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## APPLICATION OF THE REPRESENTATIONS OF SYMMETRIC GROUPS TO CHARACTERIZE SOLUTIONS OF GAMES IN PARTITION FUNCTION FORM

A different perspective from the more “traditional” approaches to studying solutions of games in partition function form has been presented. We provide a decomposition of the space of such games under the action of the symmetric group, for the cases with three and four players. In particular, we identify all the irreducible subspaces that are relevant to the study of linear symmetric solutions. We then use such a decomposition to derive a characterization of the class of linear and symmetric solutions, as well as of the class of linear, symmetric and efficient solutions.

**Keywords:** *games in partition function form, value, representation theory, symmetric group*

### 1. Introduction

The problem of distributing the surplus generated by a collection of people who are willing to cooperate with one another is well captured by cooperative game theory. It is assumed that a game is characterized by giving the value of each possible coalition from a set of players. Several models describe the value of a coalition by means of a real valued characteristic function, which is defined on the set of all subsets of the set of players. However, in the case of an economy with externalities, one cannot easily recommend a distribution of the joint gains, as it depends on the organizational structure which has been formed. In this context, Lucas and Thrall [8]

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introduced a new formulation for the theory of cooperative games in terms of partition functions. They assumed that players divide into coalitions, forming a partition of the set of players. According to this model, a partition function assigns a value to each pair consisting of a coalition and a partition which includes that coalition. The advantage of this model is that it takes into account both internal factors (a coalition itself) and external factors (the coalition structure) that may affect cooperation outcomes and allows us to analyze cooperation problems more deeply.

There have been many papers dealing with solutions of games in partition function form. The first author that proposed a concept for the value of this type of game was Myerson [10], and then Bolger [20] derived a class of linear, symmetric and efficient values for games in partition function form. More recently, Albizuri et al. [1], Macho-Stadler [9], Ju [7], Pham Do and Norde [11], and De Clippel and Serrano [3] apply an axiomatic approach to find a value.

In this paper, linear symmetric solutions of games in partition function form have been studied for the cases of three and four players with the innovative use of basic representation theory to describe the group of permutations of the set of players presenting a different perspective from the more “traditional” approaches.

Very roughly speaking, representation theory is a general tool for studying abstract algebraic structures by representing their elements as linear transformations of vector spaces. This is useful, since every permutation may be thought of as a linear map<sup>2</sup> which presents the information in a more clear and concise way. It is a beautiful mathematical subject which has many applications, ranging from the number theory and combinatorics to geometry, probability theory, quantum mechanics and quantum field theory. It was recently used by Hernández-Lamonedá et al. [5] to study solutions of games in characteristic function form, where they propose representation theory as a natural tool for research in cooperative game theory.

Briefly, what we do is to derive a direct sum decomposition of the space of games in partition function form and the space of payoffs into “elementary pieces”. According to this decomposition, any linear symmetric solution, when restricted to any such elementary piece, is either zero or a multiple of a single scalar. Therefore, all linear symmetric solutions may be written as a sum of trivial maps.

Having a global description of all linear and symmetric solutions, it is easy to understand the restrictions imposed by the efficiency axiom. We then use such a decomposition to provide, in a very economical way, a characterization of the class of linear symmetric solutions and a general expression for all linear, symmetric and efficient solutions.

The paper is organized as follows. In the next section, we first recall the main basic features of games in partition function form. A decomposition of the space of

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<sup>2</sup>A precise definition will be provided in Sec. 3.

three player games in partition function form is introduced in Section 3. In the same section, we show an application of this decomposition by giving characterizations of linear symmetric solutions. In Section 4, we discuss the decomposition for the case of four player games and Section 5 concludes the paper. Long proofs are presented in the Appendix.

To finish this introduction, we give a comment on the methods employed in this paper. Although it is true that the characterization results could be proved without any explicit mention of basic representation theory with regard to symmetric groups, we feel that by doing that we would be withholding valuable information from the reader. This algebraic tool, we believe, sheds new light on the structure of the space of games in partition function form and their solutions. Part of the purpose of the present paper is to share this viewpoint with the reader.

To make the paper as self contained as possible, we have included an Appendix with some facts we need regarding basic representation theory.

## 2. Framework and notation

In this section, we give some concepts and notation related to  $n$ -person games in partition function form, as well as a brief subsection containing preliminaries related to integer partitions, since they are a key subject in subsequent derivations.

### 2.1. Games in partition function form

Let  $N = \{1, 2, \dots, n\}$  be a fixed nonempty finite set, and let the members of  $N$  be interpreted as players in some game. Given  $0N$ , let  $CL$  be the set of all coalitions (nonempty subsets) of  $N$ ,  $CL = \{S \mid S \subseteq N, S \neq \emptyset\} = 2^N \setminus \{\emptyset\}$ . Let  $PT$  be the set of partitions of  $N$ , so

$$\{S_1, S_2, \dots, S_m\} \in PT \text{ iff } \bigcup_{i=1}^m S_i = N, \quad S_j \neq \emptyset \forall j, \quad S_j \cap S_k = \emptyset \forall j \neq k$$

Also, let  $ECL = \{(S, Q) \mid S \in Q \in PT\}$  be the set of embedded coalitions, that is the set of coalitions together with specifications as to how the other players are aligned.

For the sake of concision, we often denote by  $SQ$  the embedded coalition  $(S, Q)$ , and omit braces and commas in the description of subsets (for example:  $12\{12, 3\}$ )

instead of  $(\{1, 2\}, \{\{1, 2\}, \{3\}\})$ . Additionally, we will denote the cardinality of a set by its corresponding lower-case letter, for instance  $n = |N|$ ,  $s = |S|$ ,  $q = |Q|$ .

**Definition 1.** A mapping

$$w: ECL \rightarrow \mathcal{R}$$

that assigns a real value,  $w(S, Q)$ , to each embedded coalition  $(S, Q)$  is called a game in partition function form. The set of games in partition function form with player set  $N$  is denoted by  $G$ , i.e.,

$$G = G^{(n)} = \{w \mid w: ECL \rightarrow \mathcal{R}\}$$

The value  $w(S, Q)$  represents the payoff of coalition  $S$ , given that the coalition structure  $Q$  forms. In this kind of game, the value of some coalition depends not only on what the players of such a coalition can jointly obtain, but also on the way in which the other players are organized. We assume that, in any game situation, the universal coalition  $N$  (embedded in  $\{N\}$ ) will actually form, so that the players will have  $w(N, \{N\})$  to divide among themselves. But we also anticipate that the actual allocation of this value will depend on all the other potential values  $w(S, Q)$ , as they influence the relative bargaining strengths of the players.

Given  $w_1, w_2 \in G$  and  $c \in \mathcal{R}$ , we define the sum  $w_1 + w_2$  and the product  $cw_1$ , in  $G$ , in the usual way, i.e.

$$(w_1 + w_2)(S, Q) = w_1(S, Q) + w_2(S, Q) \quad \text{and} \quad (cw_1)(S, Q) = cw_1(S, Q)$$

respectively. It is easy to verify that with these operations  $G$  is a vector space.

A solution is a function  $\varphi: G \rightarrow \mathcal{R}^n$ . If  $\varphi$  is a solution and  $w \in G$ , then we can interpret  $\varphi_i(w)$  as the payoff which player  $i$  should expect from the game  $w$ .

