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ON SOME PROPERTIES OF COMPOSITE DISTRIBUTED LAG MODELS

The analysis presented in this paper is focused on basic properties of discrete distributed lag models. Such models are commonly used to model dynamic systems in various applications. In the presented considerations, time-varying distributed lags have been analyzed. Composite distributed lag models analyzed in this paper result from the summation or superposition of component distributed lag models. The analysis is restricted to models with a lag distribution, whose mean lag and variance exist. The paper presents relationships between the mean values and variances of the lag distributions of composite distributed lag models and of the component distributed lag models, as well as the relationships between the variance of the random term of composite distributed lag models and the variance of the random term of the component distributed lag models.

Keywords: *economic modeling*, *distributed lags*, *lag distributions*

1. Introduction

Distributed lag models describe a particular kind of relationship between dependent and independent variables. Generally, a change in the value of the independent variable causes a change in the value of the dependent variable in the same and later periods. That change in the dependent variable is often distributed over time. Distributed lag models constitute parts of many models used to describe dynamic systems. Their history in economics started with the works of Fisher [3]; significant contributions were made by, for example, Koyck [12], Almon [1], Jorgensen [9], Griliches [8], and Dhrymes [5]. In most cases, distributed lag models assume constant parameters. However, there are also studies tackling models with coefficients changing in time [13–15, 4, 6, 7].

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This paper focuses on the properties of composite models which result from combining distributed lag models in the form of a sum or superposition of component distributed lag models*. Under certain assumptions, such models also belong to the class of distributed lag models.

A distributed lag model facilitates distinction between two important factors: the lag mechanism which distributes the impact of a change in the independent variable on the dependent variable over time, and other circumstances usually represented by the random term. In the case of distributed lag models with time-constant lag structures, it is conventionally accepted that the lag mechanism is deterministic. In the case of time-varying distributed lag structures, the changes in the lag structure can be attributed to various causes: seasonal, structural, technological, as well as random. However, in this paper the sources of variability in the lag structure are assumed to be exogenous. This class of distributed lag models is important in the description of systems whose components can be described by distributed lag models and where it is necessary to account for significant changes in the lag mechanism. Supply systems are a relevant example; they consist of elements arranged either in series or in parallel.

The basic concepts of distributed lag models are presented in Chapter 2. In Section 3.1, the properties of the sums of distributed lag models are presented, while Section 3.2 contains considerations concerning the superposition of distributed lag models.

The use of lag operators and generating functions simplifies notation and facilitates transformations. The presentation of these notions in this paper is limited. Those who are interested in more advanced topics should consult other authors, as for example, Koźniewska [10], Kenkel [11] and Dhrymes [5].

Some variables used in this paper are described using one or two subscripts. Whenever the risk of ambiguity appears, indexes are separated by comas.

2. Basic concepts

A discrete time distributed lag model is written in the form of the following expression [5, 8, 13]:

$$y_t = \sum_{i=0}^{\infty} v_{ti}\, x_{t-i} + \varepsilon_t \tag{1}$$

*In electrical circuits such configurations of components are called parallel and series connections.

where: x_t – independent variable in period t, y_t – dependent variable in period t, v_{tt} – lag coefficients of the lag structure satisfying conditions: $v_{ti} \geq 0$, $i = 0, 1, 2, ...$, ε_t – time independent random variable with expected value

$$E(\varepsilon_t) = 0$$

with finite variance

$$D^2(\varepsilon_t) = \sigma^2$$

and for $k = ..., -2, -1, 1, 2, ...,$

$$\mathrm{cov}(\varepsilon_t, \varepsilon_{t-k}) = 0$$

The sequence V_t, given for each time period which consists of the lag coefficients v_{ti}, $i = 0, 1, 2, ...$, is called the lag structure. Throughout this paper, it is assumed that the so-called long-term multiplier a_t defined as

$$a_t = \sum_{i=0}^{\infty} v_{ti}$$

is finite. For non-trivial lag structures, it is also positive (because it is assumed that all $v_{ti} \geq 0$, $i = 0, 1, 2, ...,$).

If model (1) fulfills the above assumption, it can be written in the following form:

$$y_t = a_t \sum_{i=0}^{\infty} w_{ti} x_{t-i} + \varepsilon_t \tag{2}$$

where the coefficients w_{ti}, $i = 0, 1, 2, ...$, are obtained by standardizing the coefficients of the lag structure v_{ti}, $i = 0, 1, 2, ...$,

$$w_{ti} = \frac{v_{t,i}}{a_t} = \frac{v_{t,i}}{\sum_{i=0}^{\infty} v_{t,i}}$$

The sequence W_t of coefficients w_{ti}, $i = 0, 1, 2, ...$, is called the lag distribution. Further considerations are based on the assumption that there exists a mean value $M(W_t)$ and a variance $D^2(W_t)$ of the lag distribution W_t in all time periods (note that there exist lag distributions with undetermined mean values):

$$M(W_t) = \sum_{i=0}^{\infty} i w_{ti} \tag{3}$$

$$D^2(W_t) = \sum_{i=0}^{\infty} \left(i - M(W)\right)^2 w_{ti} = \sum_{i=0}^{\infty} i^2 w_{ti} - M^2(W_t) \tag{4}$$

Lag operators and generating functions simplify the notation and facilitate transformations.

