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A SEMI-MARKOV MODEL OF THE VARIABILITY OF POWER GENERATION FROM RENEWABLE SOURCES

The paper presents a new approach to modeling the variability of power generation from a renewable source such as wind or flowing water. The force of the power generating agent is assumed to change according to the semi-Markov process with finite state space. For the purpose of its construction, the range of possible values expressing the agent's force is divided into a finite number of subintervals. It is natural to assume that the time during which the agent's force remains within one such interval, and the probabilities of transitions to neighboring intervals depend to some extent on the agent's earlier behavior. The model's accuracy is determined by the number of subintervals used and the assumed degree to which the agent's force depends on its history. This degree is expressed by the number of the most recently entered subintervals relevant to predicting the agent's future behavior. According to the presupposed accuracy level, an appropriately complex state-space and a diagram of the inter-state transitions for the modeled process have been constructed. Subsequently, it is demonstrated how certain parameters of this process, related to forecasting power generation, can be calculated by means of the calculus of the Laplace transforms.

Keywords: *renewable power generation, forecasting of environmental conditions, semi-Markov modeling, Laplace transform*

1. Introduction

There exist a considerable number of models for predicting natural phenomena in the context of electrical power generation. Due to their nature, such phenomena can only be predicted with some degree of uncertainty, hence their modeling is based on various mathematical ways of expressing non-deterministic, uncertain information. Admittedly, deterministic models are also widely used, e.g. in weather forecasting, but their implementation based mainly on systems of partial differential equations requires

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a substantial amount of computing power and memory space. In addition, they are only suitable for short term prediction – covering a period of up to one week. Therefore, non-deterministic prediction offers a plausible alternative to deterministic forecasting, since it is not much less accurate – as long as certain average values are calculated over medium or long time periods, and significantly less resource consuming. Traditionally, prediction of the former type is carried out with the use of probabilistic or statistical tools (cf. [2, 3, 5, 8, 9]). However, methods employing fuzzy logic, the theory of possibility, neural networks or wavelet theory have become increasingly popular in recent years (cf. [4, 6]). An overview of the forecasting methodology is presented in Fig. 1.

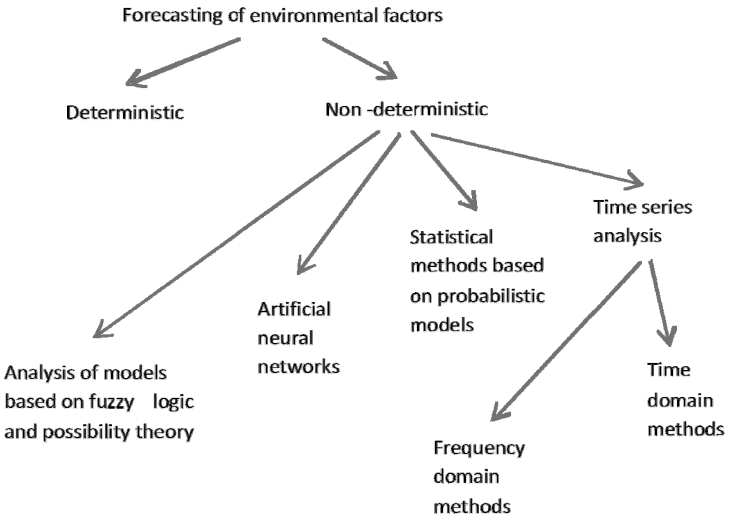

Fig. 1. An overview of forecasting methodology

This paper is concerned with the issue of medium or long term forecasting of the behavior of renewable energy sources such as wind or flowing water, based on probabilistic modeling. Short-term forecasting is not addressed herein, as a plethora of well-established methods have been developed for this purpose (e.g. deterministic modeling, time series analysis). A novel stochastic model is presented to describe the variability of a power generating agent over time. The fluctuations of its force are described by a stochastic process with a finite state space. In order to construct this space, the range of the force of the agent is divided into a finite number of subintervals (e.g. corresponding to energy production levels) denoted $I_1, \ldots, I_m$. It is assumed that the time during which the agent's force remains within one such interval, and the probabilities of transitions to neighboring intervals depend on the agent's earlier behavior, i.e. on the sequence of intervals in which its force had been before it entered the current interval. The model's accuracy is determined by the total number of these intervals (m)

and the assumed degree of dependence of the force on its history (d) defined as the number of the most recently entered intervals relevant to predicting the agent's future behavior. A state of the process is determined by the current interval and the d most recently entered ones, thus it incorporates information relevant to the future evolution of the agent's force. Further details are given in the next chapter.

