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# THE BASIS OF FINANCIAL ARITHMETIC FROM THE VIEWPOINT OF UTILITY THEORY

The main goal of this paper is to present a modern axiomatic approach to financial arithmetic. An axiomatic theory of financial arithmetic was first proposed by Peccati, who introduced an axiomatic definition of future value. This theory has been extensively developed in recent years. The proposed approach to financial arithmetic is based on the concept of the utility of a financial flow. This utility function is defined as a linear extension of a multicriterion comparison determined by an individual's time preference and capital preference. The present value is then defined to be the utility of the financial flow. Therefore, the law of the diminishing marginal utility of wealth has been considered as an additional feature of the present value. The future value is defined as the inverse of the utility function. This definition is a generalization of Peccati's one. The net present value is given as the unique additive extension of the utility of the financial flow. Moreover, the synergy effect and the diversification effect will be discussed. At the end, the axiomatic definition of the present value will be specified in three ways.

Key words: *synergy effect of capital, diversification, financial arithmetic, Gossen's first law, utility*

## 1. Introduction

The subject of financial arithmetic is dynamic estimation of the value of money. The fundamental assumption of financial arithmetic is the certainty that the nominal value of money in circulation increases with time. In general, this assumption is justified by the analysis of the quantity of money equation proposed by Irving Fisher [2]. This analysis is based on the additional assumption that the total amount of money is constant. This is a typical normative assumption. Therefore, the value of money con-

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sidered in financial arithmetic is called the normative value of money. The growth process of the normative value of money is called the process of capital appreciation. On the other hand, economic-financial practice generally results in the increase in the amount of money being faster than the increase in production volume. Thus we observe a decrease in the real value of money. This means that the normative value of money cannot be identified with the real value of money. This raises a question about the essence of the concept of normative value. One consequence of this question is another question about the essence of the basic functions of financial arithmetic.

The answer to this question was sought through the development of an axiomatic theory for financial arithmetic. For any set of moments $\Theta \subseteq [0, +\infty[$ Peccati [17] defined the future value using a function $FV$: $\Theta \times \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

, $$\forall C \in \mathbb{R} : FV(0, C) = C \tag{1}$$

$$\forall t_{1,} t_2 \in \Theta \forall C \in \mathbb{R}^+ : t_1 < t_2 \Rightarrow FV(t_1, C) < FV(t_2, C) \tag{2}$$

$$\forall t \in \Theta \ \forall C_1, C_2 \in \mathbb{R} : FV(t, C_1 + C_2) = FV(t_1, C_1) + FV(t_1, C_2) \tag{3}$$

The certainty of the growth of the nominal value of money is described above by axiom (2). The present value is defined as the function $PV : \Theta \times \mathbb{R} \to \mathbb{R}$, uniquely determined by the identity

$$FV(t, PV(t, C)) = C \tag{4}$$

Thus a coherent theory of financial arithmetic was created. This approach has been extensively studied, inter alia, in [18, 19]. Current knowledge on the consequences of an axiomatic approach to the concept of future value is presented in [10]. On the other hand, Peccati's theory did not explain the phenomenon of growth in the nominal value of money.

In recent years, the concept of the utility of a financial flow played an important role in research on behavioral finance. This problem is discussed, for example, by Frederick et al. [7], Takahashi [25], Dacey et al. [4], Zauberman et al. [26], Kim et al. [13], Eer et al. [6], Killeen [12], Kontek [14], Doyle [5], Piasecki [20–22] and Han et al. [8]. Using this approach. we can present the notion of normative value in the context of the utility function. This approach sheds new light on the fundamental variables of financial arithmetic.

The main goal of the paper was to explain, in terms of the utility theory, basic functions of financial arithmetic: the future value, present value and net present value. The definitions obtained in this way have been compared with the axiomatic definitions of present and future values given by Peccati [17]. In order to show the usefulness of this new theory, some properties of basic functions of financial arithmetic have been discussed. The law of the diminishing marginal utility of wealth will be considered as an additional feature of the present value. Moreover, the synergy effect and the diversification effect will be discussed.