The lag operator L is a transformation* with the following properties:

$$Lx_t = x_{t-1} \tag{5}$$

$$L^2 x_t = LLx_t = Lx_{t-1} = x_{t-2} \tag{6}$$

$$L^k x_t = x_{t-k}, \quad k = ..., -2, -1, 0, 1, 2, ... \tag{7}$$

$$L^k L^l = L^{k+1}, \qquad k, l = ..., -2, -1, 0, 1, 2, ... \tag{8}$$

$$L^0 x_t = 1x_t = x_t \tag{9}$$

$$L^k L^{-k} = L^{k-k} = L^0 = 1, \qquad k = ..., -2, -1, 0, 1, 2, ... \tag{10}$$

$$(c_1 L^k + c_2 L^l)\, x_t = c_1 x_{t-k} + c_2 x_{t-1}, \qquad k, l = ..., -2, -1, 0, 1, 2, ... \tag{11}$$

where: x_t – independent variable, real number, 1 – identity operator, such, that $1x_t = x_t$, c_1 and c_2 – given scalar numbers.

Further considerations are based on the assumption that the lag operator L influences only the independent variable, and not the lag coefficients. In consequence, these considerations are less general, particularly in the case of the superposition of distributed lag models, though this assumption significantly facilitates analysis. However, it does not exclude cases where the values of the lag coefficients depend on their values from earlier periods.

The polynomial operator $C_t(L)$ based on the set of coefficients $C_t = c_{t0}, c_{t1}, c_{t2}, ...$, can be expressed in the following form:

$$C_t(L) = \sum_{i=0}^{\infty} c_{ti} L^i$$

where c_{ti}, $i = 0, 1, 2, ...$, are non-negative coefficients given for every time-period t.

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*The symbol L is used here as in [4]. However, in some sources the symbol B (backward shift operator) appears, while in papers related to technological applications, the symbol z^{-1} is used.

Sums and products of the polynomial operators based on two sets of coefficients $(c_{t0}^{(1)}, c_{t1}^{(1)}, c_{t2}^{(1)}, ...)$ and $(c_{t0}^{(2)}, c_{t1}^{(2)}, c_{t2}^{(2)}, ...)$ have the following properties:

$$C_t^{(1)}(L) + C_t^{(2)}(L) = \sum_{i=0}^{\infty} c_{ti}^{(1)} L^i + \sum_{i=0}^{\infty} c_{ti}^{(1)} L^i = \sum_{i=0}^{\infty} \left(c_{ti}^{(1)} + c_{ti}^{(2)} \right) L^i$$
$$C_t^{(1)}(L) C_t^{(2)}(L) = \sum_{i=0}^{\infty} c_{ti}^{(1)} L^i \sum_{i=0}^{\infty} c_{ti}^{(2)} L^i = \sum_{i=0}^{\infty} c_{ti} L^i \tag{12}$$

where

$$c_{ti} = \sum_{j=0}^{i} c_{tj}^{(1)} c_{t,i-j}^{(2)}, \qquad i = 0, 1, 2, \ldots \tag{13}$$

is a convolution.

It is important to note that the assumption concerning time independence of the lag coefficients results in the product of polynomial lag operators being commutative:

$$C_t^{(1)}(L) C_t^{(2)}(L) = \sum_{i=0}^{\infty} c_{ti}^{(1)} L^i \sum_{i=0}^{\infty} c_{ti}^{(2)} L^i = \sum_{i=0}^{\infty} c_{ti}^{(2)} L^i \sum_{i=0}^{\infty} c_{ti}^{(1)} L^i = C_t^{(2)}(L) C_t^{(1)}(L)$$

Using polynomial lag operators, relationship (1) can be rewritten as:

$$y_t = \left(\sum_{i=0}^{\infty} v_{ti} L^i \right) x_t + \varepsilon_t = V_t(L) x_t + \varepsilon_t$$

or

$$y_t = a_t W_t(L) x_t + \varepsilon_t = a_t \left(\sum_{i=0}^{\infty} w_{ti} L^i \right) x_t + \varepsilon_t \tag{14}$$

where $V_t(L) = \sum_{i=0}^{\infty} v_{ti} L^i$ is a polynomial lag operator built on the lag structure V_t.

The generating function $V_t(\theta)$ built on the lag structure V_t is defined as follows:

$$V_t(\theta) = \sum_{i=0}^{\infty} v_{ti} \theta^i \tag{15}$$

where: v_{ti}, $i = 0, 1, 2, ...$, are the coefficients of the lag structure V_t, and the variable θ is a real number.