Now, the group of permutations of  $N$ ,  $S_n = \{\theta: N \rightarrow N \mid \theta \text{ is bijective}\}$ , acts on  $CL$  and on  $ECL$  in the natural way; i.e., for  $\theta \in S_n$ :

$$\theta(S) = \{\theta(i) \mid i \in S\}$$

$$\theta(S_1, \{S_1, S_2, \dots, S_j\}) = (\theta(S_1), \{\theta(S_1), \theta(S_2), \dots, \theta(S_j)\})$$

Also,  $S_n$  acts on the space of payoff vectors,  $\mathcal{R}^n$ :

$$\theta(x_1, x_2, \dots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \dots, x_{\theta(n)})$$

Next, we define the usual linearity, symmetry and efficiency axioms, which solutions are required to satisfy in the framework of cooperative game theory.

**Axiom 1. Linearity.** The solution  $\varphi$  is linear if  $\varphi(w_1 + w_2) = \varphi(w_1) + \varphi(w_2)$  and  $\varphi(cw_1) = c\varphi(w_1)$ , for all  $w_1, w_2 \in G$  and  $c \in \mathcal{R}$ .

**Axiom 2. Symmetry.** The solution  $\varphi$  is said to be symmetric if and only if  $\varphi(\theta w) = \theta\varphi(w)$  for every  $\theta \in S_n$  and  $w \in G$ , where the game  $\theta w$  is defined as

$$(\theta w)(S, Q) = w[\theta^{-1}(S, Q)]$$

**Axiom 3. Efficiency.** The solution  $\varphi$  is efficient if  $\sum_{i \in N} \varphi_i(w) = w(N, \{N\})$  for all  $w \in G$ .

Myerson [10] proceeds axiomatically and proposes a value that extends the well known Shapley value [12] which is defined for TU games. His proposal satisfies the axioms of linearity, symmetry, efficiency and the “null” player property that states that players who have no effect on the outcome should neither receive nor pay anything.

The Myerson value of a player is given by

$$\psi_i(w) = \sum_{(S, Q) \in ECL} (-1)^{q-1} (q-1)! \left( \frac{1}{n} - \sum_{T \in Q \setminus \{S\}, i \notin T} \frac{1}{(q-1)(n-t)} \right) w(S, Q)$$

## 2.2. Integer partitions

A partition of a nonnegative integer is a way of expressing it as an unordered sum of other positive integers, and it is often written in tuple notation. Formally:

**Definition 2.**  $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_l]$  is a partition of  $n$  iff  $\lambda_1, \lambda_2, \dots, \lambda_l$  are positive integers and  $\lambda_1 + \lambda_2 + \dots + \lambda_l = n$ . Two partitions which only differ in the order of their elements are considered to be the same partition.

The set of all partitions of  $n$  will be denoted by  $\Pi(n)$ , and, if  $\lambda \in \Pi(n)$ ,  $|\lambda|$  is the number of elements of  $\lambda$ .

For example, the partitions of  $n = 4$  are  $[1, 1, 1, 1]$ ,  $[2, 1, 1]$ ,  $[2, 2]$ ,  $[3, 1]$ , and  $[4]$ . Sometimes we will abbreviate this notation by dropping the commas, so  $[2, 1, 1]$  becomes  $[211]$ .

If  $Q \in PT$ , there is a unique partition  $\lambda_Q \in \Pi(n)$ , associated with  $Q$ , where the elements of  $\lambda_Q$  are exactly the cardinalities of the elements of  $Q$ . In other words, if  $Q = \{S_1, S_2, \dots, S_m\} \in PT$ , then  $\lambda_Q = [s_1, s_2, \dots, s_m]$ .

For a given  $\lambda \in \Pi(n)$ , we represent by  $\lambda^\circ$  the set of numbers determined by the  $\lambda_i$  and for  $k \in \lambda^\circ$ , we denote by  $m_k$  the multiplicity of  $k$  in the partition  $\lambda$ . So, if  $\lambda = [4, 2, 2, 1, 1, 1]$ , then  $\lambda^\circ = \{1, 2, 4\}$  and  $m_1 = 3$ ,  $m_2 = 2$ ,  $m_4 = 1$ .

### 3. Representations

Precise definitions and some proofs for this section may be found in the Appendix at the end of the paper. Nevertheless, for the sake of easier reading, we repeat a few definitions here, sometimes in a less rigorous but more accessible, manner.

The group  $S_n$  acts naturally on the space of games in partition function form,  $G$ , via linear transformations (i.e.,  $G$  is a representation of  $S_n$ ). That is to say, each permutation  $\theta \in S_n$  corresponds to a linear, invertible transformation, which we still call  $\theta$ , of the vector space  $G$ , namely

$$(\theta w)(S, Q) = w[\theta^{-1}(S, Q)]$$

for every  $\theta \in S_n$ ,  $w \in G$  and  $(S, Q) \in ECL$ .

Moreover, this assignment preserves multiplication (i.e., is a group homomorphism) in the sense that the linear map corresponding to the product of the two permutations  $\theta_1\theta_2$  is the product (or composition) of the maps corresponding to  $\theta_1$  and  $\theta_2$ , in that order.

Similarly, the space of payoff vectors,  $R^n$ , is a representation of  $S_n$ :

$$\theta(x_1, x_2, \dots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \dots, x_{\theta(n)})$$

**Definition 3.** Let  $X_1$  and  $X_2$  be two representations of the group  $S_n$ . A linear map  $T: X_1 \rightarrow X_2$  is said to be  $S_n$  equivariant if  $T(\theta x) = \theta T(x)$ , for every  $\theta \in S_n$  and every  $x \in X_1$ .

**Remark 1.** Notice that, in the language of representation theory, what we call a linear symmetric solution is a linear map  $\varphi: G \rightarrow R^n$  that is  $S_n$  equivariant.

### 3.1. Decomposition of $G^{(3)}$

**Definition 4.** Let  $Y$  be a subspace of the vector space  $X$ .

- $Y$  is invariant (with respect to the action of  $S_n$ ) if for every  $y \in Y$  and every  $\theta \in S_n$ , we have

$$\theta y \in Y$$

- $Y$  is irreducible if  $Y$  itself has no invariant subspaces other than  $\{0\}$  and  $Y$  itself.

We begin with the decomposition of  $\mathcal{R}^n$  into irreducible representations which is easier, and then proceed to do the same thing for  $G$ . That is to say, we wish to write  $\mathcal{R}^n$  as a direct sum of subspaces, each invariant with respect to all permutations in  $S_n$ , in such way that the summands cannot be further decomposed (i.e., they are irreducible).

For this, set  $\mathbf{1} = (1, 1, \dots, 1) \in \mathcal{R}^n$  and

$$U_n = \langle \mathbf{1} \rangle \quad \text{and} \quad V_n = \{z \in \mathcal{R}^n \mid z \cdot \mathbf{1} = 0\}$$

The spaces  $U_n$  and  $V_n$  are usually called the “trivial” and “standard” representations, respectively. Notice that  $U_n$  is a trivial subspace in the sense that every permutation acts like the identity transformation.

Every permutation fixes each element of  $U_n$ , so, in particular, it is an invariant subspace of  $\mathcal{R}^n$ . Being one dimensional, it is automatically irreducible. Its orthogonal complement,  $U_n^\perp$ , consists of all vectors such that the sum of their coordinates is zero. Clearly, if we permute the coordinates of any such vector, its sum will still be zero. Hence,  $V_n$  is also an invariant subspace.

**Proposition 1.** The decomposition of  $\mathcal{R}^n$  under  $S_n$  into irreducible subspaces is:

$$\mathcal{R}^n = U_n \oplus V_n$$

**Proof.** First, it is clear that  $U_n \cap V_n = \{0\}$ <sup>3</sup>. We now prove that  $\mathcal{R}^n = U_n + V_n$ .

