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COMPARISON OF THE GINI AND ZENGA INDEXES USING SOME THEORETICAL INCOME DISTRIBUTIONS ABSTRACT

The most common measure of inequality used in scientific research is the Gini index. In 2007, Zenga proposed a new index of inequality that has all the appropriate properties of an measure of equality. In this paper, we compared the Gini and Zenga indexes, calculating these quantities for the few distributions frequently used for approximating distributions of income, that is, the lognormal, gamma, inverse Gauss, Weibull and Burr distributions. Within this limited examination, we have observed three main differences. First, the Zenga index increases more rapidly for low values of the variation and decreases more slowly when the variation approaches intermediate values from above. Second, the Zenga index seems to be better predicted by the variation. Third, although the Zenga index is always higher than the Gini one, the ordering of some pairs of cases may be inverted.

Keywords: *inequality, Gini index, Zenga index*

1. Introduction

During the last few decades, inequalities have played an important role in many branches of social science, mainly sociology and economics, being one of the key issues in the discourse regarding the well-being of societies and individuals (see, e.g. [6, 11, 13]). Thus, there appears a question of how to measure these inequalities appropriately. There exist many inequality indexes in the literature, e.g. the most popular ones, namely, the Gini index, Theil index and Atkinson measure [11]. Out of these, the Gini index [5] is most frequently used and also best known to non-scientists. Recently, a new inequality index was proposed by Zenga [15]. It has all the properties

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that are usually required for inequality measures (e.g. the Pigou–Dalton transfer principle or income scale independence [2, 12]). In order to decide which measure of inequality is most suitable for a given subject, it would be helpful to investigate and compare the properties of different indexes.

The aim of this paper was to compare the values of the Gini and Zenga indexes by applying them to some theoretical distributions. As our main concern will be economic inequalities, we will concentrate on distributions that are commonly used as approximations of real income distributions [4, 10], that is, the lognormal, gamma, inverse Gauss, Weibull and Burr distributions. We will present how these inequality indexes depend on skewness and variation. In four cases (lognormal, gamma, inverse Gauss, Weibull distributions) skewness is a unique, monotonic function of the variation, thus the relation between any index and either skewness or variation may be easily recovered from the relation between the index and the other. In the case of the Burr distribution, a fixed value of skewness may correspond to various values of the variation and vice versa. Therefore, we should examine how these inequality indexes depend on the variation and skewness separately for this case. In what follows, we will also briefly analyze some examples of the generalized Burr distribution of the second kind and one instance of a bimodal distribution, to draw more conclusions regarding how inequality measures depend on the variation and skewness.

The paper is organized as follows. In the next section, we briefly introduce the concept of the Zenga index. Section 3 is devoted to the methodology of the research, that is, the quantities and relations used in the subsequent sections have been defined. In sections 4–8 we examine the Zenga index and its relation to the Gini index for five different classes of distributions. Section 9 presents an in depth overview of the combined results from sections 4–8, while the final section contains some conclusions and a summary of the paper.

2. The Zenga index of inequality

In a series of papers, Zenga (e.g. [7, 9, 14, 15]) proposed a measure of inequality based on the inequality curve $I(p)$, which is defined in terms of the lower and upper arithmetic means of a distribution.

This idea of measuring income inequality consists of comparing the arithmetic means of the incomes of two groups, called the lower and upper groups. The division of the ordered data into two groups is made by choosing a point of division. At one extreme, the lower group consists of only the lowest observation. At the other extreme, the upper group consists of only the greatest income. Let

$$\left\{ \left(x_j, n_j \right) : j = 1, \ldots, s; \qquad 0 \le x_1 < x_2 < \ldots < x_s; \qquad \sum_{j=1}^{s} n_j = N \right\}$$

denote the frequency distribution of a nonnegative random variable *X*. Next, let us divide this distribution into two parts, the lower and upper group, respectively: $\left\{ \left(x_1, n_1 \right), \left(x_2, n_2 \right), \ldots, \left(x_j, n_j \right) \right\}$, $\left\{ \left(x_j, n_j \right), \left(x_{j+1}, n_{j+1} \right), \ldots, \left(x_s, n_s \right) \right\}$. For each point of division, it is possible to define the lower mean $\bar{M}(p_j)$ and upper mean $\overset{+}{M}(p_j)$ of the divided distribution as follows:

$$\bar{M}\left(p_j \right) = \frac{1}{N_j} \sum_{i=1}^{j} x_i n_i, \qquad j = 1, \ \ldots, \ s \tag{1}$$

$$\overset{+}{M}\left(p_j \right) = \frac{1}{N - N_{j-1}} \sum_{i=j}^{s} x_i n_i, \qquad j = 1, \ldots, s \tag{2}$$

where: $N_j = \sum_{i=1}^{j} n_i$ and $p_j = \frac{N_j}{N}$.

Comparing $\bar{M}\left(p_j \right)$ and $\overset{+}{M}(p_j)$ using the index $U\left(p_j \right) = \bar{M}\left(p_j \right) / \overset{+}{M}\left(p_j \right)$, we get a point measure of the uniformity of the distribution. $U\left(p_j \right) \times$ 100 gives the lower mean as a percentage of the upper mean. The point inequality index is defined in terms of $U\left(p_j \right)$ as:

$$I\left(p_j \right) = 1 - U\left(p_j \right) \tag{3}$$

The synthetic inequality measure proposed by Zenga [15] is the following weighted arithmetic mean of the point measures $I(p_j)$:

$$Z = \sum_{j=1}^{s} I\left(p_j \right) \frac{n_j}{N} \tag{4}$$

Both $U(p_j)$ and $I(p_j)$ take values between 0 and 1 inclusively. In particular:

$$U\left(p_1 \right) = \frac{x_1}{M}, \qquad U\left(p_s \right) = \frac{M}{x_s}, \qquad I\left(p_1 \right) = 1 - \frac{x_1}{M}, \qquad I\left(p_s \right) = 1 - \frac{M}{x_s}$$

where M is the mean of all the observations. The Zenga index takes the value 0 in the case of no inequality. The shape of the $I\left(p_j\right)$ curve as a function of p_j is not constrained by fixed points (0, 0) and (1, 1), as in the case of the Lorenz curve [3, 8] (also, see [15] for many illustrative examples). It has been proven that the Zenga index is characterized by all the main properties that any inequality measure should satisfy (see e.g. [2]).

3. Comparison of the Zenga and Gini indexes

In what follows, we compare the values of the Gini and Zenga indexes, calculated for some (theoretical) income distributions. These distributions are standard distributions used to simulate distributions of real incomes, that is, the lognormal, Gamma, inverse Gauss, Weibull and Burr distributions.

Our aim here is to investigate the dependence of both the Gini and Zenga indexes on the variation and skewness for each family of distributions. For distributions characterized by a single shape parameter (i.e. the lognormal, gamma, inverse Gauss and Weibull distributions), the skewness can be obtained from the variation and vice versa and thus there should be a unique function describing each of the inequality indexes on the variation (and skewness), while in the case of distributions with two shape parameters (the Burr distribution here) we can expect, in general, different relations between these inequality indexes and the variation (for constant and different values of skewness) and vice versa. We will then examine the (eventual) differences in how these inequality indexes depend on the variation. We will also examine for each distribution the inter-dependence between the Zenga index and the Gini index.

We will adopt the following quantities as measures of variation and skewness. To measure variation, we will use the classical coefficient of variation:

$$V=\frac{\sqrt{\operatorname{Var}\left(X\right)}}{E\left(X\right)}$$

where $\operatorname{Var}\left(X\right)$ denotes the variance and $E\left(X\right)$ is the expected value of the random variable X.

We will measure the skewness in terms of the third standardized moment, that is:

$$As=\frac{E\left(\left(X-E\left(X\right)\right)^{3}\right)}{\operatorname{Var}\left(X\right)^{3/2}}$$

where $E(\cdot)$ again denotes the expected value.