The generating function $W_t(\theta)$ of the lag distribution is represented by the expression

$$W_t(\theta) = \sum_{i=0}^{\infty} w_{ti} \theta^i \tag{16}$$

where: w_{ti}, $i = 0, 1, 2, ...$, are the coefficients of the lag distribution W_t.

In further considerations, the following properties of the generating functions are important:

- for $\theta = 1$ the value of the generating function $W_t(\theta)$ is equal to one:

$$W_t(1) = 1 \tag{17}$$

- the ith derivative of $W_t(\theta)$ with respect to θ for $\theta = 1$ is equal to the ith coefficient of the lag distribution multiplied by $i!$:

$$\frac{d^i W_t(1)}{d\theta^i} = i!\, w_{ti}, \qquad i = 0, 1, ... \tag{18}$$

- the mean value $M(W_t)$ of the lag distribution W_t is given by

$$M(W_t) = \frac{dW_t(1)}{d\theta} \tag{19}$$

- the variance $D^2(W_t)$ of the lag distribution W_t is given by

$$D^2(W_t) = \frac{d^2 W_t(1)}{d\theta^2} + \frac{dW_t(1)}{d\theta} - \left[\frac{dW_t(1)}{d\theta}\right]^2 \tag{20}$$

or

$$D^2(W_t) = \frac{d^2 W_t(1)}{d\theta^2} + M(W_t) - \left[M(W_t)\right]^2$$

3. Composite distributed lag models

In the analysis of composite distributed lag models, two types of relations between the component distributed lag models are considered:

• sums (configured in parallel),
• superposition (serially configured).

In the ensuing analysis, the deterministic properties of composite distributed lags will be considered first, followed by random ones.

3.1. Sum of distributed lag models

A sum of distributed lag models occurs whenever a certain variable y_t is a sum of n variables $y_t^{(j)}$, $j = 1, 2, ..., n$, all depending on the same independent variable x_t:

$$\begin{aligned} y_t &= y_t^{(1)} + y_t^{(2)} + \ldots + y_t^{(n)} \\ &= \sum_{i=0}^{\infty} v_{ti}^{(1)} x_{t-i} + \sum_{i=0}^{\infty} v_{ti}^{(2)} x_{t-i} + \ldots + \sum_{i=0}^{\infty} v_{ti}^{(n)} x_{t-i} + \sum_{j=0}^{n} \varepsilon_t^{(j)} \\ &= \sum_{i=0}^{\infty} \left(\sum_{j=1}^{n} v_{ti}^{(j)} \right) x_{t-i} + \varepsilon_t = \sum_{i=0}^{\infty} v_{ti} x_{t-i} + \varepsilon_t \end{aligned} \tag{21}$$

where all the random components $\varepsilon_t^{(j)}$, $j = 0, 1, 2, ..., n$, are independently distributed and have expected values equal to zero, $E(\varepsilon_t^{(j)}) = 0$, and finite variances $D^2(\varepsilon_t^{(j)}) = \left(\sigma^{(j)}\right)^2 < \infty$, and

$$\varepsilon_t = \sum_{j=1}^{n} \varepsilon_t^{(j)}, \qquad v_{ti} = \sum_{j=1}^{n} v_{ti}^{(j)} \tag{22}$$

Equations (21) and (22) imply that the model (1) can be interpreted as a sum of simple lag models of the type: $y_t^{(j)} = v_{tj}^{(j)} x_{t-j} + \varepsilon_t^{(j)}$, $j = 0, 1, 2, ..., n$.

If the component distributed lag models $y_t^{(j)}$, $j = 1, 2, ..., n$ (Eq. (21)) have lag distributions, then the long-term multiplier of the resultant model is equal to

$$a_t = \sum_{j=1}^{n} a_t^{(j)} = \sum_{j=1}^{n} \sum_{i=0}^{\infty} v_{ti}^{(j)} \tag{23}$$

while the coefficients of the resultant lag distribution W_t (summing the component distributed lag models with lag distributions $W_t^{(j)}$, $j = 1, 2, ..., n$) are given by the following relationship:

$$w_{ti} = \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} w_{ti}^{(j)} \tag{24}$$

where $w_{ti}^{(j)}$, i = 0, 1, 2, ..., are the coefficients of the component lag distributions $W_t^{(j)}$, j = 1, 2, ..., n. From Eq. (24), it follows that the lag distribution W_t can be written in the following form:

$$W_t = \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} W_t^{(j)}$$

Relationship (23) is derived from the definition of the long-term multiplier.

Note that the value of the long-term multiplier a_t is the sum of the long-term multipliers $a_t^{(j)}$, j = 1, 2, ..., n, of the component distributed lag models, while the value of the resultant ith lag coefficient w_{ti}, i = 0, 1, 2, ..., is the weighted average of all the ith coefficients of the component distributed lag models, where the values of the weight coefficients are determined by the shares of the jth long-term coefficients in the value of the resultant long term-coefficient $a_t^{(j)}/a_t$, j = 1, 2, ..., n.