## 2. Ordered financial flows

Let the set of moments $\Theta \subseteq [0, +\infty[$ be given. In this particular case, it may be interpreted as the set of possible capitalization moments or the non-negative half-line depicting time. In financial market analysis, each payment is represented by a financial instrument described as financial flow $(t, C)$, where the symbol $t \in \Theta$ denotes the moment of the flow and the symbol $C \in \mathbb{R}$ denotes the nominal value of this flow. Each of these financial flows can be either an executed receivable or matured liability. The nominal value of any receivable is non-negative. The debtor's liabilities are always the creditor's receivables. In this situation, the value of each liability is equal to the negative of the value of corresponding receivable.

In the first step, we confine our discussion to the set $\Phi^+ = \Theta \times [0, +\infty[$ of all receivables $(t, C)$. Investors determine their preferences on the set of receivables. These preferences have some common characteristics.

Regarding the economic theory on preferences, von Mises [16] presented a time-preference rule. This rule says that taking into account the ceteris paribus principle, an economic agent will satisfy their needs as quickly as possible. In other words, when an economic agent is faced with two goals characterized by the same subjective value, then it prefers the one which can be achieved in less time. In this particular case, this means that the investor, when comparing two payments of equal nominal value, always prefers the earlier payment. This relation can be described by means of the preorder $\succcurlyeq_T$, defined as follows

$$\forall (t_1, C_1), (t_2, C_2) \in \Phi^+ : (t_1, C) \succcurlyeq_T (t_2, C) \Leftrightarrow t_1 \leq t_2 \tag{5}$$

On the other hand, it is obvious that each economic agent is guided by the rule of capital preference. This rule means that, taking into account the ceteris paribus principle, an economic agent prefers the larger of two amounts. So when an economic agent is faced with two economic items available at the same time, he selects the one which is characterized by a greater subjective value. In our particular case, this means that the investor, when comparing two simultaneously available payments, always selects the higher payment. This relation can be described by means of the preorder $\succcurlyeq_C$, defined as follows

$$\forall (t, C_1), (t, C_2) \in \Phi^+ : (t, C_1) \succcurlyeq_C (t, C_2) \Leftrightarrow C_1 \geq C_2 \tag{6}$$

Simultaneously taking into consideration both the above preorders leads to the final determination of a creditor's preferences $\succcurlyeq$ on the receivables set $\Phi^+$, as a multicriterion comparison

$$\forall (t_1, C_1), (t_2, C_2) \in \Phi^+ : (t_1, C_1) \succcurlyeq (t_2, C_2) \Leftrightarrow t_1 \leq t_2 \wedge C_1 \geq C_2 \tag{7}$$

There exists a utility function $U : \Phi^+ \to [0, +\infty[$ fulfilling the condition

$$\forall (t_1, C_1), (t_2, C_2) \in \Phi^+ : (t_1, C_1) \succcurlyeq (t_2, C_2) \Rightarrow U(t_1, C_1) \geq U(t_2, C_2) \tag{8}$$

The next step will discuss the set $\Phi^- = \Theta \times ]-\infty, 0]$ of all liabilities. Each debtor's liability $(t, C)$ corresponds to an appropriate creditor's receivable $(t, -C)$. Each profit achieved by the creditor is the debtor's expense. It results from this relationship that a debtor's preference defined on the set of all liabilities is the inverse relation to the creditor's preferences defined on the set of appropriate receivables. Hence, the debtor's preferences $\succcurlyeq$ on the set $\Phi^-$ of all liabilities are defined by the equivalency relation

$$\forall (t_1, C_1), (t_2, C_2) \in \Phi^- : \ (t_1, C_1) \succcurlyeq (t_2, C_2) \Leftrightarrow (t_2, -C_2) \succcurlyeq (t_1, -C_1) \tag{9}$$

Comparison of equations (7) and (9) leads to the final determination of the debtor's preferences $\succcurlyeq$ on the set $\Phi^-$ of all liabilities, as a multicriterion comparison

$$\forall (t_1, C_1), (t_2, C_2) \in \Phi^- : \ (t_1, C_1) \succcurlyeq (t_2, C_2) \Leftrightarrow t_1 \geq t_2 \wedge C_1 \geq C_2 \tag{10}$$