It is well known (see e.g. [1]) that one of the most essential properties of an income distribution is the existence of only a small number of finite moments (e.g. the Burr distribution). Hence, the parametric space in which our investigations of the inequality indexes are possible is reduced (as the existence of the first moment is necessary to define both the Gini and Zenga indexes). Measuring variation and skewness in terms of the classical measures defined above reduces the parametric space even more. However, as we do not claim to investigate the whole range of the parameter space thoroughly, we will still use these classical quantities to make the results easily interpretable (the orders of magnitude of these parameters for typical income distributions are known).

As for calculating the Gini and Zenga indexes, and, correspondingly, the Lorenz curve and $I(p)$ curve, we performed numerical calculations according to the general formulas. For any continuous distribution D with probability density function $f(x)$, cumulative probability density function $F(x)=\int_{-\infty}^{x} f(y)dy$, and mean value $m=\int_{-\infty}^{\infty} yf(y)dy$, the Gini index may be calculated using:

$$G=\frac{1}{m}\int_{-\infty}^{\infty} F(x)\big(1-F(x)\big)dx \tag{5}$$

As for the Zenga index, the continuous counterpart of (4) reads as follows:

$$Z=\int_{-\infty}^{\infty} I(x)f(x)dx \tag{6}$$

where

$$I(x)=1-\frac{\int_{-\infty}^{x} yf(y)dy}{\frac{\int_{x}^{\infty} yf(y)dy}{1-F(x)}}$$

The Lorenz curve is expressed as a function of the cumulative distribution:

$$L\big(F(x)\big)=\frac{1}{m}\int_{-\infty}^{F^{-1}(x)} xf(x)dx \tag{7}$$

Having calculated the Lorenz curve, one may obtain the $I(p)$ curve using:

$$I(p) = \frac{p - L(p)}{p\left(1 - L(p)\right)} \tag{8}$$

with $p \in \langle 0, 1 \rangle$.

Because of the instability of numerical integration procedures in the case of some of the parameters considered here, we have also compared the results obtained with values computed using an *R* code, using discrete data generated from the distributions in question.

In the following sections, we present results for the distributions mentioned above. We present the density, mean value, variation and skewness for each distribution. Then we examine how the Gini and Zenga indexes depend on variation and skewness for given ranges of these measures. These ranges are, on one hand, chosen to reproduce appropriate magnitudes of the values of the variability and skewness of real incomes, and, on the other hand, restricted according to the existence of moments. Then we will present the inter-dependence between the Zenga index and the Gini index. As the Zenga index is always greater or equal to the Gini index [15]: $Z \geq G$, and both of them range between 0 and 1, $0 \leq G \leq 1$, $0 \leq Z \leq 1$, thus the points $Z(G)$ are in the upper triangle of the unit square (above the diagonal $(0, 0) \rightarrow (1, 1)$) [15]. For models with one scale parameter (here, the lognormal, Gamma, inverse Gauss and Weibull distributions), we can generally expect a unique $Z(G)$ curve (each point on this curve corresponds to a different value of the shape parameter). For distributions with two shape parameters (here, the Burr distribution) separate curves could, in principle, appear, each curve corresponding to a fixed value of one parameter and the points on such a curve corresponding to a value of the other parameter. One of our aims here is to investigate the character of the interdependence between the Gini and Zenga indexes.

4. The lognormal distribution

The lognormal distribution with parameters *m* and σ is characterized by the following probability density function:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{\left(\ln(x) - \mu\right)^2}{2\sigma^2}\right]$$

where μ is the scale parameter and σ – shape parameter.

The mean value for this distribution is: $m_{LN} = e^{\mu+\sigma^2/2}$, skewness: $A_{LN} = \sqrt{e^{\sigma^2}-1} \times \left(2+e^{\sigma^2}\right)$, and variation: $V_{LN} = \sqrt{e^{\sigma^2}-1}$. Both skewness and variation depend only on the scale parameter σ, thus the variation may be expressed as a function of skewness: $As_{LN} = V_{LN}\left(V_{LN}^2+3\right)$. Moreover, zero variation is obtained if and only if the skewness equals zero. The limit of zero skewness and variation is obtained when $\sigma^2 \to 0$; the opposite extreme of infinite skewness and variation is obtained when $\sigma^2 \to \infty$.

The shape of the probability density function (PDF) of the lognormal distribution for a few different values of skewness and variation (for fixed mean value, $m_{LN} = 3000$) are presented in Fig. 20a. Figure 1 presents how the Zenga and Gini indexes depend on skewness and variation.

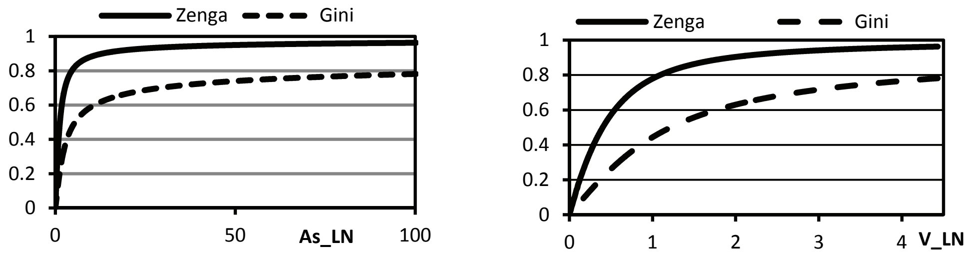


Fig. 1. Dependence of the Zenga and Gini indexes on skewness and variation for the log-normal distribution. Source: authors' calculations

As the only way of obtaining zero variation is by having all the values equal, it is obvious that when the variation equals 0, the inequality indexes should also equal 0. On the other hand, there is no general reason for inequality to be equal to 0 when the skewness is zero. However, in the case of the lognormal distribution, zero skewness also means zero variation, which means that the inequality indexes also equal zero. In the extreme limits of zero/infinite variation it is expected that both (Gini and Zenga) inequality indexes will take the values 0/1, respectively. Thus one can expect two fixed points on the $Z(G)$ curve, that is, (0, 0) and (0, 1). The $Z(G)$ curve for the values of σ considered here is presented in Fig. 2. We see that it has a clearly concave character.

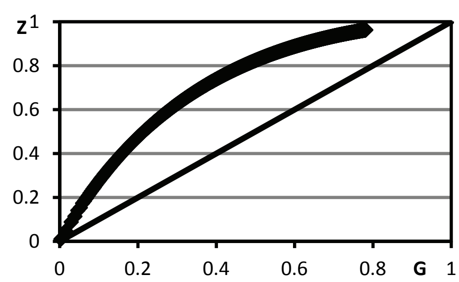
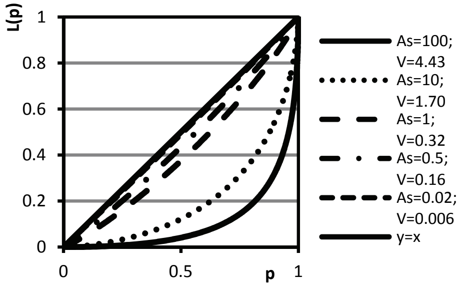


Fig. 2. Dependence of the Zenga index on the Gini index for the lognormal distribution Source: authors' calculations

Figure 3 presents the Lorenz curves and the $I(p)$ curves for a few chosen values of skewness and variation (these are the same values as those chosen for presenting $PDF(x)$ in Fig. 20a). It may be observed that inequality grows as skewness and variation grow.



