right\}$$

s. t.

$$\sum_{i=1}^{m}r_{ij}x_{ij}=x_j,\qquad j=1,...,n+1$$

$$\sum_{j=1}^{n+1}x_{ij}=a_i,\qquad i=1,...,m$$

$$x_{ij}=0,\qquad i=1,...,m,\qquad j=1,...,n$$

$$x_{ij}\geq 0,\qquad i=1,...,m,\qquad j=1,...,n$$

The KKT conditions for this problem may be written in the following form: (i = 1, ..., m, j = 1, ..., n):

$$c_{ij}+r_{ij}f'_{+j}\left(x_j\right)\geq u_i,\qquad x_{ij}=0$$

$$c_{ij}+r_{ij}f'_{-j}\left(x_j\right)\leq u_i\leq c_{ij}+r_{ij}f'_{+j}\left(x_j\right),\qquad x_{ij}>0$$

The following version of the equalization method is convergent to a KKT point (see the discussion at the end of this section). When deriving the step length, we adopt the idea also used, e.g. in the method of stepwise analysis of variables, presented in [22]. Namely, the amounts of the good delivered are changed in such a way that for each variable x_j there exists either an interval $\left[b_j^s,b_j^{s+1}\right], s=1,...,S_j,$ an interval $\left[0,b_j^1\right]$ or an interval $\left[b_j^{S_j},\infty\right)$ to which this variable belongs before and after the change. However, we avoid the following fundamental disadvantage of the method of stepwise analysis of variables. Using this method, at each step the demand constraints must be satisfied. This implies in turn that at each iteration, the solution has the form of a so called A-forest, i.e. a forest with an extra arc in every tree (see e.g. [1, Chapter 15]), which is difficult to handle. In the version of the equalization method presented, we do not have to adopt this structure.

Algorithm 1. Equalization method for the stochastic generalized transportation problem with discrete distribution of demand.

1. Initialization. Derive the initial solution using the formula:

$$x_{ij} = \begin{cases} a_i, \ j = n+1 \\ 0, \ j \neq n+1 \end{cases}$$

Compute the sums of deliveries to each of the destination points:

$$x_j = \begin{cases} \sum_{i=1}^{m} a_i, \ j = n+1 \\ 0, \ j \neq n+1 \end{cases}$$

Derive the initial values of the partial derivatives:

$$k_{ij}^{-} = c_{ij} + r_{ij} f'_{-j}(0), \qquad i = 1, ..., m, \qquad j = 1, ..., n$$

$$k_{ij}^{+} = c_{ij} + r_{ij} f'_{+j}(0), \qquad i = 1, ..., m, \qquad j = 1, ..., n$$

$$k_{i,\,n+1}^{-} = k_{i,\,n+1}^{+} = 0, \qquad i = 1, ..., m$$

Go to step 2.

2. Checking optimality. For each i compute:

$$v_i = \min\left\{k_{ij}^{+} \,\middle|\, j = 1, ..., n+1\right\}$$

$$w_i = \max\left\{k_{ij}^{-} \,\middle|\, j = 1, ..., n+1, \ x_{ij} > 0\right\} - v_i$$

Let $j^{**}(i)$ be the index j, for which $k_{ij}^{+} = v_i$ and let $j^{*}(i)$ be the index j, for which $k_{ij}^{-} - v_i = w_i$. Compute

$$\alpha = \max\left\{ w_i \,\middle|\, i = 1, ..., m \right\}$$

If $\alpha = 0$, then STOP. The solution obtained is optimal. Otherwise, let i^* be the index i, for which $w_i = \alpha$ and go to step 3.

3. Changing the solution. Let

$$\lambda^- = \begin{cases} x_{j^*} - \max\left\{ b_{j^*}^s \,\middle|\, b_{j^*}^s < x_{j^*} \right\}, & x_{j^*} \ge b_{j^*}^1 \\ x_{j^*}, & x_{j^*} < b_{j^*}^1 \end{cases}$$

Let

$$\lambda^+ = \begin{cases} \min\left\{ b_{j^{**}}^s \,\middle|\, b_{j^{**}}^s > x_{j^{**}} \right\} - x_{j^{**}}, & x_{j^{**}} \le b_{j^{**}}^{s_{j^{**}}} \\ \infty, & x_{j^{**}} > b_{j^{**}}^{s_{j^{**}}} \end{cases}$$

Let

$$\lambda^* = \min\left\{ \frac{\lambda^-}{r_{i^* j^*}}, \frac{\lambda^+}{r_{i^* j^{**}}} \right\}$$