There exists a utility function defined on the set of all liabilities. From a financial point of view, any receivable is more useful than any liability. Thus we can say that

$$\forall (t_1, C_1), (t_2, C_2) \in \Phi^+ \times \Phi^- : U(t_1, C_1) \geq U(t_2, C_2) \tag{11}$$

Fulfillment of this condition can be obtained via the assumption that the utility of any liability is non-positive*. Therefore, we can say that there exists a utility function $U : \Phi^- \to \left]-\infty, 0\right]$ fulfilling the condition

$$\forall (t_1, C_1), (t_2, C_2) \in \Phi^- : \ (t_1, C_1) \succcurlyeq (t_2, C_2) \Rightarrow U(t_1, C_1) \geq U(t_2, C_2) \tag{12}$$

To sum up the previous discussion, we can conclude that financial preferences $\succcurlyeq$ are determined on the set $\Phi = \left[0, +\infty\right[ \times \mathbb{R} \to \mathbb{R}$ of all financial flows. This relation is determined by the union of multicriterion comparisons in the following way

$$\begin{aligned} \forall (t_1, C_1), (t_2, C_2) \in \Phi : (t_1, C_1) \succcurlyeq (t_2, C_2) \Leftrightarrow (t_1 \geq t_2 \wedge 0 \geq C_1 \geq C_2) \\ \vee (t_1 \leq t_2 \wedge C_1 \geq C_2 \geq 0) \vee (C_1 \geq 0 \geq C_2) \end{aligned} \tag{13}$$

This preorder is not linear. There exists a utility function $U : \Phi \to \mathbb{R}$ fulfilling the condition

$$\forall (t_1, C_1), (t_2, C_2) \in \Phi : (t_1, C_1) \succcurlyeq (t_2, C_2) \Rightarrow U(t_1, C_1) \geq U(t_2, C_2) \tag{14}$$

Regarding this function, we will also assume that its values are determined by the receivables utility function (8) or by the liabilities utility function (12). Defined in this way, the utility function can be subjective [4]. This indicates the possibility that the financial preferences model introduced above may be apllied to behavioral finance.

## 3. Utility of financial flow

Let us now examine the properties of the utility functions described above. Comparison of the domains and co-domains of the utility functions described in (8) and (12) implies that

$$\forall t \in \Theta : U(t, 0) = 0 \tag{15}$$

*The notion of negative utility was discussed in the papers [1, 3] and [23].

The preorder $\succcurlyeq$ determines the strict order $\succ$ defined on the set $\Phi$ according to the equivalence

$$\begin{aligned}&\forall(t_1,C_1),(t_2,C_2)\in\Phi:(t_1,C_1)\succ(t_2,C_2)\\&\Leftrightarrow\big(\big((t_1,C_1)\succcurlyeq(t_2,C_2)\big)\wedge\sim\big((t_2,C_2)\succcurlyeq(t_1,C_1)\big)\big)\end{aligned}\tag{16}$$

This strict order is determined by the union of multicriterion comparisons

$$\begin{aligned}&\forall(t_1,C_1),(t_2,C_2)\in\Phi:(t_1,C_1)\succ(t_2,C_2)\\&\Leftrightarrow(t_1\geq t_2\wedge 0\geq C_1>C_2)\vee(t_1>t_2\wedge 0\geq C_1\geq C_2)\\&\vee(t_1\leq t_2\wedge C_1>C_2\geq 0)\vee(t_1<t_2\wedge C_1\geq C_2\geq 0)\end{aligned}\tag{17}$$

On the other hand, we have

$$\forall(t_1,C_1),(t_2,C_2)\in\Phi(t_1,C_1):(t_1,C_1)\succ(t_2,C_2)\Rightarrow U(t_1,C_1)>U(t_2,C_2)\tag{18}$$

Comparing equations (16), (17) and (18), we find that utility is an increasing function of the flow of nominal value. Thus we can say that

$$\forall t\in\Theta\forall C_1,C_2\in\mathbb{R}:C_1>C_2\Rightarrow U(t,C_1)>U(t,C_2)\tag{19}$$