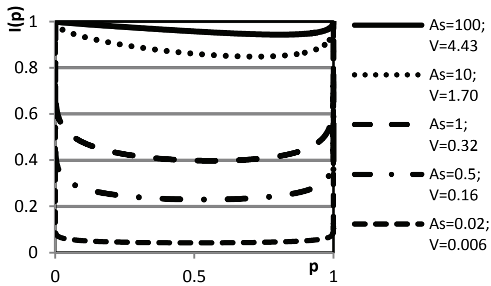


Fig. 3. The Lorenz and $I(p)$ curves for the log-normal distribution for a few chosen values of skewness and variation. Source: authors' calculations

For the Lorenz curve, the equality line is $y = x$ and the larger the deviation from that line (the closer to the line $y = 0$ for $x \in \langle 0, 1\rangle$, $y = 1$ for $x = 1$ – the complete inequality line), the larger inequality is. As for the $I(p)$ curve, the equality line is $y = 0$ for $x \in \langle 0, 1\rangle$, $y = 1$ for $x \in \{0, 1\}$. The larger the deviation from that line (the closer to the line $y = 1$, the complete inequality line), the larger inequality is. From the points on the $I(p)$ curve we can find a division of the population into two groups such that the relative difference between the mean wages in these two groups is minimized.

5. The Gamma distribution

The Gamma distribution with parameters α and β is characterized by the following probability density function:

$$f(x) = \frac{1}{\beta^{\alpha}} x^{\alpha-1} \frac{\exp(-x/\beta)}{\Gamma(\alpha)}$$

where β is the scale parameter and α – shape parameter.

The mean value for this distribution is $m_G = \alpha\beta$, skewness $As_G = 2/\sqrt{\alpha}$ and variation $V_G = 1/\sqrt{\alpha}$. Variation may be expressed via the skewness as $V_G = 0.5As_G$. As in the case of the lognormal distribution, it is not possible to obtain zero skewness with non-zero variation. Thus it may be expected that the measures of inequality will tend to zero as skewness tends to zero. This limit is obtained by letting $\alpha \to \infty$. At the opposite extreme, we will restrict ourselves to skewness in the range up to 2, as for $\alpha > 1$, the Gamma distribution has no mode and thus is not applicable to modeling real incomes. Examples of the probability density function of the Gamma distribution for a few chosen values of skewness and variation are presented in Fig. 20b.

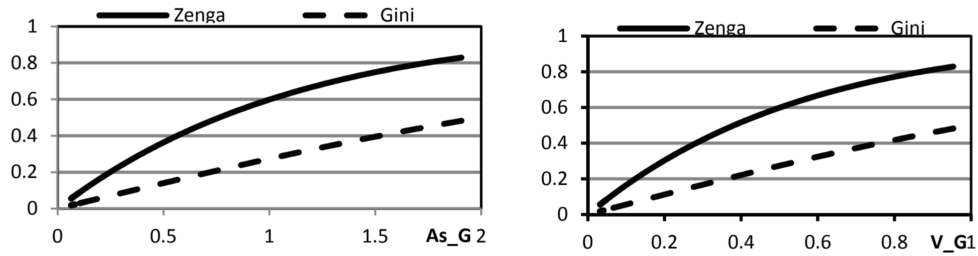


Fig. 4. Dependence of the Zenga and Gini indexes on skewness and variation for the Gamma distribution. Source: authors' calculations

Figure 4 shows how the Zenga and Gini indexes depend on skewness and variation, respectively. Similarly to the case of the lognormal distribution, both measures of inequality start from zero for zero values of skewness and variation, and grow as these characteristics increase. Note that both plots are identical in shape: as variation is a simple multiple of skewness. Hence, both plots are the same, only the scale of the horizontal axis is different. Similarly to the case of the lognormal distribution, the Zenga index increases faster. The $Z(G)$ curve for the values of α considered here is presented in Fig. 5. Again, it has a concave character.

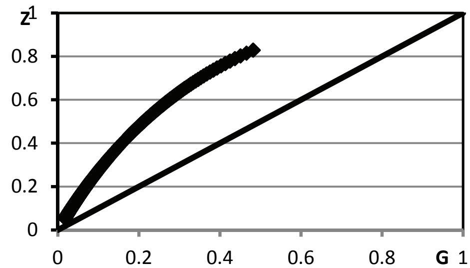


Fig. 5. Dependence of the Zenga index on the Gini index for the Gamma distribution. Source: authors' calculations

Figure 6 presents the Lorenz and $I(p)$ curves for a few chosen values of skewness and variation (these are the same values as those chosen for presenting $PDF(x)$ in Fig. 20b). As in the case of the lognormal distribution, it may be observed that inequality grows as skewness and variation increase, and – from the $I(p)$ curves – that this relation results from the overrepresentation of large incomes as the skewness increases.

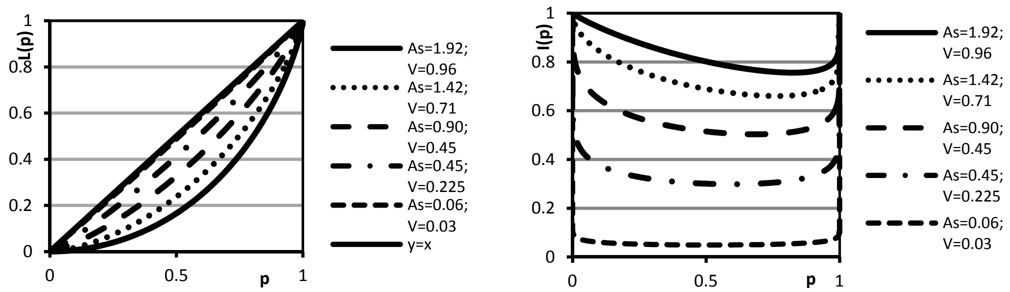


Fig. 6. The Lorenz and $I(p)$ curves for the Gamma distribution for a few chosen values of skewness and variation. Source: authors' calculations

6. The inverse Gauss distribution

The inverse Gauss distribution with parameters μ and λ is characterized by the following probability density function:

$$f(x)=\left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)$$

The mean value of this distribution is $m_{IG}=\mu$, skewness – $As_{IG}=3\sqrt{\mu/\lambda}$ and variation $V_{IG}=\sqrt{\mu/\lambda}$. Variation may be expressed via the skewness as $V_{IG}=(1/3)As_{IG}$.

Thus, as in the two previous cases, zero skewness is obtained only when the variation is zero. It may be expected that for both zero skewness and zero variation, the inequality indexes will be equal to zero; at the opposite extreme, for large values of skewness and variation, the Zenga and Gini indexes are expected to tend to their maximum value of 1. Figure 20c presents some examples of inverse Gauss distributions, for a few chosen values of skewness and variation. In Figure 7, one may observe the expected – as described in the previous section – behavior of the inequality indexes. As in the case of the Gamma distribution, both plots are identical in shape, with only the scale of the horizontal axis differing: since the variation of the inverse Gauss distribution is also a simple multiple of skewness. The $Z(G)$ curve for the values of the parameters considered here is presented in Fig. 8. It has a concave character. Finally, Figure 9 presents the Lorenz curves and $I(p)$ curves for a few values of skewness and variation, the same values as those for which the shapes of the distributions are illustrated in Fig. 20c.