1. If  $z \in (U_n + V_n)$ , then  $z \in \mathcal{R}^n$ , since  $(U_n + V_n)$  is a subspace of  $\mathcal{R}^n$ .

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<sup>3</sup>Here,  $\mathbf{0} = \{0, 0, \dots, 0 \in \mathcal{R}^n\}$

2. For  $z \in \mathbf{R}^n$ , let  $\bar{z} = \frac{1}{n} \sum_{i=1}^n z_i$ . Note that  $z$  can be written as

$$z = (\bar{z}, \bar{z}, \dots, \bar{z}) + (z_1 - \bar{z}, z_2 - \bar{z}, \dots, z_n - \bar{z})$$

and so,

$$z \in (U_n + V_n)$$

Finally, since  $U_n$  is one dimensional, then it is irreducible. To check that  $V_n$  is also irreducible, an induction argument that can be found in Hernández-Lamonedá et al. [5] may be used.

Thus, this result tells us that  $\mathbf{R}^n$ , as a vector space with group of symmetry  $S_n$ , can be written as an orthogonal sum of the subspaces  $U_n$  and  $V_n$ , which are invariant under permutations and cannot be further decomposed.

The decomposition of  $G$  is carried out in three steps. For a given  $\lambda \in \Pi(n)$ , let  $Q_\lambda = \{Q \in PT \mid \lambda_Q = \lambda\}$ . For each  $\lambda \in \Pi(n)$ , define the subspace of games

$$G_\lambda = \{w \in G \mid w(S, Q) = 0, \text{ if } Q \notin Q_\lambda\}$$

Thus,

$$G = \bigoplus_{\lambda \in \Pi(n)} G_\lambda$$

whereas, for  $k \in \lambda^\circ$ , define the subspace

$$G_\lambda^k = \{w \in G_\lambda \mid w(S, Q) = 0, \text{ if } |S| \neq k\}$$

Then each  $G_\lambda$  has a decomposition  $G_\lambda = \bigoplus_{k \in \lambda^\circ} G_\lambda^k$  and so we obtain the following decomposition of  $G$ :

$$G = \bigoplus_{\lambda \in \Pi(n)} \bigoplus_{k \in \lambda^\circ} G_\lambda^k = \bigoplus_{\lambda \in \Pi(n)} \bigoplus_{k \in \lambda^\circ} G_\lambda^k \quad (1)$$

Each subspace  $G_\lambda^k$  is invariant under  $S_n$  and the decomposition is orthogonal with respect to the invariant inner product on  $G$  given by

$$\langle w_1, w_2 \rangle = \sum_{(S,Q) \in ECL} w_1(S, Q)w_2(S, Q)$$

Here, invariance of the inner product means that each permutation  $\theta \in S_n$  is not only a linear map on  $G$  but an orthogonal map with respect to this inner product. Formally,  $\langle \theta w_1, \theta w_2 \rangle = \langle w_1, w_2 \rangle$  for every  $w_1, w_2 \in G$ .

**Example 1.** For the case  $N = \{1, 2, 3\}$ ,  $\dim G^{(3)} = 10$  and it decomposes as follows:

$$G^{(3)} = G_{[1,1,1]}^1 \oplus G_{[2,1]}^1 \oplus G_{[2,1]}^2 \oplus G_{[3]}^3$$

The next goal is to get a decomposition of each subspace of games  $G_\lambda^k$  into irreducible subspaces and hence obtain a decomposition of  $G^{(3)}$ .

The following games play an important role in describing the decomposition of the space of three player games:

$$u_\lambda^k(S, Q) = \begin{cases} 1 & \text{if } Q \in Q_\lambda, |S| = k \\ 0 & \text{otherwise} \end{cases}$$

Notice that  $G_{[n]}^n = Ru_{[n]}^n$ .

Also, for each  $\lambda \in \Pi(n) \setminus \{[n]\}$ ,  $k \in \lambda^\circ$  and  $z \in V_n$ ; let  $z^{\lambda,k} \in G_\lambda^k$  be given by

$$z^{\lambda,k}(S, Q) = \begin{cases} \sum_{i \in S} z_i & \text{if } Q \in Q_\lambda, |S| = k \\ 0 & \text{otherwise} \end{cases}$$

**Definition 5.** Suppose  $X_1$  and  $X_2$  are two representations of the group  $S_n$ , i.e., we have two vector spaces  $X_1$  and  $X_2$  where  $S_n$  acts using linear maps. We say that  $X_1$  and  $X_2$  are isomorphic, if there is a linear map between them which is 1–1 and onto and commutes with the respective  $S_n$  actions. Formally, there is an invertible linear map  $T: X_1 \rightarrow X_2$  such that  $T(\theta x) = \theta T(x)$  for each  $\theta \in S_n$  and  $x \in X_1$ . We then write  $X_1; X_2$ .

For our purposes,  $X_1$  will be an irreducible subspace of  $G$  and  $X_2$  an irreducible subspace of  $R^n$ .

Isomorphic representations are essentially “equal”; not only are they spaces of the same dimension, but the actions equivalent under some linear invertible map between them.

The next proposition provides us a decomposition of the space of three player games into irreducible subspaces.

**Proposition 2.** For  $\lambda \in \Pi(3) \setminus \{[3]\}$ ,

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k$$

where  $U_\lambda^k = \langle u_\lambda^k \rangle$ ;  $U_3$  and  $V_\lambda^k = \{z^{\lambda,k} \mid z \in V_3\}$ ;  $V_3$ . The decomposition is orthogonal. (See proof in the Appendix).

**Remark 2.** From the above Proposition, it is not difficult to verify that

$$V_\lambda^k = \left\{ w \in G_\lambda^k \mid \sum_{\substack{(S,Q) \in ECL \\ |S|=k, Q \in \mathcal{Q}_\lambda}} w(S,Q) = 0 \right\}$$

Proposition 2 gives us a decomposition of the space of three player games that is a key ingredient in our subsequent analysis.

Set  $U_G = U_{[1,1,1]}^1 \oplus U_{[2,1]}^1 \oplus U_{[2,1]}^2 \oplus U_{[3]}^3$ . This is a subspace of games whose value on a given embedded coalition  $(S, Q)$ , depends only on the cardinality of  $S$  and on the structure of  $Q^4$ . According to Proposition 12,  $U_G$  is the largest subspace of  $G^{(3)}$  where  $S_3$  acts trivially<sup>5</sup>. Let  $V_G = V_{[1,1,1]}^1 \oplus V_{[2,1]}^1 \oplus V_{[2,1]}^2$ , then

$$G^{(3)} = U_G \oplus V_G$$

Thus, given a game  $w \in G^{(3)}$ , from the above we may decompose it as  $w = u + v$ , where in turn  $u = \sum a_{\lambda,k} u_\lambda^k$  and  $v = \sum z_{\lambda,k}^{\lambda,k}$ . This decomposition is very well suited to study the image of  $w$  under any linear symmetric solution. This results from the following version of Schur’s well known Lemma<sup>6</sup>.

<sup>4</sup>Such games may be thought of as a counterpart of symmetric games in TU games.

<sup>5</sup>i.e.,  $\theta w = w$  for each  $\theta \in S_3$  and  $w \in U_G$ .

<sup>6</sup>See the Appendix for a precise statement of Schur’s theorem.