If

$$\lambda^* > x_{i^* j^*}$$

then set

$$\lambda^* := x_{i^* j^*}$$

Change the solution according to the formulae:

$$x_{i^* j^*} := x_{i^* j^*} - \lambda^*, \qquad x_{i^* j^{**}} := x_{i^* j^{**}} + \lambda^*$$

$$x_{j^*} := x_{j^*} - r_{i^* j^*}\lambda^*, \qquad x_{j^{**}} := x_{j^{**}} + r_{i^* j^{**}}\lambda^*$$

Derive the new values of the derivatives $k_{ij^*}^-, k_{ij^{**}}^-, k_{ij^*}^+, k_{ij^{**}}^+$ for $i = 1, ..., m$ and go back to step 2.

As previously mentioned, the above method is convergent to a KKT point. In fact, we can easily observe that the procedure is even finite, as there are only finitely many solutions that can be reached using the algorithm. It is straightforward to see that at each step, the value of the objective function decreases. This implies that the algorithm stops after finitely many steps.

On the other hand, the construction of the algorithm does not guarantee that the solution obtained is globally optimal (not every KKT point is a global optimum in this case, as the derivatives are not continuous). Numerical evidence shows, however, that the discrepancy between the solution obtained by the algorithm and the global optimum can be ignored (see the next section).

Observe that the reasoning does not change even if the distributions of demand do not have a finite support (e.g. geometric or Poisson). The supply constraints guarantee that only a finite number of possible values of demand are taken into consideration.

4. Numerical experiments

In order to check the efficiency of the algorithm, a number of randomly generated test problems were solved. Unit transportation costs were chosen uniformly at random from the interval [5, 10), unit surplus costs from the interval [1, 2), shortage costs from the interval [5, 10), multipliers from the interval [0.8, 0.9), and the supply capacities from the interval [10, 20). The distribution of demand was generated for each destination point in the following way. First, the number of possible values of the demand was chosen uniformly at random from the set {10, ..., 20}. Then given the number of variants, the possible values of demand were chosen as follows: the smallest one uniformly at random from the interval [0.5, 1.5), with each successive value in the support being greater than the previous one by a number chosen from the same interval. The probabilities were chosen from the interval [0.1, 1), and then scaled in such a way that their sum was equal to 1. The algorithm was implemented in Java SE and run on a PC with Intel(R) Core(TM) i7-2670 QM CPU @2.20 GHz processor. 1000 randomly generated problems were solved for each of the following sizes of problem: (m, n) = (10, 10), (10, 20), (50, 50), (50, 100), (100, 100), (100, 200), (250, 250), (250, 500), which together gives 8000 problems. The solution times in milliseconds (AVG – average, DEV – standard deviation, MIN – minimum, MAX – maximum) are presented in Table 1. In the last line, the average accuracy (ACC, the average deviation of the objective function from the exact optimum) is also given.

As can be seen, the algorithm solves relatively big problems (over 100 000 variables) very quickly (in less than 1 s). In the case of smaller problems (tens of thousand of variables), the solution time is of the order of tens of milliseconds. The smallest problems (hundreds of variables) were solved in much less than a millisecond. The differences between the values of the solutions obtained by this method and the global optima are minimal.

Table 1. The solution times [ms])

Size of problem	10×10	10×20	50×50	50×100	100×100	100×200	250×250	250×500
AVG	0.032	0.082	2.032	5.758	12.657	39.541	173.783	616.969
DEV	0.008	0.015	0.218	0.426	1.082	2.152	9.960	21.022
MIN	0.009	0.042	1.090	4.540	9.300	34.300	156.000	547.000
MAX	0.058	0.133	2.810	7.500	18.700	48.400	250.000	702.000
ACC	0.01%	0.01%	0.02%	0.03%	0.02%	0.03%	0.02%	0.04%

5. Conclusions

The stochastic generalized transportation problem with discrete demand was presented in this paper. An effective method of solving this type of problem was described. By using one-sided derivatives, we were able to apply a variant of the equalization method, although the objective function is not differentiable at some points of the domain.

In spite of making some assumptions about the form of the model, the method described may be also applied to solve other types of problems from the same family. In particular, one may solve in this way problems with any discrete distribution of demand (including those with infinite supports, like the geometric or Poisson distribution).

An even more general type of problem consists of those with any piecewise linear cost functions related to the destination points. The only additional condition is that all these functions are convex.

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