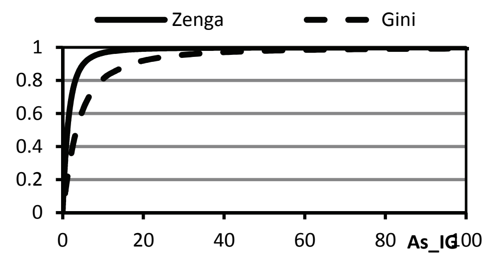


Fig. 7. Dependence of the Zenga and Gini indexes on skewness
and variation for the inverse Gaussian distribution. Source: authors' calculations

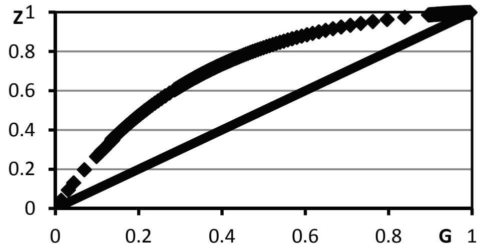


Fig. 8. Dependence of the Zenga index
on the Gini index for the inverse Gauss distribution.
Source: authors' calculations

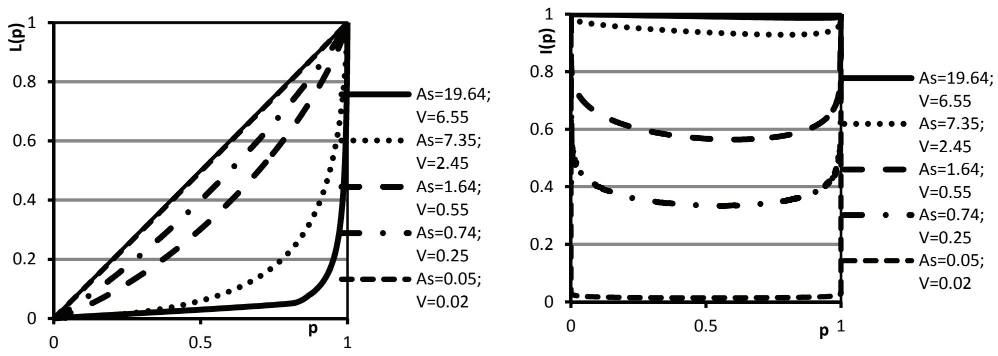


Fig. 9. The Lorenz and $I(p)$ curves for the inverse Gaussian distribution
for a few chosen values of skewness and variation. Source: authors' calculations

7. The Weibull distribution

The next distribution which happens to be used to model incomes is the Weibull distribution. Like the three previous distributions, it is in fact a two-parameter family of distributions. The Weibull distribution with parameters (α, β) is characterized by the following probability density function:

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left(-\left(\frac{x}{\beta}\right)^{a}\right)$$

The mean value is

$$m_W = \beta\Gamma\left(1+\frac{1}{\alpha}\right)$$

skewness:

$$As_W = \frac{2\Gamma^3\left(1+\frac{1}{\alpha}\right) - 3\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(1+\frac{2}{\alpha}\right) + \Gamma\left(1+\frac{3}{\alpha}\right)}{\left(\Gamma\left(1+\frac{2}{\alpha}\right) - \Gamma^2\left(1+\frac{1}{\alpha}\right)\right)^{3/2}}$$

and variation

$$V_W = \sqrt{\frac{\Gamma\left(1+\frac{2}{\alpha}\right)}{\Gamma^2\left(1+\frac{1}{\alpha}\right)} - 1}$$

where $\Gamma(z)$ denotes the gamma function.

From the above relations it may be noticed that – although variation cannot be expressed by skewness in a simple analytical form – they both depend on the same parameter. A detailed analysis shows that – as in the three previous cases – variation is a unique function of skewness (see Fig. 10). However, in contrast to the previous cases, zero variation is not obtained simultaneously with zero skewness (see the insert in Fig. 10). Zero variation is obtained in the limit $\alpha \to \infty$: $\lim_{\alpha\to\infty} V_W = 0$; in this limit, the skewness does not vanish: $\lim_{\alpha\to\infty} As_W = -1.14$. On the other hand, the skewness is zero when $\alpha = \alpha_0 \approx 3.602$; with this value of α, the variation equals $V_W(\alpha = \alpha_0) = 0.308$.

As we are interested in modeling incomes, we will restrict ourselves to right-hand-side skewness. Moreover, the Weibull probability density function has no mode for $\alpha \leq 1$. Thus we will restrict our analysis to the following range of the parameter α:

$\alpha \in \langle 1, 3.602\rangle$, and this corresponds to right-hand-side skewness with a maximum value of 2: $As_W \in \langle 0, 2)$.

In the whole range of the values of the parameter αconsidered here, we have positive (i.e. non-zero) variation ($V_W \in \langle 0.308, 1)$). Hence, in this case, as opposed to the three previous cases, we do not expect the inequality indexes to equal zero when the skewness is zero (for nonzero variation, by definition there is some inequality).

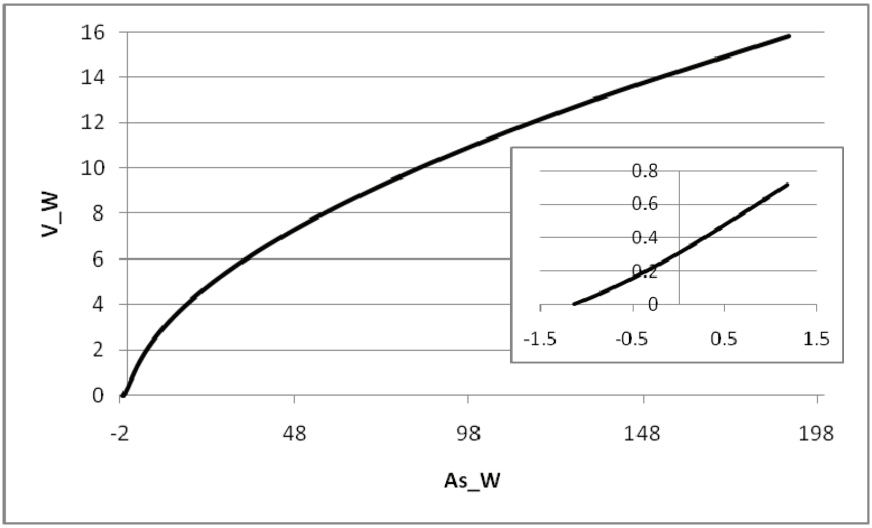


Fig. 10. The dependence of variation on the skewness for the Weibull distribution. Source: authors' calculations

Figure 20d presents the probability density function of the Weibull distribution for a few (positive or zero) values of skewness and variation. The special case of zero skewness and nonzero variation is also included here.

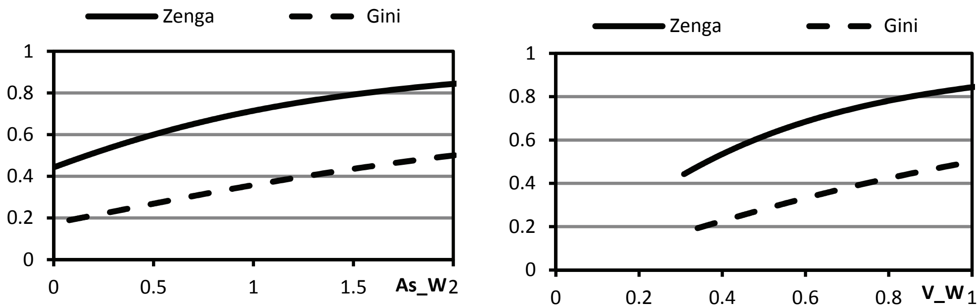


Fig. 11. Dependence of the Zenga and Gini indexes on skewness and variation for the Weibull distribution. Source: authors' calculations

Figure 11 illustrates how the Zenga and Gini indexes depend on skewness and variation. As expected, the inequality measures have nonzero values for zero skewness, which may be clearly seen in Fig. 11. At the opposite extreme, for large values of skewness and variation, both measures of inequality tend to the maximum value. The $Z(G)$ curve for the whole range of values of the parameters considered here is presented in Fig. 12.