**Theorem 1 (Schur's Lemma).** Any linear symmetric solution

$$\varphi : G^{(3)} = U_G \oplus V_G \rightarrow R^3 = U_3 \oplus V_3$$

satisfies

a)  $\varphi(U_G) \subset U_3$ ,

b)  $\varphi(V_G) \subset V_3$ .

Moreover,

- for each  $\lambda \in \Pi(3)$  and  $k \in \lambda^\circ$ , there is a constant  $\alpha_{\lambda,k} \in R$  such that, for each  $u \in U_\lambda^k$ ,

$$\varphi(u) = \alpha_{\lambda,k}(1, 1, 1) \in U_3$$

- for each  $\lambda \in \Pi(3) \setminus \{[3]\}$  and  $k \in \lambda^\circ$ , there is a constant  $\beta_{\lambda,k} \in R$  such that, for each  $z^{\lambda,k} \in V_\lambda^k$ ,

$$\varphi(z^{\lambda,k}) = \beta_{\lambda,k}z \in V_3$$

For many purposes, it suffices to use merely the existence of the decomposition of the game  $w \in G^{(3)}$ , without having to worry about the precise form of each component. Nevertheless, it will be useful to compute each component. Thus we give a formula for computing them.

**Proposition 3.** Let  $w \in G^{(3)}$ . Then

$$w = \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} a_{\lambda,k} u_\lambda^k + \sum_{\substack{\lambda \in \Pi(3) \setminus \{[3]\} \\ k \in \lambda^\circ}} z_{\lambda,k}^{\lambda,k} \tag{2}$$

where

- 1)  $a_{\lambda,k}$  is the average of the values  $w(S, Q)$  with  $Q \in Q_\lambda$  and  $|S| = k$ :

$$a_{\lambda,k} = \frac{\sum_{\substack{(S,Q) \in ECL \\ Q \in Q_\lambda, |S|=k}} w(S, Q)}{\left| \{(S, Q) \in ECL \mid Q \in Q_\lambda, |S| = k\} \right|}$$

2) For each  $\lambda \in \Pi(3) \setminus \{[3]\}$  and  $k \in \lambda^\circ$  :

$$z_{\lambda,k} = (-1)^{|\lambda|} k(|\lambda| - 1) \psi(w_{\lambda,k})$$

where  $\psi$  denotes Myerson's value and  $w_{\lambda,k}$  is the  $G_\lambda^k$  component of  $w$  (i.e.,  $w_{\lambda,k}(S, Q) = w(S, Q)$  if  $Q \in Q_\lambda$ ,  $|S| = k$ , and  $w_{\lambda,k}(S, Q) = 0$  otherwise).

**Proof.** We start by computing the orthogonal projection of  $w$  onto  $U_G$ . Notice that  $\{u_\lambda^k\}$  is an orthogonal basis for  $U_G$ , and that

$$\|u_\lambda^k\|^2 = |\{(S, Q) \in ECL \mid Q \in Q_\lambda, |S| = k\}|$$

Thus, the projection of  $w$  onto  $U_G$  is

$$\sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} \frac{\langle w, u_\lambda^k \rangle}{\langle u_\lambda^k, u_\lambda^k \rangle} u_\lambda^k$$

and thus

$$a_{\lambda,k} = \frac{\langle w, u_\lambda^k \rangle}{\langle u_\lambda^k, u_\lambda^k \rangle} = \frac{\sum_{\substack{(S,Q) \in ECL \\ Q \in Q_\lambda, |S|=k}} w(S, Q)}{|\{(S, Q) \in ECL \mid Q \in Q_\lambda, |S| = k\}|}$$

Now, for  $\lambda \in \Pi(3) \setminus \{[3]\}$ , let  $w_{\lambda,k}$  be defined as above. It follows that

$$w_{\lambda,k} = a_{\lambda,k} u_\lambda^k + z_{\lambda,k}^{\lambda,k}$$

Applying Myerson's value, we obtain

$$\psi(w_{\lambda,k}) = a_{\lambda,k} \psi(u_\lambda^k) + \psi(z_{\lambda,k}^{\lambda,k}) = \frac{1}{(-1)^{|\lambda|} k(|\lambda| - 1)} z_{\lambda,k}$$

because  $\psi(u_\lambda^k) = 0$  for  $\lambda \in \Pi(3) \setminus \{[3]\}$  from the assumption of efficiency.  $\psi(z_{\lambda,k}^{\lambda,k})$

$= \beta_{\lambda,k} z_{\lambda,k}$  by Schur's Lemma, and the precise value of  $\beta_{\lambda,k} = \frac{1}{(-1)^{|\lambda|} k(|\lambda| - 1)}$  is easy to

compute.

**Remark 3.** The use of Myerson’s value in order to compute  $z_{\lambda,k}$  is a matter of personal taste. One could use one’s own favorite linear symmetric solution – as long as it is non-zero on each  $V_{\lambda}^k$  – to compute them.

### 3.2. Applications

Now we show how to get characterizations of solutions easily by using the decomposition of a game given by (2) in conjunction with Schur’s Lemma. We start by providing a characterization of all linear symmetric solutions  $\varphi: G^{(3)} \rightarrow \mathbb{R}^3$  in the following proposition:

**Proposition 4.** Linear symmetric solutions  $\varphi: G^{(3)} \rightarrow \mathbb{R}^3$  are of the form

$$\varphi_l(w) = \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} \left[ \gamma_{\lambda,k} \sum_{\substack{(S,Q) \in ECL \\ S \ni i, |S|=k \\ Q \in Q_\lambda}} w(S,Q) + \delta_{\lambda,k} \sum_{\substack{(S,Q) \in ECL \\ S \ni i, |S|=k \\ Q \in Q_\lambda}} w(S,Q) \right] \quad (3)$$

for some real numbers

$$\left\{ \gamma_{\lambda,k} \mid \lambda \in \Pi(3), k \in \lambda^\circ \right\} \cup \left\{ \delta_{\lambda,k} \mid \lambda \in \Pi(3) \setminus \{[3]\}, k \in \lambda^\circ \right\}$$

**Proof.** Let  $\varphi: G^{(3)} \rightarrow \mathbb{R}^3$  be a linear symmetric solution. By the previous proposition,  $w \in G^{(3)}$  decomposes as

$$w = \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} a_{\lambda,k} u_{\lambda}^k + \sum_{\substack{\lambda \in \Pi(3) \setminus \{[3]\} \\ k \in \lambda^\circ}} z_{\lambda,k}^{\lambda,k}$$

Without loss of generality, we take  $i = 1$ , then

$$\varphi_1(w) = \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} a_{\lambda,k} \varphi_1(u_{\lambda}^k) + \sum_{\substack{\lambda \in \Pi(3) \setminus \{[3]\} \\ k \in \lambda^\circ}} \varphi_1(z_{\lambda,k}^{\lambda,k})$$