If each component distributed lag model has lag distribution $W_t^{(j)}$ and finite mean value $M(W_t^{(j)})$, j = 1, 2, ..., n, then the mean value $M(W_t)$ of the resultant lag distribution W_t can be expressed by the following formula:

$$M(W_t) = \frac{a_t^{(1)}}{a_t} M(W_t^{(1)}) + \frac{a_t^{(2)}}{a_t} M(W_t^{(2)}) + \ldots + \frac{a_t^{(n)}}{a_t} M(W_t^{(n)}) \tag{25}$$

The above relationship is derived from the definition of the mean value of the lag distribution:

$$M(W_t) = \frac{\sum_{i=1}^{\infty} i \sum_{j=1}^{n} v_{ti}^{(j)}}{\sum_{i=0}^{\infty}\sum_{j=1}^{n} v_{ti}^{(j)}} = \frac{\sum_{j=1}^{n}\sum_{i=1}^{\infty} i a_t^{(j)} w_{ti}^{(j)}}{\sum_{j=1}^{n}\sum_{i=0}^{\infty} v_{ti}^{(j)}} = \frac{\sum_{j=1}^{n} a_t^{(j)} \sum_{i=1}^{\infty} i w_{ti}^{(j)}}{\sum_{j=1}^{n} a_t^{(j)}} = \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M(W_t^{(j)})$$

The mean value of the resultant lag distribution (Eq. (25)) is the weighted average of the mean values of the component lag distributions, where the corresponding weight coefficients are determined by the shares of the jth long-term coefficients in the value of the resultant long term-coefficient $a_t^{(j)}/a_t$, j = 1, 2, ..., n.

The relationship between the variance of the resultant lag distribution $D^2(W_t)$ and the variances $D^2(W_t^{(j)})$ of the component lag distributions is given by the following formula:

$$D^2(W_t) \geq \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} D^2(W_t^{(j)}) \qquad \text{`(26)}$$

The proof of the above relationship is based on the fact that inequality (26) can be expressed in the following way:

$$D^2(W_t) = \sum_{i=0}^{\infty} i^2 w_{ti} - \left(\sum_{i=0}^{\infty} i w_{ti} \right)^2 = \sum_{i=0}^{\infty} i^2 \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} w_{ti}^{(j)} - \left(\sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M(W_t^{(j)}) \right)^2$$

or

$$\begin{aligned} D^2(W_t) &= \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} \left[D^2(W_t^{(j)}) + M^2(W_t^{(j)}) \right] - \left(\sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M(W_t^{(j)}) \right)^2 \\ &= \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} D^2(W_t^{(j)}) + \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M^2(W_t^{(j)}) - \left(\sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M(W_t^{(j)}) \right)^2 \end{aligned}$$

Note also that

$$\sum_{i=0}^{\infty} i^2 \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} w_{ti}^{(j)} = \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} \sum_{i=0}^{\infty} i^2 w_{ti}^{(j)} = \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} \left[D^2(W_t^{(j)}) + M^2(W_t^{(j)}) \right]$$

On the basis of the above equation, it suffices to show that the following inequality is true:

$$\sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M^2(W_t^{(j)}) \geq \left[\sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M(W_t^{(j)}) \right]^2$$

The proof is based on the Cauchy–Schwarz inequality, where for any finite sequence of real numbers c_j and b_j, $j = 1, \ldots, n$, the following relation holds (see [2]):

$$\sum_{j=1}^{n} c_j^2 \sum_{j=1}^{n} b_j^2 \geq \left(\sum_{j=1}^{n} c_j b_j \right)^2$$

Defining the parameters c_j and b_j, $j = 1, \ldots, n$, as $c_j = \sqrt{\frac{a_t^{(j)}}{a_t} M(W_t^{(j)})}$, $b_j = \sqrt{a_t^{(j)}/a_t}$, and substituting them into the considered inequality, we obtain

$$\sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M^2(W_t^{(j)}) \sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} \geq \left[\sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M(W_t^{(j)}) \right]^2$$

From the assumption

$$\sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} = 1$$

it follows that

$$\sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M^2(W_t^{(j)}) \geq \left[\sum_{j=1}^{n} \frac{a_t^{(j)}}{a_t} M(W_t^{(j)}) \right]^2$$

thus completing the proof of inequality (26).

In the case of the sum of just two distributed lag models, relationship (26) takes the following form:

$$D^2(W_t) = D^2\left(\frac{a_t^{(1)}}{a_t} W_t^{(1)} + \frac{a_t^{(2)}}{a_t} W_t^{(2)} \right)$$

$$= \frac{a_t^{(1)}}{a_t} D^2(W_t^{(1)}) + \frac{a_t^{(2)}}{a_t} D^2(W_t^{(2)}) + \frac{a_t^{(1)} a_t^{(2)}}{a_t^{\,2}} \left[M(W_t^{(1)}) - M(W_t^{(2)}) \right]^2$$

The proof of the above relationship makes use of Eqs. (5)–(26).