Thus, for any fixed moment $t\in\Theta$ we can determine the inverse function $U_0^{-1}(t,\cdot):\mathbb{R}\to\mathbb{R}$. In addition, comparing (17) and (18) we obtain

$$\forall C>0\forall t_1,t_2\in\Theta:t_1<t_2\Rightarrow U(t_1,C)>U(t_2,C)\tag{20}$$

$$\forall C<0\forall t_1,t_2\in\Theta:t_1<t_2\Rightarrow U(t_1,C)<U(t_2,C)\tag{21}$$

One conventional issue is the calibration of the value of the utility function. Here, we assume that the utility of an immediate financial flow is equal to the nominal value of this flow. This assumption is written as a boundary condition

$$\forall C\in\mathbb{R}:U(0,C)=C\tag{22}$$

All these properties of the utility function will be used to study the properties of the basic models of financial arithmetic.

## 4. Future and present values

For the preorder $\succcurlyeq$ defined by the equivalence (13), we determine its linear closure $\sqsupseteq$ in following way:

$$\forall (t_1, C_1),(t_2, C_2) \in \Phi : (t_1, C_1) \sqsupseteq (t_2, C_2) \Leftrightarrow U(t_1, C_1) \geq U(t_2, C_2) \tag{23}$$

This preorder defines the relation $\equiv$ of the equivalence of financial flows. We have

$$\forall (t_1, C_1),(t_2, C_2) \in \Phi : (t_1, C_1) \equiv (t_2, C_2) \Leftrightarrow U(t_1, C_1) = U(t_2, C_2) \tag{24}$$

If two financial flows are equally useful, then we consider them as equivalent. If two financial flows are equivalent, then the first one is called the equivalent of the second one. The nominal value of any equivalent financial flow is defined to be the normative value of such a flow.

Analysis of conditions (19) and (20) leads us to formulate the appreciation principle. This principle states that the normative value of a receivable must be increasing in time $t$ at which this receivable will be paid. In this way, utility theory confirms the usefulness of the fundamental axiom of financial arithmetic stating that the nominal value of money received must increase with time.

The concept of normative value described above can be included in the framework of a formal model. Let there be a given immediate financial flow with nominal value $C \in \mathbb{R}$. This financial flow uniquely corresponds to the pair $(0, C)$. At any time $t \in \Theta$, the normative value of such a flow is equal to $C_t$. In accordance with the definition (24) of the equivalence of financial flows and boundary condition (22), we obtain the identity

$$C = U(0, C) = U(t, C_t) \tag{25}$$

Under condition (19), for a fixed moment $t \in \Theta$ we uniquely determine the normative value

$$C_t = FV(t, C) = U_0^{-1}(t, C) \tag{26}$$

The function $FV : \Theta \times \mathbb{R} \to \mathbb{R}$ defined in this way is called the future value. In the general case, this function has the following properties: (1), (2) and

$$\forall t \in \Theta : FV(t, 0) = 0 \tag{27}$$

$$\forall (t_1, C)(t_2, C) \in \Phi^- : \ t_1 < t_2 \Rightarrow FV(t_1, C) > FV(t_2, C) \tag{28}$$

$$\forall (t, C_1)(t, C_2) \in \Phi : C_1 < C_2 \Rightarrow FV(t, C_1) < FV(t, C_2) \tag{29}$$

It is easy to check that condition (29) is a generalization of condition (3). This means that the future value function defined in this section is a generalization of the future value proposed by Peccati [17]. The future value function can be represented by the identity

,
$$FV(t, C) = C\, s(t, C) \tag{30}$$

where the function $s : \Phi \to [0, +\infty[$ is called the appreciation factor. The appreciation factor is an increasing function in time fulfilling the boundary condition:

$$s(0, C) = 1 \tag{31}$$

The appreciation factor describes the process of the relative appreciation of capital. If this factor is an increasing function of the value of capital, then we are dealing with the effect of capital synergy. The effect of capital synergy means that an increase in the value of capital causes an increase in the relative speed of appreciation.