Now, from Schur’s Lemma and using Proposition 3 again, we have

$$\begin{aligned}
\varphi_1(w) &= \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} a_{\lambda,k} \alpha_{\lambda,k} + \sum_{\substack{\lambda \in \Pi(3) \setminus \{\{3\}\} \\ k \in \lambda^\circ}} \beta_{\lambda,k} (z_{\lambda,k})_1 \\
&= \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} \alpha'_{\lambda,k} \sum_{\substack{(S,Q) \in ECL \\ Q \in \mathcal{Q}_\lambda, |S|=k}} w(S,Q) + \sum_{\substack{\lambda \in \Pi(3) \setminus \{\{3\}\} \\ k \in \lambda^\circ}} \beta'_{\lambda,k} \psi_1(w_{\lambda,k})
\end{aligned}$$

where

$$\alpha'_{\lambda,k} = \frac{\alpha_{\lambda,k}}{\left| \{(S,Q) \in ECL \mid Q \in \mathcal{Q}_\lambda, |S|=k\} \right|}$$

and

$$\beta'_{\lambda,k} = \beta_{\lambda,k} (-1)^{|\lambda|} k (|\lambda| - 1)$$

Notice that

$$\psi_1(w_{[\{111\},1]}) = -\frac{1}{3} w(1\{1,2,3\}) + \frac{1}{6} w(2\{1,2,3\}) + \frac{1}{6} w(3\{1,2,3\})$$

$$\psi_1(w_{[211],1}) = \frac{2}{3} w(1\{1,23\}) - \frac{1}{3} w(2\{2,13\}) - \frac{1}{3} w(3\{3,12\})$$

$$\psi_1(w_{[21],2}) = \frac{1}{6} w(12\{3,12\}) + \frac{1}{6} w(13\{2,13\}) - \frac{1}{3} w(23\{1,23\})$$

Finally, set

$$\gamma_{111,1} = \alpha'_{111,1} - \frac{1}{3} \beta'_{111,1}, \quad \gamma_{211,1} = \alpha'_{211,1} + \frac{2}{3} \beta'_{211,1}, \quad \gamma_{211,2} = \alpha'_{211,2} + \frac{1}{6} \beta'_{211,2}, \quad \gamma_{313} = \alpha'_{313},$$

$$\delta_{111,1} = \alpha'_{111,1} + \frac{1}{6} \beta'_{111,1}, \quad \delta_{211,1} = \alpha'_{211,1} - \frac{1}{3} \beta'_{211,1}, \quad \text{and} \quad \delta_{211,2} = \alpha'_{211,2} - \frac{1}{3} \beta'_{211,2}$$

Thus,

$$\varphi_1(w) = \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} \left[ \gamma_{\lambda,k} \sum_{\substack{(S,Q) \in ECL \\ S \ni 1, |S|=k \\ Q \in \mathcal{Q}_\lambda}} w(S,Q) + \delta_{\lambda,k} \sum_{\substack{(S,Q) \in ECL \\ S \ni 1, |S|=k \\ Q \in \mathcal{Q}_\lambda}} w(S,Q) \right]$$

We should mention that a similar formula for linear and symmetric solutions of games in partition function form was obtained by Hernández-Lamonedá et al. [6].

**Corollary 1.** The space of all linear and symmetric solutions on  $G^{(3)}$  has dimension

$$\left| \left\{ \gamma_{\lambda,k} \mid \lambda \in \Pi(3), k \in \lambda^\circ \right\} \cup \left\{ \delta_{\lambda,k} \mid \lambda \in \Pi(3) \setminus \{[3]\}, k \in \lambda^\circ \right\} \right| = 7$$

Once we have such a global description of all linear symmetric solutions, we can understand restrictions imposed by other conditions or axioms, for example, the efficiency axiom.

**Proposition 5.** Let  $\varphi: G^{(3)} \rightarrow \mathcal{R}^3$  be a linear symmetric solution. Then  $\varphi$  is efficient if and only if

- 1)  $\varphi_i(u_\lambda^k) = 0$ , for all  $\lambda \in \Pi(3) \setminus \{[3]\}$  and all  $k \in \lambda^\circ$ ; and
- 2)  $\varphi_i(u_{[3]}^3) = \frac{1}{3}$

**Proof.** First of all,  $(U_{[3]}^3)^\perp$  is exactly the subspace of games  $w$  where  $w(N, \{N\}) = 0$ . Of these games, those in  $V_G$  trivially satisfy  $\sum_{i \in N} \varphi_i(w) = 0$ , since (by Schur's Lemma)  $\varphi(V_G) \subset V$ .

Thus, efficiency need only be checked in  $U_G$ . Since  $u_\lambda^k$  is fixed by every permutation in  $S_3$ , we have

$$\sum_{i \in N} \varphi_i(u_\lambda^k) = 3\varphi_i(u_\lambda^k)$$

and so  $\varphi$  is efficient if and only if  $3\varphi_i(u_\lambda^k) = u_\lambda^k(N, \{N\}) = 1$  (if  $\lambda = [3]$ ).

Recall that  $U_G$  is a subspace of games whose value for a given embedded coalition  $(S, Q)$  depends only on the cardinality of  $S$  and the structure of  $Q$ . The next corollary characterizes the solutions to these games in terms of linearity, symmetry and efficiency. It turns out that among all linear symmetric solutions, the egalitarian solution is characterized as the unique efficient solution on  $U_G$ . Formally:

**Corollary 2.** Let  $\varphi: G^{(3)} \rightarrow \mathcal{R}^3$  be a linear, symmetric and efficient solution. Then for all  $w \in U_G$

$$\varphi_i(w) = \frac{w(N, \{N\})}{3}$$

In other words, all linear, symmetric and efficient solutions (e.g., Myerson’s value) coincide with the egalitarian solution when restricted to these type of games,  $U_G$ .

Now, another immediate application is to provide a characterization of all linear, symmetric and efficient solutions<sup>7</sup>.

**Theorem 2.** The solution  $\varphi : G^{(3)} \rightarrow \mathcal{R}^3$  satisfies linearity, symmetry and efficiency axioms if and only if it is of the form

$$\varphi_i(w) = \frac{w(N, \{N\})}{3} + \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} \delta_{\lambda,k} \left[ \sum_{\substack{(S,Q) \in ECL \\ S \ni i, |S|=k \\ Q \in \mathcal{Q}_\lambda}} (n-k)w(S, Q) - \sum_{\substack{(S,Q) \in ECL \\ S \ni i, |S|=k \\ Q \in \mathcal{Q}_\lambda}} kw(S, Q) \right] \quad (4)$$

for some real numbers  $\{\delta_{\lambda,k} \mid \lambda \in \Pi(3) \setminus \{[3]\}, k \in \lambda^\circ\}$ .

**Proof.** Let  $\varphi : G^{(3)} \rightarrow \mathcal{R}^3$  be a linear, symmetric and efficient solution, and  $w \in G^{(3)}$ . Then, by Proposition 3, Schur’s Lemma and Proposition 5:

$$\begin{aligned} \varphi_i(w) &= \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} a_{\lambda,k} \varphi_i(u_\lambda^k) + \sum_{\substack{\lambda \in \Pi(3) \setminus \{[3]\} \\ k \in \lambda^\circ}} \varphi_i(z_{\lambda,k}^{\lambda,k}) \\ &= a_{[3],3} \varphi_i(u_{[3]}^3) + \sum_{\substack{\lambda \in \Pi(3) \setminus \{[3]\} \\ k \in \lambda^\circ}} \beta(z_{\lambda,k})_i \\ &= \frac{w(N, \{N\})}{3} + \sum_{\substack{\lambda \in \Pi(3) \setminus \{[3]\} \\ k \in \lambda^\circ}} \beta'_{\lambda,k} \psi_i(w_{\lambda,k}) \end{aligned}$$

where  $\beta'_{\lambda,k} = \beta_{\lambda,k}(-1)^{|\lambda|} k(|\lambda| - 1)$ . The result follows from substituting the values  $\psi_i(w_{\lambda,k})$  grouping terms, and setting  $\delta_{111,1} = -\frac{1}{6} \beta'_{111,1}$ ,  $\delta_{211,1} = \frac{1}{3} \beta'_{211,1}$ , and  $\delta_{211,2} = \frac{1}{6} \beta'_{211,2}$ .

