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ALGORITHM FOR THE STOCHASTIC GENERALIZED TRANSPORTATION PROBLEM**

The equalization method for the stochastic generalized transportation problem has been presented. The algorithm allows us to find the optimal solution to the problem of minimizing the expected total cost in the generalized transportation problem with random demand. After a short introduction and literature review, the algorithm is presented. It is a version of the method proposed by the author for the nonlinear generalized transportation problem. It is shown that this version of the method generates a sequence of solutions convergent to the KKT point. This guarantees the global optimality of the obtained solution, as the expected cost functions are convex and twice differentiable. The computational experiments performed for test problems of reasonable size show that the method is fast.

Keywords: *generalized transportation problem, stochastic programming, convex programming, equalization method*

1. Introduction

The generalized transportation problem arises in many real-life applications. It is a special case of the generalized flow problem. More about the generalized flow problem and its possible applications may be found in [1]. The problem is also briefly discussed in [9]. A polynomial algorithm for the generalized minimum cost flow problem is presented in [24]. Combinatorial algorithms for the generalized circulation problem are presented in [12]. Interesting applications of generalized flows were considered in [18]. Some issues concerning generalized networks may be also found in [11].

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The generalized transportation problem was considered e.g. in [7], [8] and [16]. In [19], a variant of the transportation problem was considered, where additional constraints of the GTP type were included. In [6], the authors considered one of its possible applications in logistic networks, where complaints are involved in the distribution process. The influence of the complaints ratio on the complexity of the resulting optimal distribution network was analyzed.

The stochastic generalized transportation problem arises when the demands are not deterministic but given as random variables with known distributions. In such a case, we are interested in minimizing the expected value of the total transportation cost. The particular case of the stochastic transportation problem has been analyzed in [3–5, 22, 23]. In all of those papers, the equalization method was considered, which is the starting point for the algorithm described in this paper. Other methods for solving the stochastic transportation problem were considered, e.g. in [13–15]. The convergence of the equalization method for nonlinear and stochastic transportation problems was proved in [3] and [4]. The convergence of the version for nonlinear GTP was proved in [2]. For the sake of completeness, the proof of the convergence of the method for SGTP is presented in section 4. It uses some of the concepts presented in [2–4]. In [20], the forest iteration method for the stochastic problem was proposed. Finally, a variant of the latter method, the A-forest iteration method for the stochastic generalized transportation problem (SGTP) was presented in [21].

In this paper, we present a new method for the SGTP. In section 2, the problem is formulated, while sections 3 and 4 contain the version of the new algorithm for the SGTP and analysis of its convergence, respectively. In section 5, computational experiments are described. Section 6 contains main conclusions.

2. Formulation of the problem

Let us start with the ordinary generalized transportation problem. A uniform good is transported from m supply points to n destination points. On the way, the amount of a good changes, i.e. the amount delivered to demand point j from supply point i is equal to $r_{ij}x_{ij}$, where x_{ij} is the amount of the good that leaves supply point i and r_{ij} is the respective *reduction ratio*. The unit transportation costs c_{ij} are constant, the demand b_j of each demand point j has to be satisfied and the supply a_i of each supply point I cannot be exceeded. Thus, the model has the following form:

$$\min f(x) = \sum_{i=1}^{m} c_{ij} x_{ij}$$

s.t.

$$\sum_{i=1}^{m} r_{ij} x_{ij} = b_j, \qquad j = 1, ..., n$$

$$\sum_{j=1}^{n} x_{ij} \le a_i, \qquad i = 1, ..., m$$

$$x_{ij} \ge 0, \qquad i = 1, ..., m, \qquad j = 1, ..., n \tag{1}$$

Observe that this is very similar to the ordinary transportation problem. The difference appears in the first constraint, where the reduction ratios are included.