2. The model's basics

As it is assumed that the time during which the agent's value remains in an interval I_x, and the number of the next interval (it can be either $x - 1$ or $x + 1$) depend on the order of several of the most recently entered intervals, the agent's changes can be modeled by a semi-Markov process with the appropriately constructed state space $Z = \{z_1, \ldots, z_n\}$. Each interval I_x corresponds to a number of states in Z, where each state contains information on the intervals entered by the agent's force before it reached I_x. The number of previously entered intervals taken into account in constructing the state space Z (denoted as d) determines, along with m, the model's degree of accuracy. In consequence, if we neglect the degenerate case $m = 2$, the cardinality of $\{z_1, ..., z_n\}$ exceeds that of $\{I_1, ..., I_m\}$, so that $n > m$.

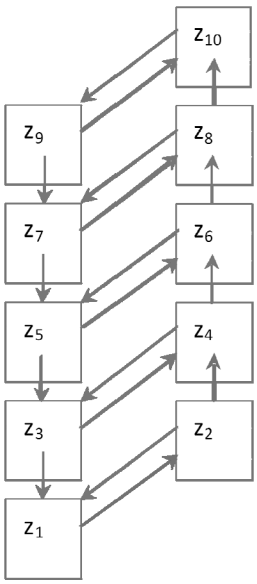

Fig. 2. The state space architecture when the degree of dependence is 1 and $m = 6$

If the degree of dependence is assumed to be 1, the future force of the agent depends on whether its value has recently increased or decreased, i.e. whether the previous interval was I_{x-1} or I_{x+1}, where I_x is the current interval. The state space of the

modeling process is composed of $n = 2m - 2$ states denoted $z_1, \ldots, z_{2m-2}$. An even-numbered state is entered if the force has most recently increased, and an odd-numbered one – if it has most recently decreased. Thus if I_{x-1} and I_x are the previous and current intervals, then the process is in the state z_{2x-2}; if the respective intervals are I_{x+1} and I_x, then the process is in the state z_{2x-1}. Figure 2 represents the considered state space along with the possible inter-state transitions for $m = 6$.

If the degree of dependence is assumed to be two, then the future force of the agent depends not only on the most recent, but also on the preceding change in value, i.e. the last two intervals are significant. The state space of the modeling process is now composed of $4m - 6$ states, which are divided into four groups: $G_1 = \{z_1, z_3, z_6, z_{10}, \ldots, z_{4m-14}\}$, $G_2 = \{z_2, z_4, z_7, z_{11}, \ldots, z_{4m-10}\}$, $G_3 = \{z_5, z_8, z_{12}, \ldots, z_{4m-9}, z_{4m-7}\}$, $G_4 = \{z_9, z_{13}, \ldots, z_{4m-8}, z_{4m-6}\}$. The states in the first group are entered if the value of the force has most recently decreased twice; in the second – if it has most recently first increased and then decreased; in the third – if it has most recently first decreased and then increased; in the fourth – if it has most recently increased twice. Figure 3 represents the considered state space along with the possible interstate transitions for $m = 6$.

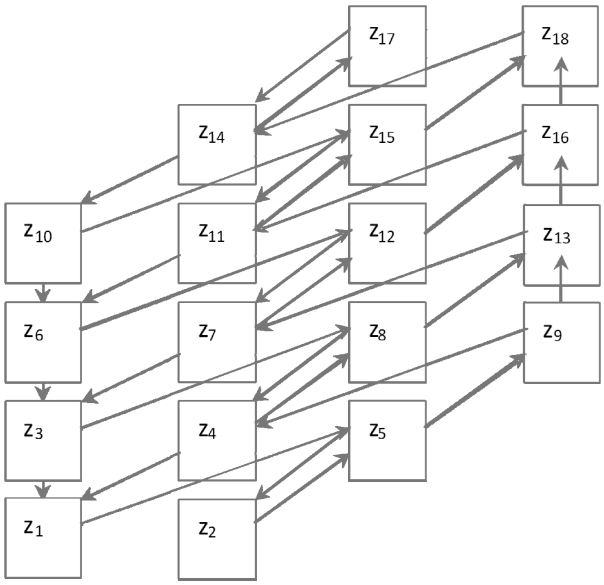

Fig. 3. The state space architecture when the degree of dependence is 2 and $m = 6$

Let $Z = \{Z_t, t \geq 0\}$ be a stochastic process with the state space $\{z_1, \ldots, z_n\}$, modeling the agent's variability. As Z is assumed to be semi-Markov, the following objects have to be defined:

$X = \{X_k, k \geq 0\}$ – the embedded Markov chain of Z, i.e. X_k is the state of Z immediately after its k-th change in state (X_0 is the initially observed state of Z).