Another object of our investigation will be a given financial flow $(t, C)$. We can determine its equivalent $(0, C_0)$. The nominal value $C_0$ of this equivalent is called the present value and is denoted by $PV(t, C)$. In accordance with the definition (24) of the equivalence of financial flows and boundary condition (22), we obtain the identity

$$C_0 = PV(t, C) = U(0, C_0) = U(t, C) \tag{32}$$

The present value of any financial flow is equal to its utility. This statement fully explains the essence of the concept of the present value. On the other hand, this interpretation of the present value is not without some formal problems, which will be discussed later. Now let us focus our attention on the formal properties of the function $PV : \Phi \to \mathbb{R}$ defined by the identity (32). We have

$$\forall C \in \mathbb{R} : PV(0, C) = C \tag{33}$$

$$\forall t \in \Theta : PV(t, 0) = 0 \tag{34}$$

$$\forall (t_1, C), (t_2, C) \in \Phi^+ : t_1 < t_2 \Rightarrow PV(t_1, C) > PV(t_2, C) \tag{35}$$

$$\forall (t_1, C),(t_2, C) \in \Phi^- : \ t_1 < t_2 \Rightarrow PV(t_1, C) < PV(t_2, C) \tag{36}$$

$$\forall (t_1, C_1),(t_2, C_2) \in \Phi : \ C_1 < C_2 \Rightarrow PV(t, C_1) < PV(t, C_2) \tag{37}$$

The present value function can be represented by the identity

$$PV(t,C) = C\, v(t,C) \tag{38}$$

where the discount factor $v : \Phi \to \left]0;1\right]$ is a decreasing function in time fulfilling the boundary condition

$$v(0,C) = 1 \tag{39}$$

Comparison of conditions (26) and (32) shows that the future value and the present value defined in this section satisfy condition (4). This means that the present value function defined here is a generalization of the present value proposed by Peccati [17].

Let us consider again the pair of equivalent financial flows $(0, C_0)$ and $(t, C_t)$. From equations (26), (30), (32) and (38) we obtain

$$C_t = FV(t, C_0) = C_0\, s(t, C_0) \tag{40}$$

$$C_0 = PV(t, C_t) = C_t\, v(t, C_t) \tag{41}$$

Comparison of the last two equations leads to the relation

$$s(t, C_0) = \frac{1}{v(t, C_t)} = \frac{1}{v\big(t, FV(t, C_0)\big)}$$

In this way, we have shown that the appreciation and discount factors determined by the same utility function satisfy the condition

$$. \qquad s(t,C) = \frac{1}{v\big(t, FV(t,C)\big)} = \frac{1}{v\big(t, Cs(t,C)\big)} \tag{42}$$

## 5. Net present value

Let the set $\Phi = \Theta \times \mathbb{R}$ of all financial flows be given. Any sequence of financial flows is called a financial investment. Thus any investment $\breve{X}$ can be described as a multiset [9] of financial flows:

$$\breve{X} = \left\{ \left( t_j, C_j \right) \in \Phi : j = 1, 2, ..., n, ... \right\} \tag{43}$$

This means that any investment may have different financial flows with identical moments of payment and identical nominal value. Each investment containing exactly one financial flow is called a simple investment. The family of all investments is denoted by the symbol $\mathbb{F}$.

A combination of a pair of investments is the single investment consisting of all the financial flows in the pair of investments. In accordance with the above, the combination $\breve{X} \uplus \breve{Y}$ of the pair of investments $\breve{X}, \breve{Y}$ is defined as the multiset sum [24]:

$$\breve{X} \uplus \breve{Y} = \left\{ \left( t, C \right) : \left( t, C \right) \in \breve{X} \vee \left( t, C \right) \in \breve{Y} \right\} \tag{44}$$