In the stochastic version, the demands b_j are not deterministic but are given as continuous random variables X_j with the density functions φ_j. The unit surplus cost $s_j^{(1)}$ and the unit shortage cost $s_j^{(2)}$ are defined for each destination point j. As the total amount of the good delivered to any destination point must be non-negative, the function describing the expected extra cost for destination j takes the form

$$f_j(x_j) = s_j^{(1)} \int_0^{x_j} (x_j - t)\varphi_j(t)dt + s_j^{(2)} \int_{x_j}^{\infty} (t - x_j)\varphi_j(t)dt \tag{2}$$

which can easily be transformed into the form

$$f_j(x_j) = s_j^{(2)} \left(E(X_j) - x_j \right) + (s_j^{(1)} + s_j^{(2)}) \int_0^{x_j} \Phi_j(t)dt \tag{3}$$

where Φ_j is the cumulative distribution function of the demand at destination j.

Finally, the SGTP has the following form:

$$\min f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} f_j(x_j)$$

s.t.

$$\sum_{i=1}^{m} r_{ij} x_{ij} = x_j, \qquad j = 1, ..., n$$

$$\sum_{j=1}^{n} x_{ij} \le a_i, \qquad i = 1, ..., m$$

$$x_{ij} \ge 0, \qquad i = 1, ..., m, \qquad j = 1, ..., n \tag{4}$$

It is straightforward to see that the first two derivatives of the expected cost functions are

$$f_j'(x_j) = -s_j^{(2)} + (s_j^{(1)} + s_j^{(2)})\Phi_j(x_j) \tag{5}$$

and

$$f_j''(x_j) = (s_j^{(1)} + s_j^{(2)})\phi_j(x_j) \tag{6}$$

so each function f_j is twice differentiable and convex. This means that the equalization method described in [2] (applicable to convex functions) may be applied.

3. Algorithm for SGTP

Let us introduce m additional variables $x_{i,n+1}$. Let $c_{i,n+1} = 0$, $r_{i,n+1} = 1$ and $f_{n+1}(x_{n+1}) = 0$ for $i = 1, \ldots, m$. Then problem (4) can be rewritten as

$$\begin{aligned}
&\min f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n+1} c_{ij}x_{ij} + \sum_{j=1}^{n+1} f_j(x_j) \\
&\text{s.t.} \\
&\sum_{i=1}^{m} r_{ij}x_{ij} = x_j, \quad j = 1, \ldots, n+1 \\
&\sum_{j=1}^{n+1} x_{ij} = a_i, \quad i = 1, \ldots, m \\
&x_{ij} \geq 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n+1
\end{aligned} \tag{7}$$

The dual problem has the form

$$\begin{aligned}
&\min g(x,u) = f(x) + \sum_{i=1}^{m} u_i\Big(a_i - \sum_{j=1}^{n+1} x_{ij}\Big) - \sum_{i=1}^{m}\sum_{j=1}^{n+1} t_{ij}x_{ij} \\
&\text{s.t.} \\
&c_{ij} + r_{ij}f_j'(x_j) - u_i - t_{ij} = 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n+1 \\
&t_{ij} \geq 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n+1
\end{aligned} \tag{8}$$

where u_i and t_{ij} are the dual variables. Thus the KKT optimality conditions are

$$c_{ij} + r_{ij} f_j'(x_j) - u_i \geq 0, i = 1, ..., m, \; j = 1, ..., n+1, \; x_{ij} = 0 \tag{9}$$

$$c_{ij} + r_{ij} f_j'(x_j) - u_i = 0, i = 1, ..., m, \; j = 1, ..., n+1, \; x_{ij} > 0. \tag{10}$$

The following version of the equalization method converges to the KKT point of problem (7), so also of problem (4) (see section 4). As it is a convex programming problem, the resulting point is an optimal solution.

Agorithm 1: The equalization method for SGTP

1. Initial solution. Set

$$x_{i,n+1} = a_i, \qquad i = 1, ..., m \tag{11}$$

$$x_{ij} = 0, \qquad i = 1, ..., m, \qquad j = 1, ..., n \tag{12}$$

Calculate the initial values of the partial derivatives:

$$k_{ij} = c_{ij} + r_{ij} f_j'(0), \qquad i = 1, ..., m, \qquad j = 1, ..., n \tag{13}$$

$$k_{i,n+1} = 0, \qquad i = 1, ..., m \tag{14}$$

Assume the accuracy level $\varepsilon > 0$. Go to step 2.