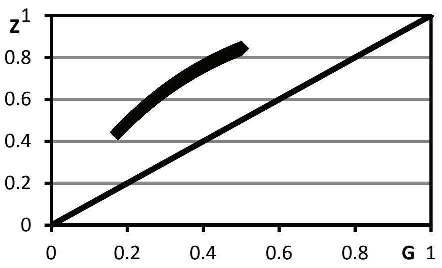


Fig. 12. Dependence of the Zenga index on the Gini index for the Weibull distribution. Source: authors' calculations

Although we have restricted ourselves to considering positive values of skewness, one subject might be worth briefly mentioning. Let us compare the values of the inequality indexes for skewness measures of the same absolute value, but of opposite signs. If inequality measures depend on the strength of skewness, they are functions of the absolute value of this measure (as the signs ascribed to right-hand-side and left-hand-side skewness are chosen arbitrarily).

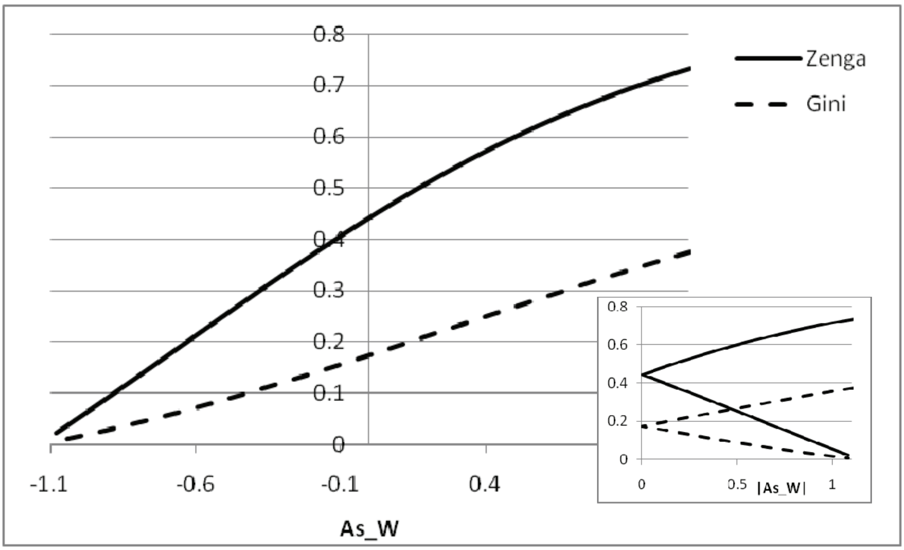


Fig. 13. Dependence of the Zenga and Gini indexes on the value of skewness and on the absolute value of skewness in the range of α where it changes sign. Source: authors' calculations

Figure 13 shows that neither index depends simply on the absolute value of skewness, but on its actual value. On the other hand, the dependence of both indexes on variation has a monotonic character over the whole range of variation, regardless of whether skewness is increasing or decreasing (see Fig. 14). This result supports the supposition that variation is a better predictor of inequality than skewness. However, as variation is still a unique function of skewness (and always has positive values), it is not possible to check whether measures of inequality will differ when the value of variation is fixed and skewness varied. This will be accomplished in the next section using another family of distributions.

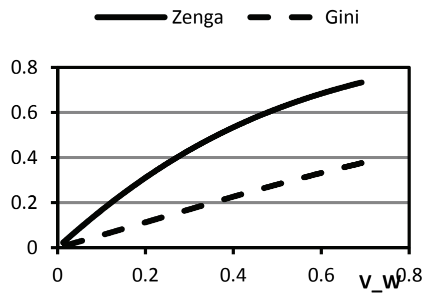


Fig. 14. Dependence of the Zenga and Gini indexes on variation in the range where skewness changes its sign. Source: authors' calculations

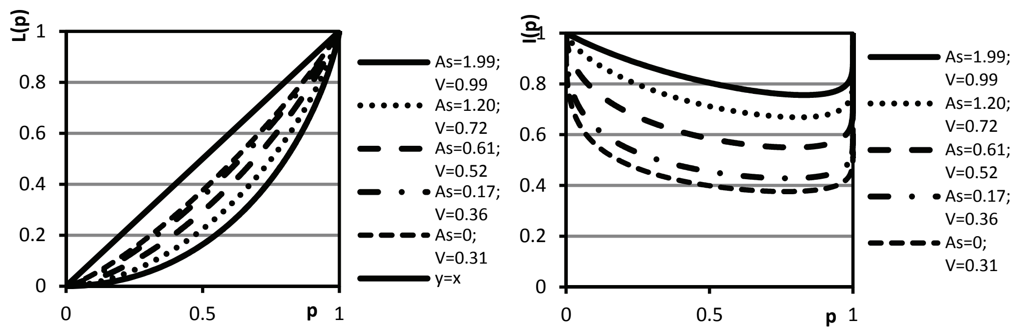


Fig. 15. The Lorenz and $I(p)$ curves for the Weibull distribution for a few chosen values of skewness and variation. Source: authors' calculations

Figure 15 presents the Lorenz and $I(p)$ curves for a few values of skewness and variation – the same as those considered in Figure 20d.

8. The Burr distribution

The last family of distributions considered in this paper is the Burr distribution, with probability density function:

$$f(x) = \frac{\alpha\tau}{\beta} \frac{\left(\frac{x}{\beta}\right)^{\tau-1}}{\left(1+\left(\frac{x}{\beta}\right)^{\tau}\right)^{\alpha+1}}$$

This distribution is characterized by one scale parameter β, and two shape parameters, α and τ. Thus, the relationship between skewness and variation cannot be described by a formula, and for any skewness there exists a continuum of possible variations and vice versa. The important characteristics of the Burr distribution with parameters (β, α, τ) may be expressed in terms of its moments. The rth moment exists if $\alpha\tau > r$. The first three moments r are given by:

$$\mu_{1B} = \frac{\alpha\beta\Gamma\left(\alpha - \frac{1}{\tau}\right)\Gamma\left(1+\frac{1}{\tau}\right)}{\Gamma(1+\alpha)}$$

$$\mu_{2B} = \frac{\alpha\beta^2\Gamma\left(\alpha - \frac{2}{\tau}\right)\Gamma\left(1+\frac{2}{\tau}\right)}{\Gamma(1+\alpha)}$$

$$\mu_{3B} = \frac{\alpha\beta^3\Gamma\left(\alpha - \frac{3}{\tau}\right)\Gamma\left(1+\frac{3}{\tau}\right)}{\Gamma(1+\alpha)}$$

where $\Gamma(z)$ denotes the Gamma function.

For $\alpha\tau > 3$ (the necessary condition for skewness to be properly defined) the characteristics of the Burr distribution are as follows:

- mean value:

$$m_B = \mu_{1B}$$

- skewness:

$$As_B = \frac{\mu_{3B} - 3\mu_{1B}\mu_{2B} + 2\mu_{1B}^3}{\left(\mu_{2B} - \mu_{1B}^2\right)^{3/2}}$$

• variation:

$$V_B = \frac{\left(\mu_{2B} - \mu_{1B}^2\right)^{1/2}}{\mu_{1B}}$$

The mode of the Burr distribution exists if $\tau > 1$. Thus we will restrict ourselves to ranges of the parameters that satisfy: $\tau > 1$, $\alpha > 3/\tau$. For this restricted range, the value of skewness tends to infinity as $\alpha\tau \to 3^+$, while the maximum value of variation $V_B^{\max}$ equals 1.73205.

In order to examine the influence of skewness and variation on inequality independently, we will fix the value of one of these measures and examine the behavior of the Zenga and Gini indexes while changing the other one of them.