In the family $\mathbb{F}$ of all investments we distinguish the subfamily $\mathbb{F}_1$ of all simple investments. Any investment can be presented as a countable multiset sum of simple investments. On the other hand, the family $\mathbb{F}_1$ of all simple investments and the set $\Phi$ of all financial flows are isomorphic. Thus the preorder $\sqsupseteq$ on the set $\Phi$ of all financial flows determines the linear preorder $\succsim$ on the set $\mathbb{F}_1$ of all simple investments. This preorder is defined by the equivalence

$$\forall \left( t_1, C_1 \right), \left( t_2, C_2 \right) \in \mathbb{F}_1 : \left\{ \left( t_1, C_1 \right) \right\} \succsim \left\{ \left( t_2, C_2 \right) \right\} \Leftrightarrow \left( t_1, C_1 \right) \sqsupseteq \left( t_2, C_2 \right) \tag{45}$$

This means that there exists a utility function $V : \mathbb{F}_1 \to \mathbb{R}$ determined by the identity

$$V\left( \left\{ \left( t, C \right) \right\} \right) = U\left( t, C \right) \tag{46}$$

Let us take into account any extension of the utility function $V : \mathbb{F} \to \mathbb{R}$. This extension is determined using the postulate that the utility function is additive [11]. This postulate is consistent with financial practice, where the value of capital is calculated as the sum of its component values. It follows that any investment utility function $V:\mathbb{F}\rightarrow\mathbb{R}$ should satisfy the following additivity condition:

$$\forall\breve{X},\breve{Y}\in\mathbb{F}:V\left(\breve{X}\uplus\breve{Y}\right)=V\left(\breve{X}\right)+\left(\breve{Y}\right)\tag{47}$$

Conditions (32), (45) and (46) are sufficient for the utility of an investment $\breve{X}\in\mathbb{F}$ to be uniquely defined as its net present value. We have

$$\forall\breve{X}\in\mathbb{F}:NPV\left(\breve{X}\right)=V\left(\breve{X}\right)=\sum\nolimits_{(t,C)\in\breve{X}}PV\left(t,C\right)\tag{48}$$

Using the utility function $NPV:\mathbb{F}\rightarrow\mathbb{R}$ determined above, we can extend the linear preorder $\succsim$ to the family $\mathbb{F}$ of all investments. We have

$$\forall\breve{X},\breve{Y}\in\mathbb{F}:\breve{X}\succsim\breve{Y}\Leftrightarrow NPV\left(\breve{X}\right)\geq NPV\left(\breve{Y}\right)\tag{49}$$

This preorder is also a linear one. This is a natural preorder applied in financial practice. This preorder defines the equivalence relation $\equiv$ on the family $\mathbb{F}$ of all investments. We have

$$\forall\breve{X},\breve{Y}\in\Lambda:\breve{X}\equiv\breve{Y}\Leftrightarrow NPV\left(\breve{X}\right)=NPV\left(\breve{Y}\right)\tag{50}$$

This relation is a generalization of the equivalence of financial flows defined by condition (24). In this way, the functions of the present value, future value and net present value have been defined in the last two sections on the basis of general utility theory. This fact will be used in discussions about the specific properties of these functions.

## 6. Gossen's first law

Gossen's first law says that the marginal utility of wealth is diminishing [2]. Now let us examine the consequence of accepting the assumption that the utility function $U:\Phi^{+}\rightarrow\left[0,+\infty\right[$ defined by equation (8) fulfils the law of the diminishing marginal utility of wealth. Then for any present value function, we can write

$$\begin{aligned}\forall\left(t,C_1\right),\left(t,C_2\right)\in\Phi^{+}\forall\alpha\in\left]0;1\right[:\alpha PV\left(t,C_1\right)+\left(1-\alpha\right)PV\left(t,C_2\right)\\<PV\left(t,\alpha C_1+\left(1-\alpha\right)C_2\right)\end{aligned}\tag{51}$$

The present value is a concave function over the set of all positive capital values. This enables us to prove the following theorem.

**Theorem 1:** The fulfillment of Gossen's first law is a necessary and sufficient condition for variability in the future value $FV: \Phi \to [0, +\infty[$ to reveal the effect of capital synergy.