$P = [p_{ij}]$, $i, j = 1, ..., n$ – the transition matrix of X, i.e. $p_{ij} = \Pr(X_k = z_j | X_{k-1} = z_i)$. It is assumed that P does not change with k, i.e. X is homogenous.

T_k – the moment of the k-th change in the state of Z.

S_{ij} – the time spent by Z in the state z_i given that the next state is z_j.

F_{ij} – the distribution function of S_{ij}, i.e.

$$F_{ij}(t) = \Pr\left(T_k - T_{k-1} \le t \middle| X_k = z_j, X_{k-1} = z_i\right) \tag{1}$$

P and F_{ij} can be obtained from the statistical analysis of the recorded values of the agent's force. The matrix P corresponding to the state space shown in Fig. 2 is presented below.

$$p = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{21} & 0 & 0 & p_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{31} & 0 & 0 & p_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{43} & 0 & 0 & p_{46} & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{53} & 0 & 0 & p_{56} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{65} & 0 & 0 & p_{68} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{75} & 0 & 0 & p_{78} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P_{87} & 0 & 0 & p_{8,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & p_{97} & 0 & 0 & p_{9,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Clearly, $p_{ii} = 0$ and $p_{i1} + \ldots + p_{in} = 1$ for $i = 1, ..., n$ – these are obvious conditions that must be fulfilled by any transition matrix.

Remark: In general, the probability of a transition from z_i to z_j depends on the amount of time spent in z_i, thus

$$\Pr\left(X_k = z_j \middle| X_{k-1} = z_i, T_k - T_{k-1} \le t\right) \ne \Pr\left(X_k = z_j \middle| X_{k-1} = z_i, T_k - T_{k-1} \le u\right) \tag{2}$$

may hold for $u \ne t$. The precise definition of p_{ij} should therefore be as follows:

$$p_{ij} = \Pr\left(X_k = z_j\right) \middle| X_{k-1} = z_i, T_k - T_{k-1} < \infty \tag{3}$$

This underscores the fact that p_{ij} is the probability of transition from z_i to z_j if no information about the sojourn time in z_i is available. However, the event $\{T_k - T_{k-1} < \infty\}$ is

irrelevant to p_{ij} as expressed by (3), because $\Pr(T_k - T_{k-1} < \infty) = 1$. It should be noted that this remark pertains to all semi-Markov processes.

3. Equations for basic parameters of the energy production process

The model constructed in the previous section is useful for determining many parameters that characterize the process of electrical power production, especially for predicting future power output. For example, it is possible to forecast the total expected energy output during a given time period, the probability that during such a period the output value will remain within certain limits, or the expected number of times it will cross these limits. It will now be shown in detail how to determine the first parameter. Let π_i be the power generated when Z is in the state z_i, i.e. when the factor's force is in the corresponding interval. Let $G_i(u, t)$ be the expected amount of energy produced in the time interval $[u, t]$, given that at the moment u the process Z enters the state z_i. It can be easily shown that the $G_i(0, t)$, $i = 1, ..., n$ satisfy the following set of equations:

$$G_i(0,t) = \pi_i t \sum_{j \neq i} p_{ij} \left[1 - F_{ij}(t)\right] + \sum_{j \neq i} p_{ij} \int_0^t \left[\pi_i u + G_j(u,t)\right] dF_{ij}(u) \tag{4}$$

Indeed, taking into account that

$$\begin{aligned} p_{ij} F_{ij}(t) &= \frac{\Pr\left(X_k = z_j, X_{k-1} = z_i\right)}{\Pr\left(X_{k-1} = z_i\right)} \frac{\Pr\left(T_k - T_{k-1} \le t, X_k = z_j, X_{k-1} = z_i\right)}{\Pr\left(X_k = z_j, X_{k-1} = z_i\right)} \\ &= \Pr\left(T_k - T_{k-1} \le t, X_k = z_j \middle| X_{k-1} = z_i\right) \end{aligned} \tag{5}$$

the first component on the right-hand side of (4) is related to the amount of energy produced in the time interval $[0, t]$, given that no change in state occurred from 0 to t. In turn, the second component is related to the amount of energy produced in that interval, given that the first change in state occurs at $u \le t$.