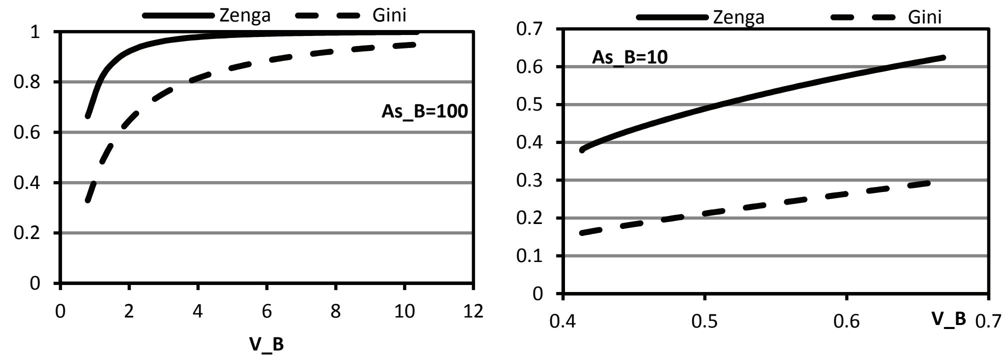


Fig. 16. Dependence of the Zenga and Gini indexes on variation for a fixed value of skewness, $As_B = 100$ and $As_B = 10$ for the Burr distribution. Source: authors' calculations

First, let us fix the value of skewness, initially at $As_B = 100$ and then $As_B = 10$. Figure 16 shows how the inequality indexes depend on variation within the range of variation for the given value of skewness. One may observe typical (as observed for the previous distributions) behavior: both curves are increasing and the Zenga index is greater than the Gini index. Figure 17 illustrates how the Zenga index depends on the Gini index, revealing its concave character for both $As_B = 100$ and then $As_B = 10$. It is interesting to note that when the points corresponding to both values of skewness are placed on one graph (Fig. 17 lower), we obtain a smooth curve with no noticeable transition between points belonging to different sets.

Now let us fix the value of variation and examine the dependence of measures of inequality on just skewness. Figure 18 presents results for variation equal to 1.5 and 1.0.

Unexpectedly, one can observe that in both cases inequality decreases (very slightly in most of the range) as skewness increases. This is in contrast to the previous distributions, where inequality is an increasing function of skewness. However, in those cases the skewness is an increasing function of variation.

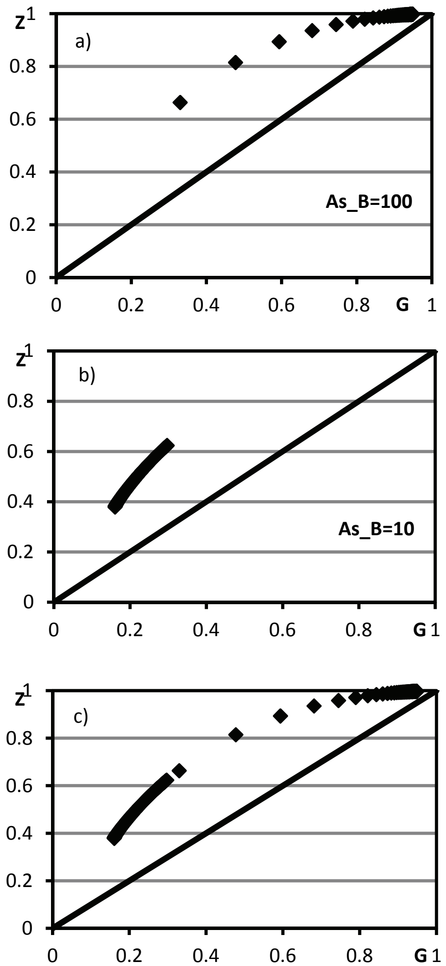


Fig. 17. Dependences of the Zenga index on the Gini index for the Burr distribution with fixed value of skewness, $As_B = 100$, $As_B = 10$, (a, b), and for both $As_B = 100$, and $As_B = 10$, (c). Source: authors' calculations

One might suspect that the positive dependence of inequality on skewness in the previous cases was simply an artifact of the positive dependence of skewness on variation. Let us also examine the dependence of the Zenga index on the Gini index (Fig. 19). The relations between these two measures of inequality are similar to the relationships observed in the previous cases. Plotting the points obtained for both values of variation on one graph reveals that although the points seems to lie on a smooth curve, it may be noticed that there is a slight change in the curve when passing from one set to the other (see Figure 19 below, where fits with quadratic functions are added for both sets of points).

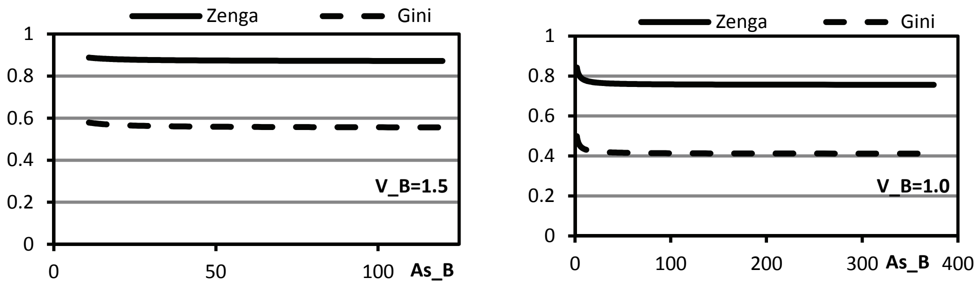


Fig. 18. Dependence of the Zenga and Gini indexes on skewness for a fixed value of variation, $V_B = 1.5$ and $V_B = 1.0$ for the Burr distribution. Source: authors' calculations

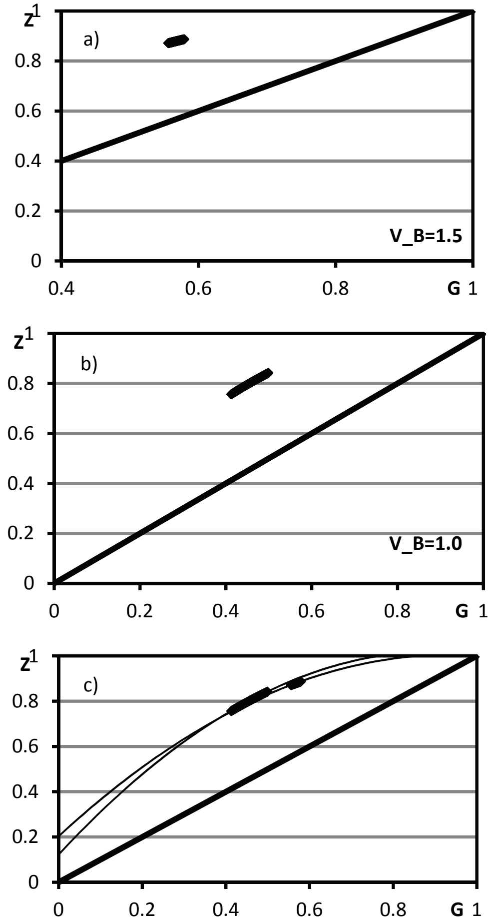


Fig. 19. Dependence of the Zenga index on the Gini index for the Burr distribution with fixed value of variation, $V_B = 1.5$, $V_B = 1.0$ (a, b), and for both $V_B = 1.5$ and $V_B = 1.0$ (c). Source: authors' calculations