**Proof:** Let a fixed moment $t \in \Theta$ be given. We substitute $C_2 = 0$ into inequality (51). For any positive value $C_3 < C_1$, we have

$$\frac{C_3}{C_1} PV(t, C_1) + \left(1 - \frac{C_3}{C_1}\right) PV(t, 0) < \left(t, \frac{C_3}{C_1} C_1 + \left(1 - \frac{C_3}{C_1}\right) 0\right)$$

which together with (37) leads to

$$v(t, C_1) = \frac{PV(t, C_1)}{C_1} < \frac{PV(t, C_3)}{C_3} = v(t, C_3) \tag{52}$$

Thus the discount factor is a decreasing function of positive nominal value. On the other hand, if the discounting factor fulfils (52) then, in accordance with (38), the functions $PV(t, C)$ and $f(C) = C - PV(t, C)$ are increasing functions of a positive nominal value. This implies that condition (51) is fulfilled. Inequality (51) is a necessary and sufficient condition for inequality (52) to hold. Comparison of equations (42) and (29) shows that the discount factor decreases iff the appreciation factor increases. This conclusion ends the proof. □

## 7. Investment diversification

An investment composed of financial flows with an identical flow moment is called a portfolio. Thus, the net total nominal value of these financial flows is called the future value of the portfolio. The common flow moment is called the maturity date of the portfolio.

Investment diversification is understood as the allocation of resources among different investments. The principle of diversification has been popularized by the portfolio theory introduced by Markowitz [15]. This principle states that diversification should be preferred. The formal model of the principle of diversification is given by the condition

$$\{(t,C_1),(t,C_2)\} \succsim \{(t,C_1+C_2)\} \tag{53}$$

First, we examine the impact of accepting the principle of diversification on the present value. In accordance with equations (48) and (49), condition (53) is equivalent to the inequality

$$PV(t,C_1)+PV(t,C_2) \geq PV(t,C_1+C_2) \tag{54}$$

If the present value fulfils the above inequality, then conditions (15) and (32) imply

$$PV(t,C)+PV(t,-C) \geq PV(t,C-C)=0 \tag{55}$$

Thus for $C>0$, we have

$$Cv(t,C)-C\cdot v(t,-C) \geq 0$$

which allows us to write

$$v(t,C) \geq v(t,-C) \tag{56}$$

This shows that, according to the principle of diversification, liabilities are discounted more strongly than receivables.

Now we examine the impact of accepting the principle of diversification on the future value.

**Theorem 2:** If condition (54) is fulfilled, then the future value $FV:\Phi \to [0,+\infty[$ satisfies the inequality

$$FV(t,\ C_1)+FV(t,\ C_2) \leq FV(t,\ C_1+C_2) \tag{57}$$

**Proof:** For $(t,C_1),(t,C_2)\in\Phi$, using conditions (26), (32) and (54), we obtain

$$PV(t,FV(t,C_1+C_2))=U(t,U_0^{-1}(t,C_1+C_2))=C_1+C_2$$
$$PV(t,FV(t,C_1))+PV(t,FV(t,C_2)) \geq PV(t,FV(t,C_1)+FV(t,C_2))$$

which together with inequality (37) gives (57). □

This result indicates that the growth rate of capital increases with the value of capital. Moreover, let us note that for $C_2 < 0$, condition (57) describes the effect of financial leverage. Using equations (15) and (26), we obtain

$$FV(t,C)+FV(t,-C)\le FV(t,C-C)=0 \tag{58}$$

Thus for $C > 0$, we have

$$C\cdot s(t,C)-Cs(t,-C)\le 0$$

which implies

$$s(t,C)\le s(t,-C) \tag{59}$$

This result indicates that the growth rate of receivables is less than or equal to the growth rate of liabilities. This is fully consistent with the interpretation of inequality (56).