$$D^2\left(\frac{a_t^{(1)}}{a_t} W_t^{(1)} + \frac{a_t^{(2)}}{a_t} W_t^{(2)} \right) = \frac{d^2}{d\theta^2}\left[\frac{a_t^{(1)}}{a_t} W_t^{(1)}(1) + \frac{a_t^{(2)}}{a_t} W_t^{(2)}(1) \right]$$

$$+ \frac{d}{d\theta}\left[\frac{a_t^{(1)}}{a_t} W_t^{(1)}(1) + \frac{a_t^{(2)}}{a_t} W_t^{(2)}(1) \right] - \left\{ \frac{d}{d\theta}\left[\frac{a_t^{(1)}}{a_t} W_t^{(1)}(1) + \frac{a_t^{(2)}}{a_t} W_t^{(2)}(1) \right] \right\}^2$$

By differentiating and using (19), one obtains:

$$D^2\left(W_t\right)=\frac{a_t^{(1)}}{a_t}\frac{d^2W_t^{(1)}(1)}{d\theta^2}+\frac{a_t^{(2)}}{a_t}\frac{d^2W_t^{(2)}(1)}{d\theta^2}+\frac{a_t^{(1)}}{a_t}M(W_t^{(1)})$$

$$+\frac{a_t^{(2)}}{a_t}M(W_t^{(2)})-\left[\frac{a_t^{(1)}}{a_t}M(W_t^{(1)})+\frac{a_t^{(2)}}{a_t}M(W_t^{(2)})\right]^2$$

By rearranging the terms, we get:

$$D^2\left(W_t\right)=\frac{a_t^{(1)}}{a_t}\frac{d^2W_t^{(1)}(1)}{d\theta^2}+\frac{a_t^{(1)}}{a_t}M(W_t^{(1)})-\frac{a_t^{(1)}}{a_t}M^2(W_t^{(1)})$$

$$+\frac{a_t^{(2)}}{a_t}\frac{d^2W_t^{(2)}(1)}{d\theta^2}+\frac{a_t^{(2)}}{a_t}M(W_t^{(2)})-\frac{a_t^{(2)}}{a_t}M^2(W_t^{(2)})$$

$$-\left[\frac{a_t^{(1)}}{a_t}M(W_t^{(1)})+\frac{a_t^{(2)}}{a_t}M(W_t^{(2)})\right]^2+\frac{a_t^{(1)}}{a_t}M^2(W_t^{(1)})+\frac{a_t^{(2)}}{a_t}M^2(W_t^{(2)})$$

$$=\frac{a_t^{(1)}}{a_t}D^2\left(W_t^{(1)}\right)+\frac{a_t^{(2)}}{a_t}D^2\left(W_t^{(2)}\right)+\frac{a_t^{(1)}}{a_t}M^2(W_t^{(1)})+\frac{a_t^{(2)}}{a_t}M^2(W_t^{(2)})$$

$$-\left(\frac{a_t^{(1)}}{a_t}\right)^2M^2(W_t^{(1)})-\left(\frac{a_t^{(2)}}{a_t}\right)^2M^2(W_t^{(2)})-2\frac{a_t^{(1)}a_t^{(2)}}{a_t^2}M(W_t^{(1)})M(W_t^{(2)})$$

$$=\frac{a_t^{(1)}}{a_t}D^2\left(W_t^{(1)}\right)+\frac{a_t^{(2)}}{a_t}D^2\left(W_t^{(2)}\right)+\frac{a_t^{(1)}}{a_t}M^2(W_t^{(1)})\left(1-\frac{a_t^{(1)}}{a_t}\right)+\frac{a_t^{(2)}}{a_t}M^2(W_t^{(2)})$$

$$\times\left(1-\frac{a_t^{(2)}}{a_t}\right)-2\frac{a_t^{(1)}a_t^{(2)}}{a_t^2}M(W_t^{(1)})M(W_t^{(2)}).$$

From the following formula:

$$\frac{a_t^{(1)}}{a_t}+\frac{a_t^{(2)}}{a_t}=1$$

we get

$$D^2(W_t) = \frac{a_t^{(1)}}{a_t} D^2\left(W_t^{(1)}\right) + \frac{a_t^{(2)}}{a_t} D^2\left(W_t^{(2)}\right) + \frac{a_t^{(1)} a_t^{(2)}}{a_t^2} M^2(W_t^{(1)})$$
$$+ \frac{a_t^{(1)} a_t^{(2)}}{a_t^2} M^2(W_t^{(2)}) - 2\frac{a_t^{(1)} a_t^{(2)}}{a_t^2} M(W_t^{(1)}) M(W_t^{(2)})$$
$$= \frac{a_t^{(1)}}{a_t} D^2\left(W_t^{(1)}\right) + \frac{a_t^{(2)}}{a_t} D^2\left(W_t^{(2)}\right) + \frac{a_t^{(1)} a_t^{(2)}}{a_t^2} \left[M(W_t^{(1)}) - M(W_t^{(2)})\right]^2$$

which was to be proved.