2. Checking optimality. For each i calculate

$$v_i = \min\{k_{ij} \mid j = 1, ..., n+1\} \tag{15}$$

and

$$w_i = \max\{k_{ij} \mid j = 1, ..., n+1, \; x_{ij} > 0\} - v_i \tag{16}$$

If

$$\left(\sum_{i=1}^{m} w_i^2 \right)^{1/2} < \varepsilon \tag{17}$$

then STOP, the desired accuracy level has been reached. Otherwise go to step 3.

3. Changing the solution. Choose i^* such that

$$w_{i*} = \max\{w_i \mid i = 1, ..., m\} \tag{18}$$

Choose j^* and j^{**} such that

$$k_{i*j*} = \max\{k_{i*j} \mid j = 1, ..., n+1, x_{i*,j} > 0\} \tag{19}$$

and

$$k_{i*j**} = \min\{k_{i*j} \mid j = 1, ..., n+1\} \tag{20}$$

Let

$$\delta^-(\lambda) = f'_{j*}(x_{j*}) - f'_{j*}(x_{j*} - \lambda) \tag{21}$$

and

$$\delta^+(\lambda) = f'_{j**}(x_{j**} + \lambda) - f'_{j**}(x_{j**}). \tag{22}$$

Let λ^* be the solution to the equation

$$\delta^-(\lambda) + \delta^+(\lambda) = w_{i*} \tag{23}$$

If

$$\lambda^* > x_{i*j*} \tag{24}$$

then set

$$\lambda^* := x_{i*j*} \tag{25}$$

Set

$$k_{ij*} := k_{ij*} - r_{ij*}\delta^-(\lambda^*), \quad i = 1, ..., m \tag{26}$$

$$k_{ij**} := k_{ij**} + r_{ij**}\delta^+(\lambda^*), \quad i = 1, ..., m \tag{27}$$

$$x_{i^*j^*} := x_{i^*j^*} - \lambda^* \tag{28}$$

$$x_{i^*j^{**}} := x_{i^*j^{**}} + \lambda^* \tag{29}$$

Go back to step 2.

Let us briefly discuss the steps of the algorithm. It will be easier to imagine the transportation tableau as used when solving the ordinary transportation problem (one of the differences is that the marginal costs in the tableau change in every iteration). The last column of the tableau is the column of the so-called fictitious destination.

In step 1, we place all the units that are supposed to be transported in the last column, see Eqs. (11), (12). This allows us to simplify the calculation of the marginal costs (Eqs. (13), (14)).

In step 2, we calculate the optimality indicators, as Eqs. (15), (16) correspond to the left hand sides of the KKT conditions (9), (10). Of course, we cannot be sure that the respective values will be equal to 0, thus we only check whether they are close enough to 0 (condition (17)).

In step 3, we change the solution if it is not optimal. Equations (18)–(20) define the search direction. In the transportation tableau this corresponds to moving the good from the most expensive to the cheapest cell in the row where the optimality indicator differs most from 0. Equations (23)–(25) provide a simple way to calculate the optimal step length. As the changes in flows relate to only two columns of the transportation tableau, only two summands in the second part of the objective function (model (4)) are involved in these changes. This means that there is no need to calculate the whole objective function at every step, but only to analyze the changes in these two summands together with some linear term. In the general case, the solution to equation (23) can be found using the Newton algorithm (we can use this method as the function on the left hand side of the equation is always differentiable). It is worth noticing that this equation has a unique root when at least one of the functions $\delta^+(\lambda)$ and $\delta^-(\lambda)$ is increasing, i.e. when the respective density function is positive. Even if this is not the case (which is highly unlikely in real-life applications), the set of the roots takes the form of an interval, as $\delta^+(\lambda)$ and $\delta^-(\lambda)$ are non-decreasing functions. There are also some special cases when we are able to derive a simple formula for the root of Eq. (23). For instance, if all the demands are uniformly distributed, then Eq. (23) is linear. On the other hand, when all the demands have exponential distributions with the same parameter, then the equation reduces to a quadratic equation with one positive and one negative root, where the negative root has to be excluded. Equations (24), (25) guarantee that the length of the step will not leave the domain. Finally, using again the fact that the objective function is separable and only two summands are involved in the operation of changing the solution, we can quickly change the solution and partial derivatives, as we can restrict ourselves to only two columns of the transportation tableau, see Eqs. (26)–(29).