## 8. Diversification neutrality

In this particular case, when assessing future financial flows, we can ignore the potential benefits achieved through investment diversification. This is known as diversification neutrality. The formal model of diversification neutrality is given by the condition

$$\{(t,C_1),(t,C_2)\}\equiv\{(t,C_1+C_2)\} \tag{60}$$

In accordance with equations (48) and (50), condition (60) is equivalent to the identity

$$PV(t,C_1)+PV(t,C_2)=PV(t,C_1+C_2) \tag{61}$$

Let us consider any present value $PV:\Phi\to\mathbb{R}$ fulfilling conditions (33) and (35). Then in [18] and [19] it was shown that condition (61) is necessary and sufficient for the present value to be given by the identity

$$PV(t,C)=C\,v(t) \tag{62}$$

where the discount factor $v:\Theta\to\,]0;1]$ is a non-increasing function of time satisfying

$$v(0) = 1 \tag{63}$$

The present value determined in this way is a linear capital function. This means that diversification neutrality is a necessary and sufficient condition to reject Gossen's first law. Such rejection only means that in assessing financial flows we ignore the effect of diminishing marginal utility.

Condition (42) implies that the present value $PV : \varPhi \to \mathbb{R}$ fulfils conditions (62) and (63), iff the future value is $FV : \varPhi \to \mathbb{R}$ given by

$$FV\left(t, C\right) = C\, s(t) \tag{64}$$

where the appreciation factor $s : \varPhi \to \left[0, +\infty\right[$ is an increasing function of time fulfilling the boundary condition:

$$s(0) = 1 \tag{65}$$

This means that diversification neutrality is a necessary and sufficient condition to reject the effect of capital synergy. Such rejection only means that in assessing financial flows we ignore the effect of capital synergy.

Condition (64) is equivalent to condition (3). Thus condition (3) is equivalent to condition (60) of diversification neutrality. This means that Peccati's definition of future value implicitly assumes diversification neutrality.

## 9. A generalized definition of present value

It has been shown that the application of „classical" financial arithmetic rules means that the effects of diminishing marginal utility, capital synergy and principle of diversification are ignored in financial flow assessment. This observation encourages us to search for useful generalizations of the definition of future value.

Generalization of Peccati's definition of future value can be achieved by replacing condition (3) by the more general inequality (29). Moreover, inequality (29) is a necessary condition for retaining the principle of diversification. Thus, a generalized future value is defined as any function $FV : \varPhi \to \mathbb{R}$ fulfilling conditions (1), (2) and (29).

On the other hand, Gossen's first law is expressed by means of the present value function. Thus it is convenient to take the definition of the present value as a formal basis for financial arithmetic. Hence, the definition of generalized future value should be replaced by an equivalent definition of generalized present value, which is defined as any function $PV : \varPhi \to \mathbb{R}$ fulfilling conditions (33), (35) and (37).

Inequality (37) is a necessary condition for retaining the principle of diversification. It was shown that the principle of diversification is a sufficient condition for taking the synergy effect into account. On the other hand, the synergy effect is equivalent to the law of diminishing marginal utility (51). These observations lead us to undertake detailed studies of three variants of the definition of generalized present value:

• generalized present value $PV:\Phi\to\mathbb{R}$ given by any function fulfilling the conditions (33), (35) and (37),

• generalized present value $PV:\Phi\to\mathbb{R}$ given by any function fulfilling the conditions (33), (35) and (54),

• generalized present value $PV:\Phi\to\mathbb{R}$ given by any function fulfilling the conditions (33), (35) and

$$\begin{aligned}&\forall(t_1,C_1),(t_2,C_2)\in\Phi^+\forall\alpha\in[0:1]:\alpha PV(t,C_1)\\&+(1-\alpha)\cdot PV(t,C_2)\le PV(t,\alpha C_1+(1-\alpha)C_2)\end{aligned}\tag{66}$$

These may be the topics of future studies of prescriptive models for financial arithmetic.

## 10. Conclusions

The relationship presented above between the utility of wealth and present value shows the logical consistency of formal economics and financial models. Finding such relationships is especially important now when we have become participants in the global financial crisis triggered by financial management in isolation from the fundamental base created by the economy.

The notion of present value is subjective in nature, because it is identical to the utility of financial flow. In this situation, we obtain a theoretical foundation for the construction of behavioral finance models using subjective evaluation to determine present value.

Note that interest in the field of financial arithmetic is increasingly going beyond the domain of theory. In this situation, financial arithmetic should be treated as a subjective extension of the theory of interest rates, which is based on objective premises. In this paper, it was shown that this extension is important.

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