Relationship (26) shows that in the general case increasing the number of summed distributed lag models causes an increase in the variance of the variable described by the resultant distributed lag model. An exception occurs when all the mean values of the summed distributed lag models are equal; in that case the variance of the lag distribution described by the resultant distributed lag model is equal to the weighted average of the variances of the lag distributions described by the component distributed lag models.

Based on the assumption that the random terms $\varepsilon_t^{(j)}$, $j = 1, ..., n$, are independent, it is obvious that

$$E(\varepsilon_t) = 0 \quad \text{and} \quad D^2(\varepsilon_t) = \sum_{j=1}^{n} D^2(\varepsilon_t^{(j)})$$

3.2. Superposition of distributed lag models

Superposition of distributed lag models occurs when an independent variable $y_t^{[n]}$ is determined by a distributed lag model whose argument is the independent variable $y_t^{[n-1]}$, which in turn is also a dependent variable with regard to a certain independent variable $y_t^{[n-2]}$, etc. Thus

$$\begin{aligned}
y_t^{[n]} &= \sum_{i=0}^{\infty} v_{t\,i}^{(n)} y_{t-i}^{[n-1]} + \varepsilon_t^{(n)} = V_t^{(n)}(L) y_{t-i}^{[n-1]} + \varepsilon_t^{(n)} \\
y_t^{[n-1]} &= \sum_{i=0}^{\infty} v_{t\,i}^{(n-1)} y_{t-i}^{[n-2]} + \varepsilon_t^{(n-1)} = V_t^{(n-1)}(L) y_{t-i}^{[n-2]} + \varepsilon_t^{(n-1)} \\
&\dots \\
y_t^{[1]} &= \sum_{i=0}^{\infty} v_{t\,i}^{(1)} x_{t-i} + \varepsilon_t^{(1)} = V_t^{(1)}(L) x_t + \varepsilon_t^{(1)}
\end{aligned} \tag{27}$$

In Equations (27), all the random components $\varepsilon_t^{(j)}$, $j = 0, 1, 2, ..., n$, are independently distributed and have expected values equal to zero, $E(\varepsilon_t^{(j)}) = 0$, and finite variances $D^2(\varepsilon_t^{(j)}) = \left(\sigma^{(j)}\right)^2 < \infty$.

On the basis of Eq. (14), recursive substitution using the equations in (27) makes it possible to present the dependent variable $y_t^{[n]}$ in the following form:

$$y_t^{[1]} = V_t^{(1)}(L)x_t + \varepsilon_t^{(1)}$$

$$y_t^{[2]} = V_t^{(2)}(L)V_t^{(1)}(L)x_t + V_t^{(2)}(L)\varepsilon_t^{(1)} + \varepsilon_t^{(2)}$$

$$y_t^{[3]} = V_t^{(3)}(L)V_t^{(2)}(L)V_t^{(1)}(L)x_t + V_t^{(3)}(L)V_t^{(2)}(L)\varepsilon_t^{(1)} + V_t^{(3)}(L)\varepsilon_t^{(2)} + \varepsilon_t^{(3)}$$

$$...$$

$$y_t^{[n]} = V_t^{(n)}(L) \times ... \times V_t^{(1)}(L)x_t$$

$$+ V_t^{(n)}(L) \times ... \times V_t^{(2)}(L)\varepsilon_t^{(1)} + V_t^{(n)}(L) \times ... \times V_t^{(3)}(L)\varepsilon_t^{(2)} + ... + V_t^{(n)}\varepsilon_t^{(n-1)} + \varepsilon_t^{(n)}$$

or in a more concise form:

$$y_t^{[n]} = V_t^{[n]}(L)x_t + \varepsilon_t^{[n]} \tag{28}$$

where the expression:

$$V_t^{[k]}(L) = \prod_{j=1}^{k} V_t^{(j)}(L)$$

denotes the polynomial operator being the product of the polynomial operators $V_t(j)(L)$, $j = 1, 2, ..., k$, $k = 1, 2, ..., n$, and the term:

$$\varepsilon_t^{[n]} = \sum_{k=1}^{n} \frac{V_t^{[n]})L)}{V_t^{[k]}(L)} \varepsilon_t^{(k)}$$

denotes the random component of the composite distributed lag model obtained by superposition of n distributed lag models.

Note that the coefficients of the polynomial operators $V_t^{[k]}(L)$, $k = 1, 2, ..., n$, correspond to lag structures $V_t^{[k]}$ (all the coefficients are non-negative) consisting of lag coefficients $v_{t-1}^{[k]}$, $i = 0, 1, 2, ...$

If there exist lag distributions $W_t^{(1)}$, $W_t^{(2)}$, ..., $W_t^{(n)}$ built on the lag structures $V_t^{(1)}$, $V_t^{(2)}$, ..., $V_t^{(n)}$, relationship (28) can be expressed in the following form:

$$y_t^{[n]} = a_t^{[n]}\, W_t^{[n]}(L)x_t + \sum_{k=1}^{n} \frac{a_t^{[n]} W_t^{[n]}(L)}{a_t^{[n]} W_t^{[k]}(L)}\, \varepsilon_t^{(k)} \tag{29}$$

where the long-term multiplier $a_t^{[k]}$ is the product of the long-term multipliers $a_t^{(i)}$, $i = 1, 2, ..., k$, $k \le n$ of the component distributed lag models.