4. The convergence of the method

Let A denote the algorithmic map of the equalization method. Let B be the map finding the search direction and let C be the map finding the next feasible solution when the search direction is already given. Let X denote the set of feasible solutions to the SGTP problem and let D denote the set of search directions. It is straightforward to check that both X and D are compact sets (moreover, D is finite for fixed m and n).

Now we are going to prove that both mappings B and C are closed.

Lemma 4.1: The algorithmic map $B : X \to D$ in the equalization method for the SGTP is closed on X.

Proof: The sets D and X are compact. Moreover, we know that B: $X \to D$ is a point-set map. It remains to show that the conditions $x^{(k)} \in X$, $x^{(k)} \to x$, $d^{(k)} \in B(x^{(k)}$, $d^{(k)} \to d$ imply that $d \in B(x)$ (see Definition 7.2.2 in [10], p. 321).

As D is finite, starting from some k, the conditions $d^{(k)} \in B(x^{(k)})$ and $d^{(k)} \to d$ are equivalent to $d \in B(x^{(k)})$ (this follows directly from the fact that for almost all k, the equality $d_k = d$ must hold). Let us choose some $\varepsilon > 0$. Let $k_{0,1}$be the index starting from which all the elements of the sequence $\{d^{(k)}\}$are equal to d. As the sequence $\{x^{(k)}\}$ is convergent, there exists an index $k_{0,2}$ such that for $k > k_{0,2}$, the inequality $\left\|x^{(k)} - x\right\| < \varepsilon$ holds. Let $k_0 = \max\left\{k_{0,1}; k_{0,2}\right\}$. Then for $k > k_0$ we have:

$$\left|x_{ij}^{(k)} - x_{ij}\right| < \varepsilon, \qquad i = 1, ..., m, \qquad j = 1, ..., n \tag{30}$$

As all the derivatives are finite in the domain (and thus the Lipschitz condition is satisfied), we have for some absolute constant L, depending only on the parameters of the problem:

$$\left|k_{ij}^{(k)} - k_{ij}\right| < L\varepsilon, \qquad i = 1, ..., m, \qquad j = 1, ..., n \tag{31}$$

From the above it follows in turn that:

$$\left|w_i^{(k)} - w_i\right| < 2L\varepsilon, \qquad i = 1,\dots,m \tag{32}$$

$$\left|\max_{i=1,\dots,m}\left\{w_i^{(k)}\right\} - \max_{i=1,\dots,m}\left\{w_i\right\}\right| < 4L\varepsilon \tag{33}$$

For ε sufficiently small the index i^* will be chosen from the same set as in the case of the solution x. Having chosen the index i^*, we obtain the following two inequalities:

$$\left|\max_{\substack{j=1,\dots,n \\ x_{i^*j}^{(k)}>0}}\left\{k_{i^*j}^{(k)}\right\} - \max_{\substack{j=1,\dots,n \\ x_{i^*j}^{(k)}>0}}\left\{k_{i^*j}\right\}\right| < L\varepsilon \tag{34}$$

and

$$\left|\min_{j=1,\dots,n}\left\{k_{i^*j}^{(k)}\right\} - \min_{j=1,\dots,n}\left\{k_{i^*j}\right\}\right| < L\varepsilon \tag{35}$$

For ε sufficiently small the indices j^* and j^{**} will be chosen from the same set as in the case of the solution x.

As the choice of i^*, j^* and j^{**} is equivalent to the choice of the search direction, we obtain $d \in B(x)$. Therefore, we have proved the lemma.

Lemma 4.2: $C : D \to X$ in the equalization method for SGTP is closed on D.

Proof: We already know that X and D are closed. Moreover $C : D \to X$ is a point-point map (so also a point-set map). It remains to show that the conditions $d^{(k)} \in D$, $d^{(k)} \to d$, $x^{(k)} = C(d^{(k)})$, $x^{(k)} \to x$ imply $x = C(d)$.