**Corollary 3.** The space of all linear, symmetric and efficient solutions on  $G^{(3)}$  has dimension

$$\left| \left\{ \delta_{\lambda,k} \mid \lambda \in \Pi(3) \setminus \{[3]\}, k \in \lambda^\circ \right\} \right| = 3$$

---

<sup>7</sup>An equivalent expression to (4) can be found in the paper by Hernández-Lamonedá et al. [6].

It is also possible to give an expression for all linear, symmetric and efficient solutions of TU games in a characteristic function form. Let

$$J^{(n)} = \{v : 2^N \rightarrow \mathbb{R} \mid v(\emptyset) = 0\}$$

be the set of all TU games in characteristic function form with  $n$  players.

**Corollary 4.** The solution  $\varphi : J^{(3)} \rightarrow \mathbb{R}^n$  satisfies the linearity, symmetry and efficiency axioms if and only if it is of the form

$$\varphi_i(v) = \frac{v(N)}{3} + \sum_{\substack{S \subseteq N \\ i \in S}} \rho_s \frac{v(S)}{s} - \sum_{\substack{S \subseteq N \\ i \notin S}} \rho_s \frac{v(S)}{n-s}$$

for some real numbers  $\{\rho_1, \rho_2\}$ .

**Proof.** Take  $w \in G$  such that  $w(S, Q) = v(S)$  for all  $(S, Q) \in ECL$  in Eq. (4).

### 4 The case $n = 4$

One may notice that all the previous results follow from the decomposition of the space of games into a direct sum of irreducible subspaces. In this part, we provide such a decomposition for four player games.

**Example 2.** For the case  $N = \{1, 2, 3, 4\}$ ,  $\dim G^{(4)} = 37$  and following from (1), it decomposes as follows:

$$G^{(4)} = G_{[1,1,1,1]}^1 \oplus G_{[2,1,1]}^1 \oplus G_{[2,1,1]}^2 \oplus G_{[3,1]}^1 \oplus G_{[3,1]}^3 \oplus G_{[2,2]}^2 \oplus G_{[4]}^4$$

Once again, we first obtain a decomposition of each subspace of games  $G_{\lambda}^k$  into irreducible subspaces and hence derive the decomposition of  $G^{(4)}$ . For this purpose, let  $I_{\lambda,k}$  be a set such that

$$I_{\lambda,k} = \begin{cases} \lambda^\circ \setminus \{k\} & \text{if } m_k = 1 \\ \lambda^\circ & \text{if } m_k > 1 \end{cases}$$

For each  $\lambda \in \Pi(n) \setminus \{[n]\}$ ,  $k \in \lambda^\circ$  and  $x \in V_n$ , define the set of games in  $G_\lambda^k$ ,  $\{x^{\lambda,k,j} \mid j \in I_{\lambda,k}\}$  as follows:

$$x^{\lambda,k,j}(S, Q) = \begin{cases} \sum_{\substack{T \in Q \\ |T|=j, T \neq S}} \sum_{i \in T} x_i & \text{if } Q \in Q_\lambda, \quad |S| = k \\ 0 & \text{otherwise} \end{cases}$$

**Proposition 6.** For  $\lambda \in \Pi(4) \setminus \{[4]\}$ ,

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k \oplus W_\lambda^k$$

where  $U_\lambda^k = \langle u_\lambda^k \rangle$ ;  $U_4$ ,  $V_\lambda^k = \bigoplus_{j \in I_{\lambda,k}} \{x^{\lambda,k,j} \mid x \in V_4\}$  and neither any  $\{x^{\lambda,k,j} \mid x \in V_4\}$ ;  $V_4$ ; nor  $W_\lambda^k$  contains any summands isomorphic to either  $U_4$  or  $V_4$ . The decomposition is orthogonal (See proof in the Appendix).

**Remark 4.** Proposition 6 does not quite give us a decomposition of  $G_\lambda^k$  into irreducible summands. The subspace  $U_\lambda^k$  is irreducible and  $V_\lambda^k$  is a direct sum of irreducible subspaces, whereas  $W_\lambda^k$  may or may not be irreducible (depending on  $\lambda$  and  $k$ ). However, as we shall see, the exact nature of this subspace plays no role in the study of linear symmetric solutions, since it lies in the kernel of any such solution.

As in the case of three player games, set  $U_G = \bigoplus_{\substack{\lambda \in \Pi(4) \\ k \in \lambda^\circ}} U_\lambda^k$ . Once again,  $U_G$  is a subspace of games, whose values  $w(S, Q)$  depend only on the cardinality of  $S$  and on the structure of  $Q$ . Set  $V_G = \bigoplus_{\substack{\lambda \in \Pi(4) \setminus \{[4]\} \\ k \in \lambda^\circ}} V_\lambda^k$  and  $W_G = \bigoplus_{\substack{\lambda \in \Pi(4) \setminus \{[4]\} \\ k \in \lambda^\circ}} W_\lambda^k$ , then:

$$G^{(4)} = U_G \oplus V_G \oplus W_G$$

**Corollary 5.** If  $\varphi: G^{(4)} \rightarrow \mathcal{R}^4$  is a linear symmetric solution, then  $\varphi(w) = 0$  for each  $w \in W_G$ .

**Proof.** Let  $\varphi: G^{(4)} = U_G \oplus V_G \oplus W_G \rightarrow \mathcal{R}^4 = U_4 \oplus V_4$  be a linear symmetric solution. Assume  $X \subset W_G$  is an irreducible summand in the decomposition of  $W_G$  (even when we do not know the decomposition of  $W_G$  as a sum of irreducible

subspaces, it is known that such a decomposition exists). Let  $p_1$  and  $p_2$  denote the orthogonal projections of  $\mathbb{R}^4$  onto  $U_4$  and  $V_4$ , respectively. Now,  $\varphi: G^{(4)} \rightarrow \mathbb{R}^4 = U_4 \oplus V_4$  may be written as  $\varphi = (p_1 \circ \varphi, p_2 \circ \varphi)$ . Denote by  $\iota: X \rightarrow G^{(4)}$  the inclusion. Then the restriction of  $\varphi$  to  $X$  may be expressed as

$$\varphi|_X = \varphi \circ \iota = (p_1 \circ \varphi \circ \iota, p_2 \circ \varphi \circ \iota)$$

Now,  $p_1 \circ \varphi \circ \iota: X \rightarrow U_4$  and  $p_2 \circ \varphi \circ \iota: X \rightarrow V_4$  are linear symmetric maps. Since  $X$  is not isomorphic to either of these two spaces, Schur's Lemma (see the Appendix for its statement) says that  $p_1 \circ \varphi \circ \iota$  and  $p_2 \circ \varphi \circ \iota$  must be zero. Since this is true for every irreducible summand  $X$  of  $W_G$ ,  $\varphi$  is zero on all of  $W_G$ .

**Remark 5.** According to Proposition 6 and the previous result, in order to study linear symmetric solutions, one only needs to look at those games inside  $U_G \oplus V_G$ .

As we have already pointed out, in the case of four player games, we can also obtain characterizations of the class of linear and symmetric solutions, as well as of the class of linear, symmetric and efficient solutions. Once again, the key is the decomposition of  $G^{(4)}$  into irreducible subspaces (Proposition 6), together with Shur's Lemma.

## 5. Concluding remarks

We have noted that the point of view we take in this article depends heavily on the decomposition of the space of  $n$ -player games into a direct sum of "special" subspaces. In the cases where  $n = 3, 4$ , it was decomposed as a direct sum of three orthogonal subspaces: a subspace containing a type of symmetric games, another subspace which we call  $V_G$ , and a subspace  $W_G$ , which only plays the role of the common kernel of every linear symmetric solution. Although  $V_G$  does not have a natural definition in terms of well known game theoretic considerations, it has a simple characterization in terms of vectors whose entries add up to zero.