Being a semi-Markov process, Z "forgets" its history at each change in state, so that $G_i(s, t) = G_i(0, t - s)$, $i = 1, ..., n$. In consequence, (4) can be transformed to:

$$G_i(0,t) = \pi_i \sum_{j \neq i} p_{ij} \left[t\left(1 - F_{ij}(t)\right) + \int_0^t u dF_{ij}(u) \right] + \sum_{j \neq i} p_{ij} \int_0^t G_j(0, t-u) dF_{ij}(u) \tag{6}$$

Let us note that

$$t\left[1-F_{ij}\left(t\right)\right]+\int_0^t udF_{ij}\left(u\right)=E\left[\min\left(S_{ij},t\right)\right] \tag{7}$$

where E is the expected value. Hence, (6) can be written in the following compact form:

$$G_i\left(0,t\right)=\pi_i\sum_{j\neq i}p_{ij}H_{ij}\left(t\right)+\sum_{j\neq i}p_{ij}\int_0^t G_j\left(0,t-u\right)dF_{ij}\left(u\right) \tag{8}$$

where

$$H_{ij}\left(t\right)=E\left[\min\left(S_{ij},t\right)\right] \tag{9}$$

As (8) is a system of integral equations, the calculus of Laplace transforms can be used to find its solution. Let

$$\Gamma_i\left(s\right)=\mathcal{L}\left\{G_i\left(0,t\right)\right\} \tag{10}$$

$$\Phi_{ij}\left(s\right)=\mathcal{L}\left\{f_{ij}\left(t\right)\right\}=\mathcal{L}^*\left\{F_{ij}\left(t\right)\right\} \tag{11}$$

$$\Psi_{ij}\left(s\right)=\mathcal{L}\left\{H_{ij}\left(t\right)\right\} \tag{12}$$

where $\mathcal{L}$ and $\mathcal{L}^*$ denote the Laplace and Laplace–Stieltjes transforms, respectively, and f_{ij} is the probability density function of S_{ij}. From the basic properties of the Laplace transform, it follows that

$$\Psi_{ij}\left(s\right)=\frac{\left[1-\Phi_{ij}\left(s\right)\right]}{s^2} \tag{13}$$

Applying $\mathcal{L}$ to both sides of (8), we obtain

$$\Gamma_i\left(s\right)=\pi_i\sum_{j\neq i}p_{ij}\Psi_{ij}\left(s\right)+\sum_{j\neq i}p_{ij}\Phi_{ij}\left(s\right)\Gamma_j\left(s\right) \tag{14}$$

Note that the integral in (8) is the convolution of G_j and f_{ij}, which, after applying the Laplace transform, becomes the product of Γ_j and Φ_{ij}. As (14) is a system of linear algebraic equations, it can be written in the following matrix form:

$$A\left(s\right)\begin{bmatrix}\Gamma_1\left(s\right)\\ \vdots\\ \Gamma_n\left(s\right)\end{bmatrix}=\begin{bmatrix}\pi_1\sum_{j\neq i}p_{1j}\Psi_{1j}\left(s\right)\\ \vdots\\ \pi_n\sum_{j\neq i}p_{nj}\Psi_{nj}\left(s\right)\end{bmatrix} \tag{15}$$

where the elements of the matrix $A(s)$ are given by:

$$a_{ij}(s) = \delta_{ij} - p_{ij}\Phi_{ij}(s) \tag{16}$$

4. Solving the equations obtained

A closed-form solution of (15) should express Γ_i, $i = 1, ..., n$ in terms of π_i, p_{ij}, Φ_{ij}, and Ψ_{ij}, where $i, j = 1, ..., n$. However, it is practically impossible to find such a solution in general. For example, in the case of the system whose state space is shown in Fig. 2, we have the following matrix $A(s)$:

$$A(s) = \begin{bmatrix}
1 & -\Phi_{12}(s) & 0 & 0 & 0 \\
-p_{21}\Phi_{21}(s) & 1 & 0 & -p_{24}\Phi_{24}(s) & 0 \\
-p_{31}\Phi_{31}(s) & 0 & 1 & -p_{34}\Phi_{34}(s) & 0 \\
0 & 0 & -p_{43}\Phi_{43}(s) & 1 & 0 \\
0 & 0 & -p_{53}\Phi_{53}(s) & 0 & 1 \\
0 & 0 & 0 & 0 & -p_{65}\Phi_{65}(s) \\
0 & 0 & 0 & 0 & -p_{75}\Phi_{75}(s) \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\cdots \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-p_{46}\Phi_{46}(s) & 0 & 0 & 0 & 0 \\
-p_{56}\Phi_{56}(s) & 0 & 0 & 0 & 0 \\
1 & 0 & -p_{68}\Phi_{68}(s) & 0 & 0 \\
0 & 1 & -p_{78}\Phi_{78}(s) & 0 & 0 \\
0 & -p_{87}\Phi_{87}(s) & 1 & 0 & -p_{8,10}\Phi_{8,10}(s) \\
0 & -p_{97}\Phi_{97}(s) & 0 & 1 & -p_{9,10}\Phi_{9,10}(s) \\
0 & 0 & 0 & -\Phi_{10,9}(s) & 1
\end{bmatrix}$$

$A(s)$ is a sparse matrix (it includes many zeros) and can be transformed into tri-diagonal form by interchanging adjacent columns (which involves the corresponding

rearrangement of the elements of $[\Gamma_1(s), ..., \Gamma_n(s)]^T$). Thus the system (15) can be solved by Gaussian elimination (GE) with lower than maximal complexity, which is $O(n^3)$. However, even the Thomas algorithm (a simplified form of GE designed especially for tri-diagonal matrices), applied to (15) with $A(s)$ as given above, results in very complicated closed formulas, which are practically impossible to derive for large n. Needless to say, these formulas become even more complicated for $d > 1$.

From the above considerations, (15) has to be solved numerically rather than analytically. For example, it is possible to find $\Gamma_i(s)$ for a number of discrete values of s along a vertical line in the space of complex numbers, thus obtaining data which allow us to compute $G_i(0, t)$ by numerical integration using the following well-known formula for the reverse Laplace transform:

$$G_i(0,t) = \frac{1}{2\pi_i} \lim_{y\to\infty} \int_{x-iy}^{x+iy} e^{st} \Gamma_i(s)\, ds \tag{17}$$

Here, x is the real part of the complex number $x + iy$.

5. Finding other parameters

We will now briefly outline how other parameters, e.g. the probability that during a given time period the generated power will remain within certain limits, or the expected number of times it will cross these limits, can be found in a way similar to that described in Sections 3 and 4.

Let $N_i(u, t)$ be the expected number of times that from u to t the generated power output crosses the lower bound of I_a from above, or the upper bound of I_b from below, where $a \le b$, given that at the moment u the process Z enters the state z_i. Let $Z_1 = \{z \in Z: a \le x(z) \le b\}$ and $Z_2 = Z \setminus Z_1$. As in the case of $G_i(0, t)$, $i = 1, ..., n$ (see Section 3), it can be shown that $N_i(0, t)$, $i = 1, ..., n$ fulfill the following set of equations:

$$N_i(0,t) = \sum_{j\neq i} p_{ij} \int_0^t \left[\gamma(i,j) + N_j(u,t)\right] dF_{ij}(u), \qquad i = 1, \ldots, n \tag{18}$$

where $\gamma(i, j) = 1$ for $i \in Z_1$ and $j \in Z_2$, otherwise $\gamma(i, j) = 0$.

Let $P_i(u, t)$ be the probability that from time u to time t the generated power output remains above the lower bound of I_a and below the upper bound of I_b, $i \in Z_1$. The following set of equations is obtained for $P_i(0, t)$, $i = 1, ..., n$:

$$P_i(0,t) = \sum_{j\neq i} p_{ij} \left[\delta(i,j) \int_0^t P_j(u,t)\, dF_{ij}(u) + \left(1 - F_{ij}(t)\right) \right], \qquad i \in Z_1 \tag{19}$$

where $\delta(i, j) = 1$ for $j \in Z_1$, $\delta(i, j) = 0$ for $j \in Z_2$. Note that in (19) the number of unknowns is equal to the number of equations, which is equal to card(Z_1). Thus a necessary condition for the existence of a unique solution is fulfilled.

Both (18) and (19) can be transformed to their "convolution" versions, analogous to (8), and then solved using transform calculus, as shown in Section 4.

6. Conclusion

A new model of the stochastic variability of a renewable energy source has been presented. This model can be used to determine the number of parameters characterizing the energy output over a medium or long time horizon. As the model is purely stochastic, a question may arise regarding its applicability, especially when the power generating agent is strongly dependent on some deterministic factor. For example, ocean tides, made use of in tidal power stations, depend on the positions of the Sun and the Moon in relation to the Earth. In such cases, the proposed model might describe the stochastic component of the mainly deterministic variability of a given agent.

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