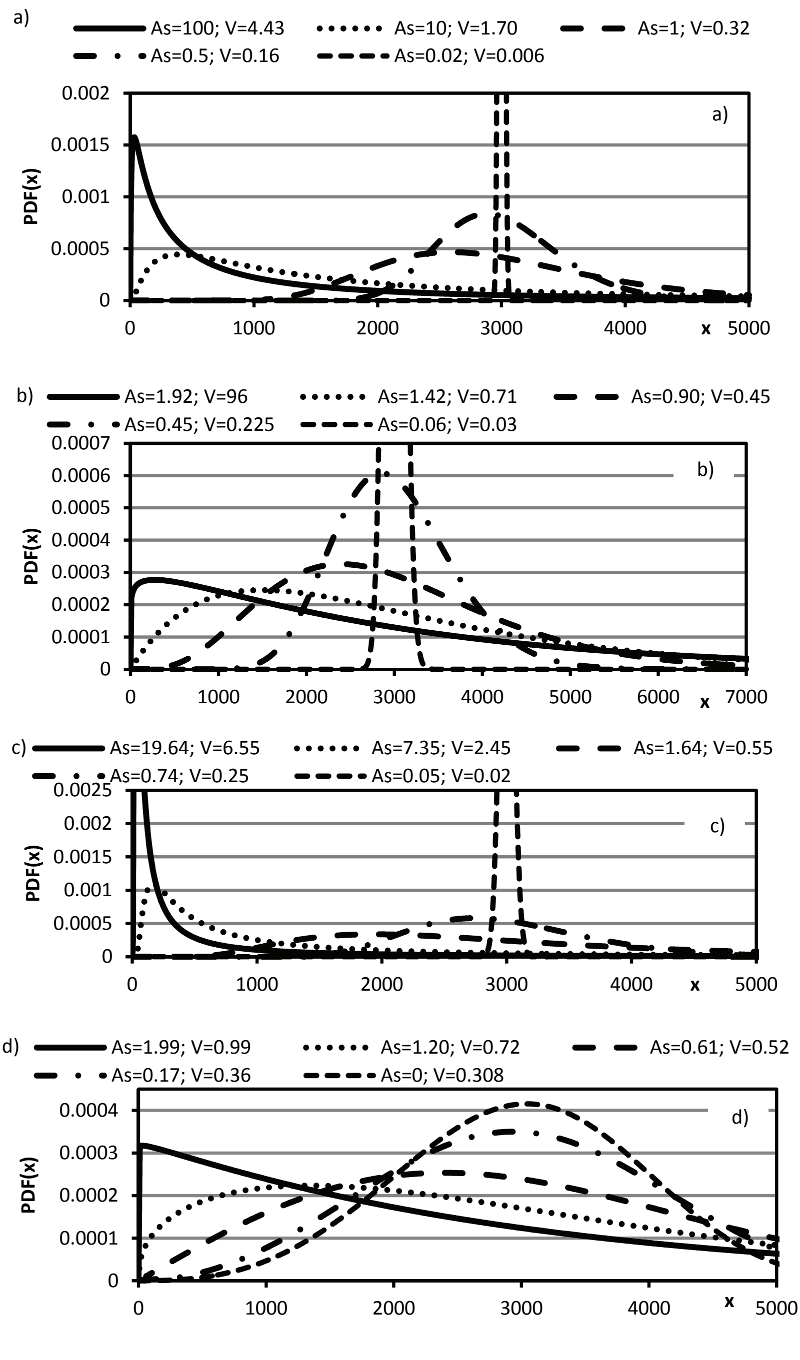


Fig. 20. Shape of the probability density function of the a) log-normal, b) Gamma, c) inverse Gaussian, d) Weibull distribution for a few values of skewness and variation. Source: authors' calculations

9. Dependence of the Zenga and Gini indexes on variation and skewness

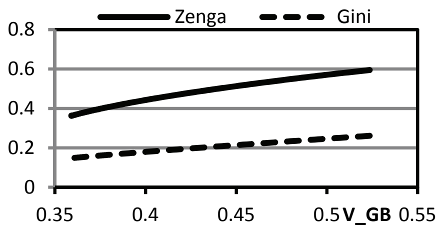
In the preceding section we made use of the fact that for the Burr distribution skewness is not uniquely determined by variation. Thus we were able to investigate how the inequality measures depend on these characteristics of a distribution separately. It appears that both the Zenga and Gini indexes are strongly dependent on variation, but not on skewness. Moreover, their dependence on skewness may be even counter-intuitive, that is to say within some ranges of the parameters they may be decreasing functions of skewness.

To explore more deeply how the Zenga and Gini indexes depend on variation and skewness, let us use two other distributions, the generalized Burr distribution and a bimodal distribution, obtained using a mixture of lognormal distributions.

The generalized Burr distribution is a 4-parameter (a, b, p, q) distribution, of which special cases are the generalized gamma, Singh–Maddala, Burr and Weibull distributions. From a practical point of view, it is impossible to explore the whole range of possible parameter values, neither is it a goal here to investigate the properties of the generalized Burr distribution. We will use in a limited way only one of the properties of this family of distributions, namely the possibility of changing both variation and skewness simultaneously and independently. In particular, we may change the values of the parameters in such a way that variation is a decreasing function of skewness. Such a case will give deeper insight into the relation between variation and skewness on one hand and measures of inequality on the other.

As an example, let us take the following values: $a = 20$, $b = 3000$, $q = 0.2$ with p ranging from 0.1 to 100. For such parameters, the skewness is a decreasing function of variation.

Figure 21 clearly shows that both the Zenga and Gini indexes are increasing functions of variation but not skewness.



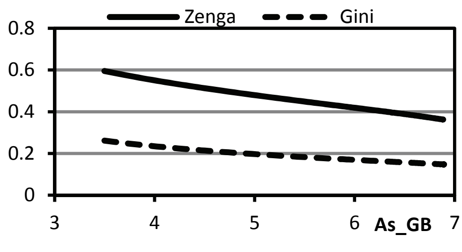


Fig. 21. Dependences of the Zenga and Gini indexes on variation (left) and skewness (right) for the generalized Burr distribution and $a = 20$, $b = 3000$, $q = 0.2$ with p ranging from 0.1 to 100. Source: authors' calculations

Continuing our investigation into the dependence of these inequality measures on variation and skewness, let us sum up all the results for each of the distributions discussed in the previous sections. Figure 22 presents the curves illustrating the Zenga and Gini indexes as functions of variation for the distributions considered above, over the whole range of variation examined in previous sections.

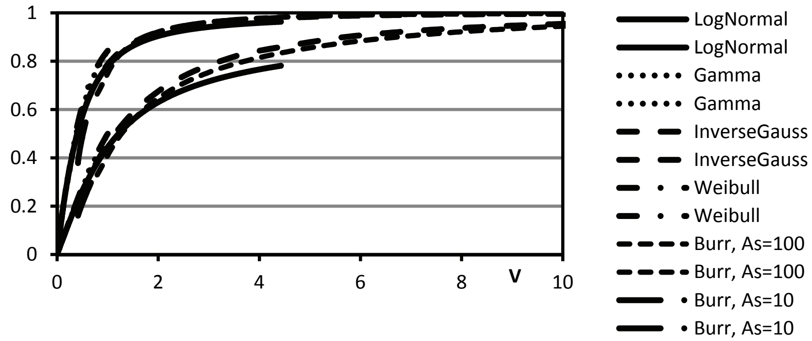


Fig. 22. Dependences of the Zenga and Gini indexes on variation for various distributions.
Source: authors' calculations

Note that some of the curves are nearly indistinguishable. However, it is clear that neither of the two inequality indexes are uniquely determined by the value of variation. One may suppose, that the Zenga index is determined by variation to a higher degree, as the maximum difference observed here (between the Burr and Weibull distributions, for variation $V \approx 0.5$) equals 0.26, while for the Gini index this maximum difference exceeds 0.32. Also, for variation $V \approx 4.4$, the difference between the values of the Zenga index for lognormal and inverse Gauss distributions does not exceed 0.02, while in the case of the Gini index this difference is more than 5 times larger, exceeding 0.1.