Characterizations of solutions follow from such a decomposition in an very economical way. It remains an open challenge to obtain the general decomposition of  $G^{(n)}$  into a direct sum of irreducible subspaces, since mathematically, the general case seems to have a much more complicated structure.

Although it is true that the characterization of these results could be proved without any explicit mention of representation theory with regard to symmetric groups, we feel that by doing that we would be withholding valuable information from

the reader. This algebraic tool, we believe, sheds new light on the structure of the space of games in partition function form and their solutions. Part of the purpose of the present paper is to share this viewpoint with the reader.

## Appendix

A reference for basic representation theory is Fulton and Harris [4]. Nevertheless, we recall all the basic facts that we need.

The symmetric group  $S_n$  acts on  $G$  via linear transformations (i.e.,  $G$  is a representation of  $S_n$ ). That is to say, there is a group homomorphism  $\rho: S_n \rightarrow GL(G)$ , where  $GL(G)$  is the group of invertible linear maps in  $G$ . This relation is given by:

$$(\theta w)(S, Q) := [\rho(\theta)(w)](S, Q) = w[\theta^{-1}(S, Q)]$$

for every  $\theta \in S_n$ ,  $w \in G$  and  $(S, Q) \in ECL$ .

**Definition 6.** Let  $H$  be an arbitrary group. A representation of  $H$  is a homomorphism  $\rho: H \rightarrow GL(X)$ , where  $X$  is a vector space and  $GL(X) = \{T: X \rightarrow X \mid T \text{ linear and invertible}\}$ .

In other words, a representation of  $H$  is a map assigning to each element  $h \in H$  a linear map  $\rho(h): X \rightarrow X$  that respects multiplication:

$$\rho(h_1 h_2) = \rho(h_1) \rho(h_2)$$

for all  $h_1, h_2 \in H$ .

One usually abuses notation and talks about the representation  $X$  without explicitly mentioning the homomorphism  $\rho$ . Thus, when applying the linear transformation corresponding to  $h \in H$  to the element  $x \in X$ , we write  $hx$  rather than  $(\rho(h))(x)$ .

The space of payoff vectors,  $\mathcal{R}^n$  is also an  $S_n$  representation:

$$\theta(x_1, x_2, \dots, x_n) := [\rho(\theta)](x_1, x_2, \dots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \dots, x_{\theta(n)})$$

**Definition 7.** Let  $X_1$  and  $X_2$  be two representations of the group  $H$ .

• A linear map  $T: X_1 \rightarrow X_2$  is said to be  $H$  equivariant if  $T(hx) = hT(x)$  for every  $h \in H$  and  $x \in X_1$ .

•  $X_1$  and  $X_2$  are said to be isomorphic  $H$ -representations,  $X_1; X_2$ , if there exists a  $H$ -equivariant isomorphism between them.

Thus, two representations that are isomorphic are, as far as all problems dealing with linear algebra on a group of symmetries, the same. They are vector spaces of the same dimension, where actions are seen to correspond according to a linear isomorphism.

**Definition 8.** A representation  $X$  is irreducible if it does not contain a nontrivial invariant subspace. That is to say, if  $Y \subset X$  is also a representation of  $H$  (meaning that  $hy \in Y, \forall h \in H$ ), then  $Y$  is either  $\{0\}$  or all of  $X$ .

**Proposition 7.** For any representation  $X$  of a finite group  $H$ , there is a decomposition

$$X = X_1^{\oplus a_1} \oplus X_2^{\oplus a_2} \oplus \dots \oplus X_j^{\oplus a_j}$$

where the  $X_i$  are distinct irreducible representations. This decomposition is unique, as are the  $X_i$  that occur and their multiplicities  $a_i$ .

This property is called “complete reducibility” and the extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique is one of the consequences of the following:

**Theorem 3 (Schur’s Lemma).** Let  $X_1, X_2$  be irreducible representations of a group  $H$ . If  $T : X_1 \rightarrow X_2$  is  $H$  equivariant, then  $T = 0$  or  $T$  is an isomorphism.

Moreover, if  $X_1$  and  $X_2$  are complex vector spaces, then  $T$  is unique up to multiplication by a scalar  $\lambda \in \mathbb{C}$ .

The previous theorem is one of the reasons why it is worth carrying around the group action when there is one. Its simplicity hides the fact that it is a very powerful tool.

Following Fulton and Harris [4], the only three irreducible representations of  $S_3$  are the trivial  $U_3$ , the standard  $V_3$  and alternating representation<sup>8</sup>  $U'$ . Thus, for an arbitrary representation  $X$  of  $S_3$  we can write

$$X = U_3^{\oplus a} \oplus U'^{\oplus b} \oplus V_3^{\oplus c} \tag{5}$$

and there is a way to determine the multiplicities  $a, b$  and  $c$ , in terms of  $\tau = (123)$  and  $\sigma = (12)$ , which generate  $S_3$ ,  $c$ , for example, is the number of independent eigenvectors

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<sup>8</sup>Here, this action is given by  $\theta x = \text{sgn}(\theta)x$ , for  $\theta \in S_3$  and  $x \in \mathbb{R}$ .

of  $\tau$  with eigenvalue  $\omega^9$  whereas  $a + c$  is the multiplicity of 1 as an eigenvalue of  $\sigma$ , and  $b + c$  is the multiplicity of  $-1$  as an eigenvalue of  $\sigma$ .

**Proposition 2.** For  $\lambda \in \Pi(3) \setminus \{[3]\}$

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k$$

where  $U_\lambda^k = \langle u_\lambda^k \rangle; U_3$  and  $V_\lambda^k = \{z^{\lambda,k} \mid z \in V_3\}; V_3$ . The decomposition is orthogonal.

**Proof.** We start by showing that  $G_\lambda^k$  has exactly 1 copy of  $U_3$  and 1 copy of  $V_3$  if  $\lambda \in \Pi(3) \setminus \{[3]\}$ .

It is clear that  $\mathbf{B} = \{u_{(S,Q)} \mid (S, Q) \in ECL\}$  form a basis for  $G^{(3)}$ , where

$$u_{(S,Q)}(T, P) = \begin{cases} 1 & \text{if } (T, P) = (S, Q) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

For  $G^{(3)}$ , it is easy to verify that  $[\tau]_{\mathbf{B}}$  has the characteristic polynomial  $p(x) = (x-1)^4 [(x-\omega)(x-\omega^2)]^3$  and  $[\sigma]_{\mathbf{B}}$  has the characteristic polynomial  $p(x) = (x-1)^7 (x+1)^3$ . From these and (5), we have  $c = 3$ ,  $a + c = 7$  and  $b + c = 3$ . Thus

$$G^{(3)} = U_3^{\oplus 4} \oplus V_3^{\oplus 3}$$

This implies directly that if  $\lambda \in \Pi(3) \setminus \{[3]\}$ , then every  $G_\lambda^k$  has exactly 1 copy of  $U_3$  and 1 copy of  $V_3$ , since  $G_{[3]}^3 = \mathbf{R}u_{[n]}^n; U_3$  and  $\dim G_\lambda^k = 3$ .