$$a_t^{[k]} = \prod_{i=1}^{k} a_t^{(i)}$$

It is worth noting that $W_t^{[n]}$ is also a lag distribution, its coefficients are non-negative and they sum to 1, which follows from the fact that for $\theta = 1$ the value of the generating function $W_t(\theta)$ is equal to 1. On the basis of (16):

$$W_t^{(i)}(1) = 1, \qquad i = 1, 2, ...$$

hence:

$$W_t^{[n]}(1) = \prod_{i=1}^{n} W_t^{(i)}(1) = 1$$

The mean value $M(W_t^{[n]})$ of the lag distribution $W_t^{[n]}$ of the superposition of n distributed lag models is the sum of the mean values $M(W_t^{(i)}), i = 1, 2, ..., n$ of the component distributed lag models:

$$M(W_t^{[n]}) = M(W_t^{(1)}) + M(W_t^{(2)}) + ... + M(W_t^{(n)}) \tag{30}$$

The proof of relationship (30) is based on the derivative of the product of functions:

$$\frac{dW_t^{[n]}(1)}{d\theta} = \left.\frac{d\left[W_t^{(1)}(\theta) \times \ldots \times W_t^{(n)}(\theta)\right]}{d\theta}\right|_{\theta=1} = \left.\left\{\prod_{i=1}^{n} W_t^{(i)}(\theta)\left[\sum_{i=1}^{n} \frac{dW_t^{(i)}(\theta)}{d\theta} \frac{1}{W_t^{(i)}(\theta)}\right]\right\}\right|_{\theta=1}$$

$$= \sum_{i=1}^{n} \frac{dW_t^{(i)}(1)}{d\theta} = \sum_{i=1}^{n} M(W_t^{(i)}) = M(W_t^{[n]})$$

The variance $D^2(W_t^{[n]})$ of the lag distribution $W_t^{[n]}$ of the distributed lag model built as the superposition of n distributed lag models is the sum of the variances $D^2(W_t^{(i)}),\ i = 1, 2, ..., n$ of those models.

$$D^2(W_t^{[n]}) = D^2(W_t^{(1)}) + D^2(W_t^{(2)}) + \ldots + D^2(W_t^{(n)}) \tag{31}$$

The proof of relationship (31) is also based on Eq. (19) using the formula for the second derivative of the generating function:

$$\frac{d^2W_t^{[n]}(\theta)}{d\theta^2} = \frac{d^2\left[W_t^{(1)}(\theta)\times\ldots\times W_t^{(n)}(\theta)\right]}{d\theta^2} = \prod_{i=1}^{n} W_t^{(i)}(\theta)$$
$$\times\left\{\sum_{i=1}^{n}\frac{d^2W_t^{(i)}(\theta)}{d\theta^2}\frac{1}{W_t^{(i)}(\theta)} + \left[\sum_{i=1}^{n}\frac{dW_t^{(i)}(\theta)}{d\theta}\frac{1}{W_t^{(i)}(\theta)}\right]^2 - \sum_{i=1}^{n}\left[\frac{dW_t^{(i)}(\theta)}{d\theta}\frac{1}{W_t^{(i)}(\theta)}\right]^2\right\}$$
$$\times\frac{d^2W_t^{[n]}(1)}{d\theta^2} = \sum_{i=1}^{n}\frac{d^2W_t^{(i)}(1)}{d\theta^2} + \left[\sum_{i=1}^{n} M\left(W_t^{(i)}\right)\right]^2 - \sum_{i=1}^{n} M^2\left(W_t^{(i)}\right)$$

Taking into account the above equation, one obtains:

$$D^2(W_t^{[n]}) = \sum_{i=1}^{n}\frac{d^2W_t^{(i)}(1)}{d\theta^2} + \sum_{i=1}^{n} M\left(W_t^{(i)}\right) - \sum_{i=1}^{n} M^2\left(W_t^{(i)}\right)$$
$$= \sum_{i=1}^{n}\left[\frac{d^2W_t^{(i)}(1)}{d\theta^2} + M\left(W_t^{(i)}\right) - M^2\left(W_t^{(i)}\right)\right] = \sum_{i=1}^{n} D^2(W_t^{(i)})$$

which was to be proved.