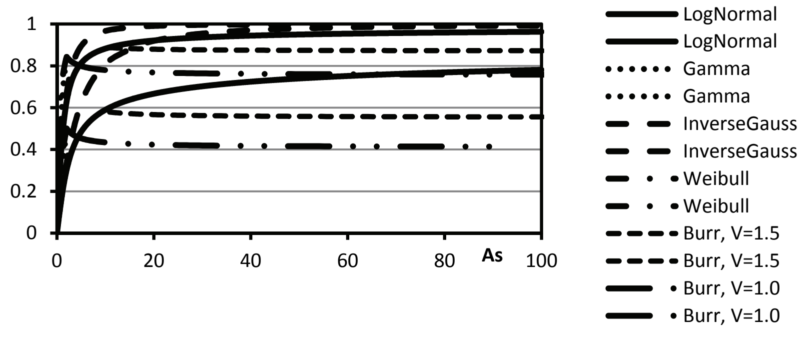


Fig. 23. Dependences of the Zenga and Gini indexes on skewness for various distributions.
Source: authors' calculations

Figure 23 illustrates the dependence of the Zenga and Gini indexes on skewness for all the distributions considered in the previous sections. One may observe that the dependence, in both cases – of the Zenga and Gini indexes – clearly varies according to the family of distributions. It may be concluded that the value of skewness is a very poor predictor of the value of the inequality index. It is a good predictor of the index only in cases where skewness is an increasing function of variation, as in the cases of the lognormal, Gamma and inverse Gauss distributions.

Concentrating on variation as a predictor of these inequality indexes, let us illustrate in a different way the aggregated results describing the dependence of the inequality measures on variation for the distributions considered. Figure 24 presents plots of the Zenga and Gini indexes against variation, without making any distinction between the different distributions considered.

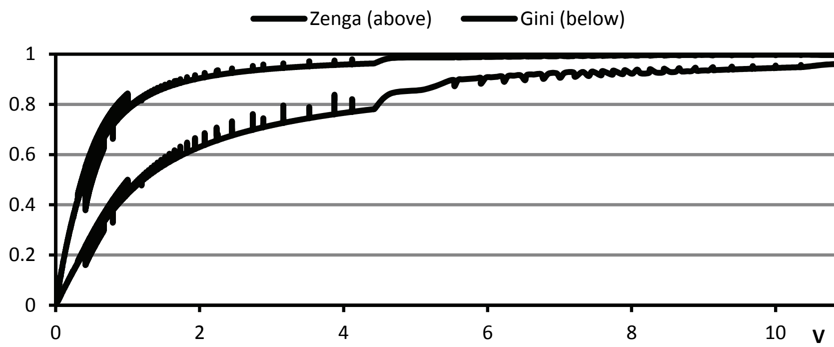


Fig. 24. Dependences of the Zenga and Gini indexes on variation for all the distributions considered. Source: authors' calculations

One may see that neither of the Zenga and Gini indexes are strictly increasing functions of variation. However, there is a very strong tendency for the indexes to increase as variation increases, even stronger for the Zenga index than for the Gini one. Fluctuations in the values of the Gini index around the trend curve are stronger than such fluctuations in the Zenga index; which is even more apparent when percentage fluctuations are taken into account.

This non-monotonic character of these inequality measures as functions of variation may also be observed when examining a bimodal distribution obtained as a mixture of two lognormal distributions. Taking a linear superposition of two lognormal distributions with parameter values: $(m, \sigma) = (8, 0.5)$ and $(9, 0.18)$ and changing their relative share in the mixture, we can change variation, skewness, and, as expected, also the values of the Zenga and Gini indexes. The dependence of these two measures of inequality on variation is presented in Fig. 25. The dependence of skewness on variation is also shown to confirm the earlier conclusion that these measures of inequality are not increasing functions of the skewness of a distribution. One may observe the non-monotonic character of both the Zenga and Gini indexes as functions of variation. However, there is a clear tendency for the inequality index to increase as the variation increases.

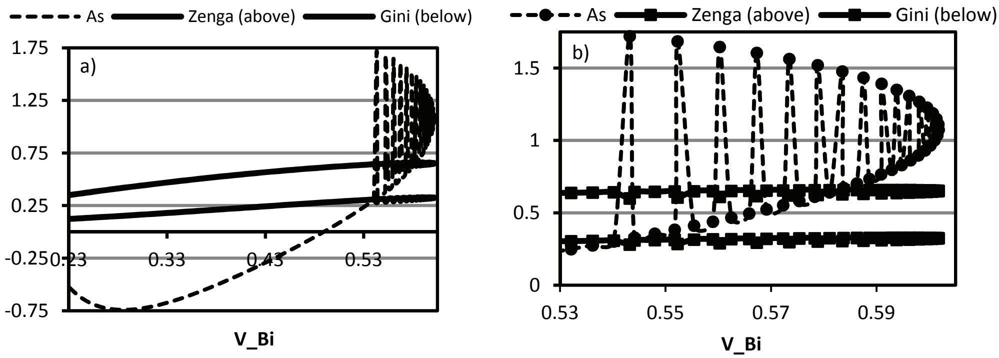


Fig. 25. The dependences of skewness, the Zenga index and Gini index with variation for a bimodal distribution. Source: authors' calculations

Figure 25b shows an enlargement of Fig. 25a. It may be clearly seen that the local minima of inequality measures do not correspond to smaller values of skewness, but rather the opposite.

10. Conclusions

The above investigations clearly do not explain the quantitative difference between the Zenga and Gini measures of inequality. One cannot be treated as a function of the other. There are three main differences that may be observed within these limited examinations of both indexes.

First, the Zenga index is less "indulgent" to inequality. Compared to the Gini index, it increases more rapidly for small deviations from perfect equality, and decreases less rapidly when a small transfer from one single owner of all the sources of income is made. In all situations the Zenga index is greater (or equal) to the Gini index. However, in the middle range of variation values the changes in the values of both indexes after such a transfer do not differ as much as when the initial level of variation is very small or very large.

Second, the Zenga index seems to be better predicted by the value of variation. However, this suggestion, since it is based only on some chosen distributions, is not conclusive, and may be falsified when more distributions and/or theoretical considerations are taken into account.

Third, although the Zenga index is always greater than the Gini one, the ordering of some pairs of distributions according to the former and the latter may be reversed. This is a consequence of the fact that the Zenga index is not a monotonic function of the Gini index. If the values of both indexes were increasing functions of variation, this would be impossible. However, as their dependence on variation is not monotonic, there is a possibility that the relation between the Zenga and Gini index is not monotonic. Figure 26 shows the dependence of the Zenga index on the values of the Gini index firstly for a collection of unimodal distributions and secondly for a bimodal distribution. Whenever the Zenga index is decreasing in the value of the Gini index, the ordering of two distributions according to these two measures of inequality is reversed.

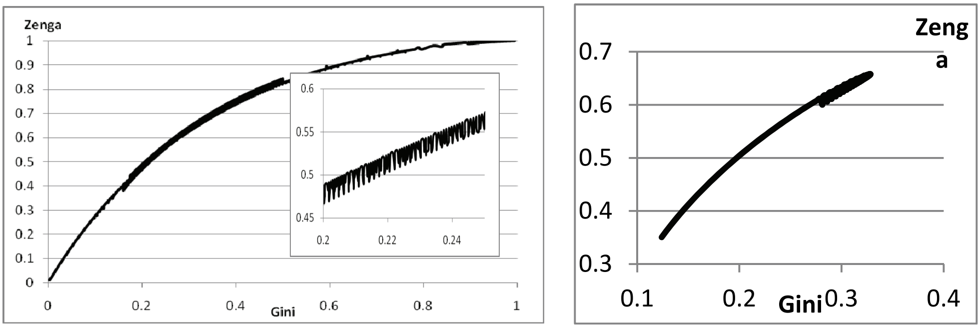


Fig. 26. Relation between the Zenga index and Gini index for a) unimodal distributions and b) bimodal distribution. Source: authors' calculations

However, it is not clear under which circumstances this order becomes inverted, and we have not yet identified any special features of distributions that the Zenga index is more sensitive to, or vice versa.

Besides the values of the Zenga and Gini indexes themselves, there is still the very important question of their interpretation and the information that can be obtained from them. Despite the fact that the Lorenz curve can be transformed into the $I(p)$ curve and vice versa, depending on the goal of research, it is more convenient to use just one of these curves.

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