Now, define the map  $T_\lambda^k : \mathbf{R}^n \rightarrow G_\lambda^k$  by  $T_\lambda^k(z) = z^{\lambda,k}$ . This map is an isomorphism between  $U_\lambda^k$  and  $U_3$  (similarly, between  $V_\lambda^k$  and  $V_3$ ), since it is linear,  $S_3$ -equivariant and 1-1. From Proposition 9, we obtain the decomposition  $\mathbf{R}^3 = U_3 \oplus V_3$ . Thus, inside  $G_\lambda^k$ , we have the images of these two subspaces:  $U_\lambda^k = T_\lambda^k(U_3)$  and  $V_\lambda^k = T_\lambda^k(V_3)$ .

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<sup>9</sup>Denoting by 1,  $\omega$ ,  $\omega^2$  the cube roots of unity.

Finally, the invariant inner product, defines an equivariant isomorphism, which in particular must preserve the decomposition. This implies the orthogonality of the decomposition.

There is a remarkably effective technique for decomposing any given finite dimensional representation into its irreducible components. The secret is *character theory*. In the analysis of the representations of  $S_3$ , the key was to study the eigenvalues of the actions of individual elements of  $S_3$ . This is the starting point of character theory. Finding individual eigenvalues, however, is difficult. Luckily, it is sufficient to consider their sum, the trace, which is much easier to compute.

**Definition 9.** Let  $\rho : H \rightarrow GL(X)$  be a representation. The character of  $X$  is the complex-valued function  $\chi_X : H \rightarrow \mathbb{C}$ , defined as:

$$\chi_X(h) = \text{Tr}(\rho(h))$$

The character of a representation is easy to compute. If  $H$  acts on an  $n$ -dimensional space  $X$ , we write each element  $h$  as an  $n \times n$  matrix according to its action expressed in some convenient basis, then sum up the diagonal elements of the matrix describing  $h$  to get  $\chi_X(h)$ . For example, the trace of the identity map of an  $n$ -dimensional vector space is the trace of the  $n \times n$  identity matrix, i.e.  $n$ . In fact,  $\chi_X(e) = \dim X$  for any finite dimensional representation  $X$  of any group.

Notice that, in particular, we have  $\chi_X(h) = \chi_X(ghg^{-1})$  for  $g, h \in H$ . So that  $\chi_X$  is constant on the conjugacy classes of  $H$ . Such a function is called a class function.

**Definition 10.** Let  $\mathbf{C}_{\text{class}}(H) = \{f : H \rightarrow \mathbb{C} \mid f \text{ is a class function on } H\}$ . If  $\chi_1, \chi_2 \in \mathbf{C}_{\text{class}}(H)$ , we define an Hermitian inner product on  $\mathbf{C}_{\text{class}}(H)$  by

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|H|} \sum_{h \in H} \overline{\chi_1(h)} \chi_2(h) \tag{7}$$

As was said, the character of a representation of a group  $H$  is really a function on the set of conjugacy classes in  $H$ . This suggests expressing the basic information about the irreducible representations of a group  $H$  in the form of a character table. This is a table with the conjugacy classes  $[h]$  of  $H$  listed across the top, usually given by a representative  $h$ , with the number of elements in each conjugacy class written above it. The irreducible representations of  $H$  are listed on the left and the value of the

character on the conjugacy class  $[h]$  is given in the appropriate cell. For example, if  $H = S_4$  and we only focus on the irreducible representations  $U_4$  and  $V_4$ , then<sup>10</sup>:

	1	6	8	6	3
$S_4$	$[e]$	$[(12)]$	$[(123)]$	$[(1234)]$	$[(12)(34)]$
$U_4$	1	1	1	1	1
$V_4$	3	1	0	-1	-1

Finally, the multiplicities of the irreducible subspaces in a representation can be calculated via the following proposition:

**Proposition 9.** If  $Z = Z_1^{\oplus a_1} \oplus Z_2^{\oplus a_2} \oplus \dots \oplus Z_j^{\oplus a_j}$ , then the multiplicity of  $Z_i$  (irreducible representation) in  $Z$ , is:

$$a_i = \langle \chi_Z, \chi_{Z_i} \rangle$$

where,  $\langle \cdot, \cdot \rangle$  is the inner product given by (7).

**Proposition 6.** For  $\lambda \in \Pi(4) \setminus \{[4]\}$ ,

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k \oplus W_\lambda^k$$

where  $U_\lambda^k = \langle u_\lambda^k \rangle$ ;  $U_4$ ,  $V_\lambda^k = \bigoplus_{j \in I_{\lambda,k}} \{x^{\lambda,k,j} \mid x \in V_4\}$  and neither any  $\{x^{\lambda,k,j} \mid x \in V_4\}$ ;  $V_4$ ; nor  $W_\lambda^k$  contains any summands isomorphic to either  $U_4$  or  $V_4$ . The decomposition is orthogonal.

**Proof.** First,  $\langle \chi_{G_\lambda^k}, \chi_{U_4} \rangle$  and  $\langle \chi_{G_\lambda^k}, \chi_{V_4} \rangle$  are the number of subspaces isomorphic to the trivial ( $U_4$ ) and standard representation ( $V_4$ ) within  $G_\lambda^k$ , respectively. The characters for each  $G_\lambda^k$  are given by<sup>11</sup>:

<sup>10</sup>In fact, there are five irreducible representations of  $S_4$ .

<sup>11</sup>In which a convenient basis is the one given in (6).

$S_4$	1	6	8	6	3
	[(1)]	[(12)]	[(123)]	[(1234)]	[(12)(34)]
$G_{[1111]}^1$	4	2	1	0	0
$G_{[211]}^1$	12	2	0	0	0
$G_{[211]}^2$	6	2	0	0	2
$G_{[31]}^1$	4	2	1	0	0
$G_{[31]}^3$	4	2	1	0	0
$G_{[22]}^2$	6	2	0	0	2

Thus from (7),  $\langle \mathcal{X}_{G_\lambda^k}, \mathcal{X}_{U_4} \rangle = 1$  for each  $G_\lambda^k$ ,

$$\langle \mathcal{X}_{G_{[1111]}^1}, \mathcal{X}_{V_4} \rangle = \langle \mathcal{X}_{G_{[211]}^2}, \mathcal{X}_{V_4} \rangle = \langle \mathcal{X}_{G_{[31]}^1}, \mathcal{X}_{V_4} \rangle = \langle \mathcal{X}_{G_{[31]}^3}, \mathcal{X}_{V_4} \rangle = \langle \mathcal{X}_{G_{[22]}^2}, \mathcal{X}_{V_4} \rangle = 1$$

and

$$\langle \mathcal{X}_{G_{[211]}^1}, \mathcal{X}_{V_4} \rangle = 2$$

The last part is to identify such copies of  $U_4$  and  $V_4$  inside  $G_\lambda^k$ . For this end, define the functions  $L_{\lambda,k,j} : \mathbb{R}^n \rightarrow G_\lambda^k$  by  $L_{\lambda,k,j}(x) = x^{\lambda,k,j}$ . These maps are isomorphisms between  $U_\lambda^k$  and  $U_4$  (similarly, between  $\{x^{\lambda,k,j} \mid x \in V_4\}$  and  $V_4$ ), since they are linear,  $S_4$ -equivariant and 1-1. Thus, inside of  $G_\lambda^k$ , we have the following images of these two subspaces:  $U_\lambda^k = L_{\lambda,k,j}(U_4)$  and  $V_\lambda^k = \bigoplus_{j \in I_{\lambda,k}} L_{\lambda,k,j}(V_4)$ .

The orthogonality of the decomposition follows again from the fact that the invariant inner product  $\langle, \rangle$  gives an equivariant isomorphism, which preserves the decomposition.

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