To analyze the random component of the composite distributed lag model resulting from the superposition of n distributed lag models, we examine the last term of equation (28):

$$\varepsilon_t^{[n]} = \sum_{k=1}^{n}\frac{V_t^{[n]}(L)}{V_t^{[k]}(L)}\,\varepsilon_t^{(k)}$$

It should be noted that the above expression can also be rewritten in the following form:

$$\varepsilon_t^{[n]} = \sum_{k=1}^{n} V_t^{\{k\}}(L)\,\varepsilon_t^{(k)} \tag{31}$$

or

$$\varepsilon_t^{[n]} = \sum_{k=1}^{n} a_t^{\{k\}} W_t^{\{k\}}(L)\, \varepsilon_t^{(k)} \tag{31a}$$

where

$$V_t^{\{k\}}(L) = \frac{V_t^{[n]}(L)}{V_t^{[k]}(L)} = V_t^{(k+1)}(L) V_t^{(k+2)}(L) \times \ldots \times V_t^{(n)}(L)$$

$$a_t^{(k)} = \frac{a_t^{[n]}}{a_t^{[k]}} = a_t^{k+1} a_t^{k+2} \times \ldots \times a_t^{(n)} \tag{32}$$

$$W_t^{\{k\}}(L) = \frac{W_t^{[n]}(L)}{W_t^{[k]}(L)} = W_t^{(k+1)}(L) W_t^{(k+2)}(L) \times \ldots \times W_t^{(n)}(L)$$

Hence, the random term $\varepsilon_t^{[n]}$ in (31) is the sum of *n* products of the *k*th long-term multiplier $a_t^{\{k\}}$, the *k*th lag operator $W_j^{(k)}(L)$, and the random term $\varepsilon_t^{(k)}$ of the *k*th component of the distributed lag model, $k = 1, 2, ..., n$.

Now, equation (31) can be written as:

$$\begin{aligned}
\varepsilon_t^{[n]} &= \sum_{k=1}^{n} V_t^{\{k\}}(L)\, \varepsilon_t^{(k)} \\
&= \sum_{i=1}^{\infty} v_t^{\{1\}} \varepsilon_{t-i}^{(k)} \\
&+ \sum_{i=1}^{\infty} v_t^{\{2\}} \varepsilon_{t-i}^{(2)} \\
&\ldots \\
&+ \sum_{i=1}^{\infty} v_t^{\{n-1\}} \varepsilon_{t-i}^{(n-1)} \\
&+ \sum_{i=1}^{\infty} v_t^{\{n\}} \varepsilon_{t-i}^{(n)}.
\end{aligned}$$

Note that $\sum_{i=1}^{\infty} v_t^{\{n\}} \varepsilon_{t-i}^{(n)} = \varepsilon_t^{(n)}$.

The expected value of the random term $\varepsilon_t^{[n]}$ is equal to:

$$\varepsilon_t^{[n]} = 0$$

because

$$E(\varepsilon_t^{[n]}) = E\left(\sum_{k=1}^{n} V_t^{\{k\}}(L)\, \varepsilon_t^{(k)} \right) = \sum_{k=1}^{n} \sum_{i=0}^{\infty} v_{ti}^{\{k\}} E(\varepsilon_{t-i}^{(k)}) = 0$$

The variance $D^2(\varepsilon_t^{[n]})$ of the random term $\varepsilon_t^{[n]}$ is equal to:

$$D^2(\varepsilon_t^{[n]}) = \sum_{k=1}^{n} a_t^{\{k\}} \left(\sigma_t^{(k)}\right)^2 = \sigma_t^{(n)} + \sum_{k=1}^{n-1} a_t^{\{k\}} \left(\sigma_t^{(k)}\right)^2$$

because

$$D^2(\varepsilon_t^{[n]}) = D^2\left(\sum_{k=1}^{n} V_t^{\{k\}}(L)\, \varepsilon_t^{(k)} \right) = \sum_{k=1}^{n} \sum_{i=0}^{\infty} v_{ti}^{\{k\}} D^2(\varepsilon_{t-i}^{(k)})$$

$$= \sum_{k=1}^{n} \sum_{i=0}^{\infty} v_{ti}^{\{k\}} (\sigma^{(k)})^2 = \sum_{k=1}^{n} a_t^{\{k\}} \left(\sigma^{(k)}\right)^2$$

and $a_t^{\{n\}} = 1$.

4. Conclusions

The mean value of the lag distribution described by a distributed lag model composed as the sum (parallel connection) of distributed lag models is equal to the weighted average of the mean values of the lag distributions of the component distributed lag models. The variance of the lag distribution described by a distributed lag model composed as the sum of distributed lag models (parallelly connected) is greater than the sum of the variances of the lag distributions of the component distributed lag models. An exception to this rule occurs when all the mean values of the component lag distributions are equal. In this case, the resulting variance is equal to the sum of the variances of the component lag distributions. The expected value of the resulting random term is equal to zero and its variance is the sum of the variances of the component lag distributions.

The mean value of the lag distribution described by a distributed lag model composed as the superposition (series connection) of distributed lag models is equal to the sum of the mean values of the component lag distributions. The variance of the lag distribution described by a distributed lag model composed as the superposition of distributed lag models is equal to the sum of the variances of the component lag distributions. The expected value of the resulting random term is equal to zero and its variance is equal to the weighted sum of the variances of the component lag distributions where the weight coefficients are the products of the component long-term multipliers.

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