Similarly as in the proof of the previous lemma, we observe that for almost all k, the equality $d_k = d$ holds. It follows that for almost all k the equality $x^{(k)} = C(d)$ also holds. This implies that almost all the elements of the sequence $\{x^{(k)}\}$ are equal to $C(d)$. This in turn, together with the convergence of the sequence to the point x, implies that $x = C(d)$. This means that C is closed.

Lemma 4.3: $A : X \to X$ is closed on X.

Proof: The map $A : X \to X$ has the form $A = CB$, where $B : X \to D$ and $C : D \to X$ are closed on X and D, respectively. Moreover, we know that D is compact. The lemma follows then directly from Corollary 1 to Theorem 7.3.2 ([10], p. 325; see also Corollary 1, [17], p. 205).

Theorem 4.4: $A : X \to X$ in the equalization method for SGTP is convergent, in the sense that either the algorithm stops after a finite number of steps or all the accumulation points of the generated sequence of solutions belong to the set of optimal solutions $\Omega = \{x : \max_{i=1,\ldots,m} \{w_i\} = 0\}$.

Proof: The set $\Omega = \{x : \max_{i=1,\ldots,m} \{w_i\} = 0\}$ is a nonempty and closed subset of X. This implies, in particular, that A is closed on the closure of Ω. Moreover, we know that the sequence of solutions generated by the algorithm is contained in X, which is a compact set. Let us define the function $\alpha(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} f_j(x_j)$, (i.e. $\alpha(x)$ is equal to the objective function of the problem). It is straightforward to see that at each step of the algorithm the value of the function strictly decreases. Thus by the convergence theorem (Theorem 7.3.4, [10], p. 326; see also global convergence theorem, [17], p. 206) the equalization method for the SGTP is convergent.

5. Computational experiments

Some test problems were randomly generated and solved by the proposed method. Two types of demand distributions were considered: uniform $U(0, u)$ and exponential $\exp(\lambda)$, where u and λ were chosen uniformly at random from the intervals $\langle 15, 20)$ and $\langle 0.5, 0.6)$, respectively. In both cases, the unit transportation costs were chosen from the interval $\langle 5, 10)$, the surplus costs from the interval $\langle 1, 2)$, shortage costs from the interval $\langle 5, 10)$, reduction ratios from the interval $\langle 0.8, 0.9)$ and the supply from each source point from the interval $\langle 10, 20)$. In the case of the problems with uniformly distributed demand, the solution to Eq. (22) was derived using the simplified formula, while in the case of the exponential distribution, the one-dimensional Newton method was used. The algorithm was implemented in Java SE and run on a standard PC with Intel® Core™ i7-2670 QM CPU @2.20 GHz. For both types of distributions, 1000 randomly generated problems of four sizes were solved: $(m, n) = (10, 10)$, $(10, 20)$, $(100, 100)$ and $(100, 200)$ – 8000 test problems in total. The running times in milliseconds (average, standard deviation, minimum and maximum) are presented in Table 1.

Table 1. Running times in milliseconds

Problem type	$U(0, u)$ 10×10	$U(0, u)$ 10×20	$U(0, u)$ 100×100	$U(0, u)$ 100×200	exp(λ) 10×10	exp(λ) 10×20	exp(λ) 100×100	exp(λ) 100×200
AVG	0.144	0.796	292.410	3774.901	2.081	6.479	1942.263	9046.705
ST DEV	0.376	2.395	288.932	2505.025	8.557	40.364	2025.620	6504.964
MIN	0.015	0.094	31.000	437.000	0.110	0.470	157.000	1249.000
MAX	6.099	32.557	2310.000	19221.000	184.221	1012.440	17347.000	50279.000

As we can see, the algorithm is very fast – the running times are less than a second in the case of smaller problems and less than a minute in the case of bigger ones.

6. Conclusion

The equalization method for the stochastic generalized transportation problem has been presented. This is a version of equalization method for the nonlinear generalized transportation problem, presented in [2]. The method converges to the KKT point, which is sufficient to reach the global optimum as the objective function is in this case convex. The computational experiments show that this method is fast. Also, simplicity is one of its advantages.

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